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Research Article

Analytical Treatment of the Generalized Hirota-Satsuma-Ito Equation Arising in Shallow Water Wave

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In the current study, an analytical treatment is studied starting from the $(2+1)$-dimensional generalized Hirota-Satsuma-Ito (HSI) equation. Based on the equation, we first establish the evolution equation and obtain rational function solutions by means of the bilinear form with the help of the Hirota bilinear operator. Then, by the suggested method, the periodic, cross-kink wave solutions are also obtained. Also, the semi-inverse variational principle (SIVP) will be utilized for the generalized HSI equation. Two major cases were investigated from two different techniques. Moreover, the improved tan $(\chi(\xi))$ method on the generalized Hirota-Satsuma-Ito equation is probed. The 3D, density, and contour graphs illustrating some instances of got solutions have been demonstrated from a selection of some suitable parameters. The existing conditions are handled to discuss the available got solutions. The current work is extensively utilized to report plenty of attractive physical phenomena in the areas of shallow water waves and so on.

1. Introduction

Nonlinear partial differential equations containing time variables are generally referred to as nonlinear evolution equations (NLEEs) which can describe the state or process changing along with times in physics, dynamics, and other nature sciences. For the past decades, a variety of methods have sprung up to obtain exact solutions of NLEEs such as the homogeneous balance method [1], the generalized auxiliary equation technique [2], the inverse scattering method [3], the homotopy perturbation method [4], the optimal Galerkin-homotopy asymptotic method [5, 6], the tan $(\phi/2)$-expansion method [7], new Kudryashov's method [8], the function expansion method [9], the standard truncated Painlevé expansion method [10], Hirota's bilinear method [11–15], He's variational principle [16, 17], binary Darboux transformation [18], Lie group analysis [19, 20], Bäcklund transformation method [21], and the multiple Exp-function method [22]. Based on the above methods, plenty of exact solutions including soliton solution [23], lump solution [24], interaction solution [25], and rational solution [26] have been derived. In [27], some novel exact analytical solutions, such as soliton wave, periodic wave, and singular, and kink-singular wave solutions to the fractional Whitham-Broer-Kaup and generalized Hirota-Satsuma coupled KdV equations were investigated.

In this paper, we consider the $(3+1)$-dimensional generalized Hirota-Satsuma-Ito (HSI) shallow water wave equation which will be read [28] as

$$\Psi_{xxxt} + 3(\Psi_x\Psi_t)_x + \delta_1\Psi_{yt} + \delta_2\Psi_{xx} + \delta_3\Psi_{xy} + \delta_4\Psi_{xt} + \delta_5\Psi_{yy} = 0, \tag{1}$$

by using the following bilinear transformation

$$\Psi = 2(\ln f)_x. \tag{2}$$

Equation (1) is transformed to the bilinear form as below:

$$\left(D_x^3D_t + \delta_1 D_yD_t + \delta_2 D_x^2 + \delta_3 D_xD_y + \delta_4 D_xD_t + \delta_5 D_y^2\right)\mathfrak{f}.\mathfrak{f} = 0, \tag{3}$$

in which

$$\begin{aligned}
D_x^2\mathfrak{f}.\mathfrak{f} &= 2\left(\mathfrak{f}\mathfrak{f}_{xx} - \mathfrak{f}_x^2\right),\\
D_y^2\mathfrak{f}.\mathfrak{f} &= 2\left(\mathfrak{f}\mathfrak{f}_{yy} - \mathfrak{f}_y^2\right),\\
D_xD_y\mathfrak{f}.\mathfrak{f} &= 2\left(\mathfrak{f}\mathfrak{f}_{xy} - \mathfrak{f}_x\mathfrak{f}_y\right),\\
D_xD_t\mathfrak{f}.\mathfrak{f} &= 2\left(\mathfrak{f}\mathfrak{f}_{xt} - \mathfrak{f}_x\mathfrak{f}_t\right),\\
D_x^3D_t\mathfrak{f}.\mathfrak{f} &= 2\left(\mathfrak{f}\mathfrak{f}_{xxxt} - \mathfrak{f}_t\mathfrak{f}_{xxx} - 3\mathfrak{f}_x\mathfrak{f}_{xxt} + 3\mathfrak{f}_{xx}\mathfrak{f}_{xt}\right).
\end{aligned} \tag{4}$$

Firstly, Hirota and Satsuma introduced the $(1+1)$-dimensional Hirota-Satsuma shallow water wave equation as a model equation describing the unidirectional propagation of shallow water waves [29], where we can write as

$$\Psi_t - \Psi_{xxt} - 3\Psi\Psi_t + 3\Psi_x\int_x^{\infty}\Psi_t dx + \Psi_x = 0 \Rightarrow \left(D_xD_t - D_x^3D_t + D_x^2\right)\mathfrak{f}.\mathfrak{f} = 0, \tag{5}$$

by applying the bilinear transformation $\Psi = 2(\ln f)_{xx}$. Also, by introducing the Hirota bilinear method in integrability of nonlinear systems, the $(2+1)$-dimensional Hirota-Satsuma shallow water wave equation [30] was studied as

$$\Psi_{xxxt} + 3(\Psi_x\Psi_t)_x + \Psi_{yt} + \Psi_{xx} = 0 \Rightarrow \left(D_x^3D_t + D_yD_t + D_x^2\right)\mathfrak{f}.\mathfrak{f} = 0, \tag{6}$$

by using the bilinear transformation $\Psi = 2(\ln f)_x$.

Chen and coauthors proposed the $(3+1)$-dimensional Hirota-Satsuma-Ito-like equation to describe the wave motion in fluid dynamics and shallow water [31]. Also, Liu et al. [32] investigated the N-soliton solution to construct the $(2+1)$-dimensional generalized Hirota-Satsuma-Ito equation, from which some localized waves such as line solitons, lumps, periodic solitons, and their interactions. Kuo and Ma [33] studied on resonant multisoliton solutions to the $(2+1)$-dimensional Hirota-Satsuma-Ito equations and the existence and nonexistence of solutions. Kaur and Wazwaz [34] used the bilinear form to the new reduced form of the $(3+1)$-dimensional generalized BKP equation and obtained lump solutions with sufficient and necessary conditions. A variety of lump solutions, generated from quadratic functions, for the $(3+1)$-dimensional BKP-Boussinesq equation have been obtained by using the Hirota bilinear form in [35]. Also, the same authors obtained the optical soliton solutions to the Schrödinger-Hirota equation [36]. The authors of [37] obtained the lump solutions by making use of Hirota bilinear form to the $(3+1)$-dimensional generalized KP-Boussinesq equation. By using Hirota's bilinear form and the extended Ansatz function method, Liu and Ye got the new exact periodic cross-kink wave solutions for the $(2+1)$-dimensional KdV equation [38]. Liu and Xiong [39] obtained abundant multiwave, breather wave, and lump solutions by using the three wave method, the homoclinic breather approach, and the Hirota bilinear method for A variable-coefficient Boiti-Leon-Manna-Pempinelli (BLMP) equation. Also, Liu and He [40] utilized the Hirota bilinear form and concluded abundant lump solutions and lump-kink solutions of the new $(3+1)$-dimensional generalized KP equation. Some three-wave solutions including kinky periodic solitary wave, periodic soliton, and kink solutions have been obtained to the $(3+1)$-dimensional BLMP equation by the extended three-wave approach and the Hirota bilinear method in [41].

The outline of this paper is organized as follows. In Section 2, the new periodic solutions and multiple wave solutions of the $(2+1)$-dimensional generalized HSI equation will be obtained by applying the Hirota bilinear method; in addition, the corresponding three-dimensional, contour, and density plots vividly show the physical structure of the periodic wave solutions. In Section 3, carrying the bilinear method to the cross-kink wave solutions will be obtained via choosing the specified function. In addition, we will plot several groups of maps to illustrate the cross-kink of the corresponding solutions by symbolic computation. We gave three cases of solitary solutions with the semi-inverse variational principle in Section 4. Finally, the improved tan $(\chi(\xi))$ method and its application are given and investigated in Section 5. A few conclusions and outlook will be given in the final section.

2. New Periodic Wave Solutions for Generalized HSI Eq

By employing Hirota operator [42] for Equation (1), we have

$$f = a_1H_1 + a_2H_2 + a_3H_3 + a_4H_4, \tag{7}$$

$$\begin{aligned}
H_1 &= \exp\,(\Omega_1x + \Omega_2y + \Omega_3t),\\
H_2 &= \exp\,(-\Omega_1x - \Omega_2y - \Omega_3t),\\
H_3 &= \cos\,(\Omega_4x + \Omega_5y + \Omega_6t),\\
H_4 &= \cosh\,(\Omega_7x + \Omega_8y + \Omega_9t),
\end{aligned} \tag{8}$$

$$\begin{aligned}
H_5 &= \sin\,(\Omega_4x + \Omega_5y + \Omega_6t),\\
H_6 &= \sinh\,(\Omega_7x + \Omega_8y + \Omega_9t),
\end{aligned} \tag{9}$$

$$u = 2\frac{\partial}{\partial x}\ln(f) = 2\,\frac{a_1\Omega_1H_1 - a_2\Omega_1H_2 - a_3H_5\Omega_4 + a_4H_6\Omega_7}{f}, \tag{10}$$

where $\Omega_i, i = 1, \cdots, 9, a_i, j = 1, \cdots, 4$, are free values. By putting (10) into Equation (3), one gets

$$\begin{cases} 2a_4^2\left(4\Omega_7^3\Omega_9 + \Omega_7^2\delta_2 + \Omega_7\Omega_8\delta_3 + \Omega_7\Omega_9\delta_4 + \Omega_8^2\delta_5 + \Omega_8\Omega_9\delta_1\right) = 0, \\ 2a_3a_4\left(\Omega_4^3\Omega_6 - 3\Omega_4^2\Omega_7\Omega_9 - 3\Omega_4\Omega_6\Omega_7^2 + \Omega_7^3\Omega_9 - \Omega_4^2\delta_2 - \Omega_4\Omega_5\delta_3 - \Omega_4\Omega_6\delta_4 - \Omega_5^2\delta_5 - \Omega_5\Omega_6\delta_1 + \Omega_7^2\delta_2 + \Omega_7\Omega_8\delta_3 + \Omega_7\Omega_9\delta_4 + \Omega_8^2\delta_5 + \Omega_8\Omega_9\delta_1\right) = 0, \\ 2a_1a_4\left(\Omega_1^3\Omega_3 + 3\Omega_1^2\Omega_7\Omega_9 + 3\Omega_1\Omega_3\Omega_7^2 + \Omega_7^3\Omega_9 + \Omega_1^2\delta_2 + \Omega_1\Omega_2\delta_3 + \Omega_1\Omega_3\delta_4 + \Omega_2^2\delta_5 + \Omega_2\Omega_3\delta_1 + \Omega_7^2\delta_2 + \Omega_7\Omega_8\delta_3 + \Omega_7\Omega_9\delta_4 + \Omega_8^2\delta_5 + \Omega_8\Omega_9\delta_1\right) = 0, \\ -2a_3a_4\left(\Omega_4^3\Omega_9 + 3\Omega_4^2\Omega_6\Omega_7 - 3\Omega_4\Omega_7^2\Omega_9 - \Omega_6\Omega_7^3 - 2\Omega_4\Omega_7\delta_2 - \Omega_4\Omega_8\delta_3 - \Omega_4\Omega_9\delta_4 - \Omega_5\Omega_7\delta_3 - 2\Omega_5\Omega_8\delta_5 - \Omega_5\Omega_9\delta_1 - \Omega_6\Omega_7\delta_4 - \Omega_6\Omega_8\delta_1\right) = 0, \\ -2a_1a_4\left(\Omega_1^3\Omega_9 + 3\Omega_1^2\Omega_3\Omega_7 + 3\Omega_1\Omega_7^2\Omega_9 + \Omega_3\Omega_7^3 + 2\Omega_1\Omega_7\delta_2 + \Omega_1\Omega_8\delta_3 + \Omega_1\Omega_9\delta_4 + \Omega_2\Omega_7\delta_3 + 2\Omega_2\Omega_8\delta_5 + \Omega_2\Omega_9\delta_1 + \Omega_3\Omega_7\delta_4 + \Omega_3\Omega_8\delta_1\right) = 0, \\ 2a_3^2\left(4\Omega_4^3\Omega_6 - \Omega_4^2\delta_2 - \Omega_4\Omega_5\delta_3 - \Omega_4\Omega_6\delta_4 - \Omega_5^2\delta_5 - \Omega_5\Omega_6\delta_1\right) = 0, \\ 2a_1a_3\left(\Omega_1^3\Omega_3 - 3\Omega_1^2\Omega_4\Omega_6 - 3\Omega_1\Omega_3\Omega_4^2 + \Omega_4^3\Omega_6 + \Omega_1^2\delta_2 + \Omega_1\Omega_2\delta_3 + \Omega_1\Omega_3\delta_4 + \Omega_2^2\delta_5 + \Omega_2\Omega_3\delta_1 - \Omega_4^2\delta_2 - \Omega_4\Omega_5\delta_3 - \Omega_4\Omega_6\delta_4 - \Omega_5^2\delta_5 - \Omega_5\Omega_6\delta_1\right) = 0, \\ 2a_1a_3\left(\Omega_1^3\Omega_6 + 3\Omega_1^2\Omega_3\Omega_4 - 3\Omega_1\Omega_4^2\Omega_6 - \Omega_3\Omega_4^3 + 2\Omega_1\Omega_4\delta_2 + \Omega_1\Omega_5\delta_3 + \Omega_1\Omega_6\delta_4 + \Omega_2\Omega_4\delta_3 + 2\Omega_2\Omega_5\delta_5 + \Omega_2\Omega_6\delta_1 + \Omega_3\Omega_4\delta_4 + \Omega_3\Omega_5\delta_1\right) = 0, \\ 8a_1a_2\left(4\Omega_1^3\Omega_3 + \Omega_1^2\delta_2 + \Omega_1\Omega_2\delta_3 + \Omega_1\Omega_3\delta_4 + \Omega_2^2\delta_5 + \Omega_2\Omega_3\delta_1\right) = 0. \end{cases} \tag{11}$$

Solving the above equations, we get the following.

Case 1.

$$\Psi_1 = \Psi_0 - \frac{\left[2a_3 \sin(\Xi_1)\Omega_4\right]}{\left[a_1 e^{-\Omega_2\delta_5 t/\delta_1 + \Omega_2 y} + a_2 e^{\Omega_2\delta_5 t/\delta_1 - \Omega_2 y} + a_3 \cos(\Xi_1) + a_4 \cosh(\Omega_9 t - \Omega_9\delta_1 y/\delta_5)\right]}, \tag{12}$$

$$\Xi_1 = \Omega_4 x - \frac{\Omega_4\left(\Omega_4^2\delta_5 + \delta_1\delta_3 - \delta_4\delta_5\right)y}{\delta_1\delta_5}, \tag{13}$$

where $a_2, a_3, a_4, \Omega_2, \Omega_4, \Omega_9$ are arbitrary values. Also, we need to satisfy the condition $\Omega_7 \neq 0, \delta_1\delta_5 \neq 0$. By assigning particular values of the parameters, we can easily observe the wave motion as periodic waves in parallel to the x axis in Figure 1.

Case 2.

$$\Psi_2 = \Psi_0 + \frac{\left[2a_1\Omega_1 e^{\Xi_2} - 2a_2\Omega_1 e^{-\Xi_2}\right]}{\left[a_1 e^{\Xi_2} + a_2 e^{-\Xi_2} + a_3 \cos(-\Omega_5\delta_5 t/\delta_1 + \Omega_5 y) + a_4 \cosh(\Omega_9 t - \Omega_9\delta_1 y/\delta_5)\right]}, \tag{14}$$

$$\Xi_2 = \Omega_1 x + \frac{\Omega_1\left(\Omega_1^2\delta_5 - \delta_1\delta_3 + \delta_4\delta_5\right)y}{\delta_1\delta_5}, \tag{15}$$

where $a_2, a_3, a_4, \Omega_1, \Omega_5, \Omega_9$ are arbitrary values. Also, we need to satisfy the condition $\Omega_1 \neq 0, \delta_1\delta_5 \neq 0$. By assigning particular values of the parameters, we can easily observe the wave motion as periodic waves in the line of $x - y$ axis in which they intersect at one point in Figure 2.

Case 3.

$$\Psi_3 = \Psi_0 + \frac{\left[2a_4 \sinh(\Xi_3)\Omega_7\right]}{\left[a_1 e^{-\Omega_2\delta_5 t/\delta_1 + \Omega_2 y} + a_2 e^{\Omega_2\delta_5 t/\delta_1 - \Omega_2 y} + a_3 \cos(\Omega_6 t - \Omega_6\delta_1 y/\delta_5) + a_4 \cosh(\Xi_3)\right]}, \tag{16}$$

$$\Xi_3 = \Omega_7 x + \frac{\Omega_7\left(\Omega_7^2\delta_5 - \delta_1\delta_3 + \delta_4\delta_5\right)y}{\delta_1\delta_5}, \tag{17}$$

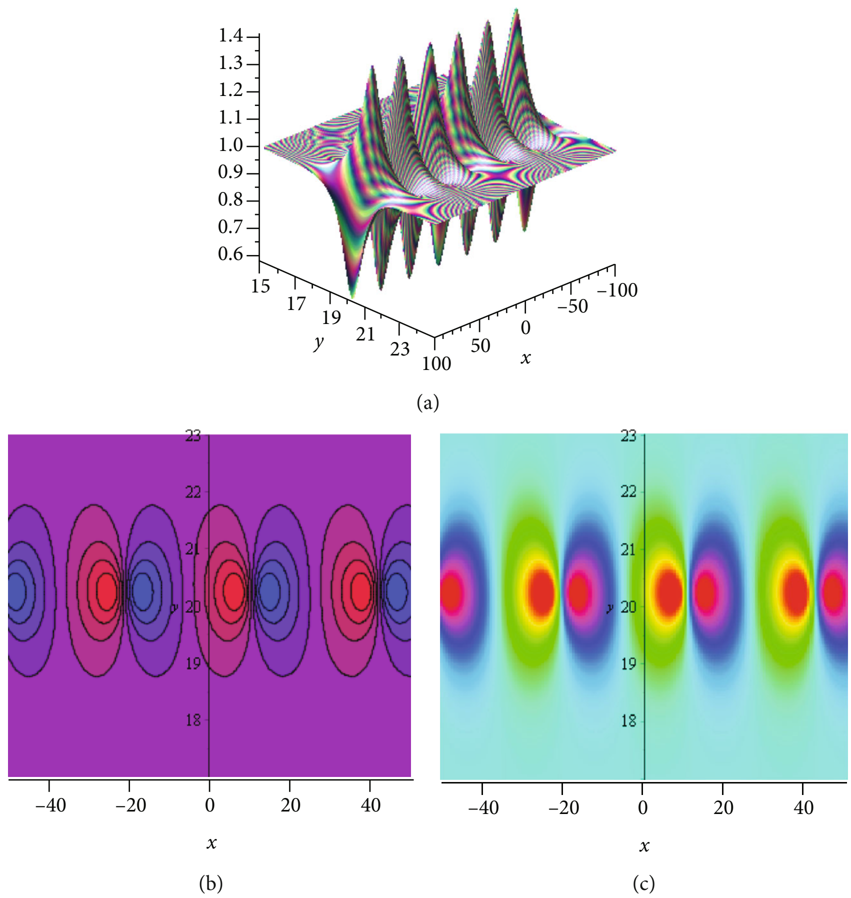


Figure 1: Plot evolution of periodic waves (12) utilizing values $a_1 = .5, a_2 = 1.5, a_3 = 2, a_4 = 1, \Omega_2 = 1.9, \Omega_4 = .2, \Omega_9 = 2, \delta_1 = 1, \delta_2 = 1.3, \delta_3 = 2, \delta_4 = 1.3, \delta_5 = 2, \Psi_0 = 1, t = 10$ and (a) 3D plot, (b) density plot, and (c) contour plot.

where $a_2, a_3, a_4, \Omega_2, \Omega_6, \Omega_7$ are arbitrary values. Also, we need to satisfy the condition $\Omega_7 \neq 0, \delta_1\delta_5 \neq 0$. By assigning particular values of the parameters, we can easily observe the wave motion as periodic waves in the line of $x - y$ axis in which they intersect at one point in Figure 3.

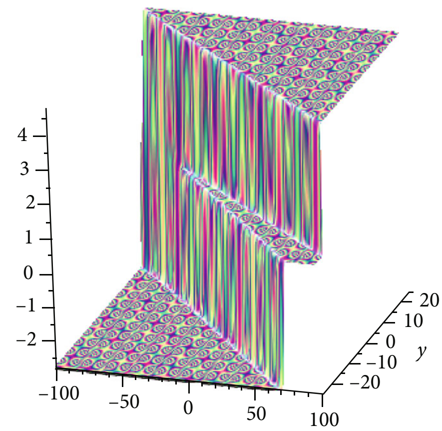
Case 4.

$$\begin{aligned}\Psi_4 &= \Psi_0 + \frac{\left[2a_1\Omega_7 e^{\Xi_4} - 2\,a_2\Omega_7 e^{-\Xi_4} + 2\,a_4 \sinh\left(\Xi_5\right)\Omega_7\right]}{\left[a_1 e^{\Xi_4} + a_2 e^{-\Xi_4} + a_3 \cos\left(\Omega_6 t - \Omega_6\delta_1 y/\delta_5\right) + a_4 \cosh\left(\Xi_5\right)\right]},\\ \Xi_4 &= \Omega_7 x + \frac{\Omega_7\left(\Omega_7{}^2\delta_5 - \delta_1\delta_3 + \delta_4\delta_5\right)y}{\delta_1\delta_5},\\ \Xi_5 &= \Omega_7 x + \frac{\Omega_7\left(\Omega_7{}^2\delta_5 - \delta_1\delta_3 + \delta_4\delta_5\right)y}{\delta_1\delta_5},\end{aligned} \tag{18}$$

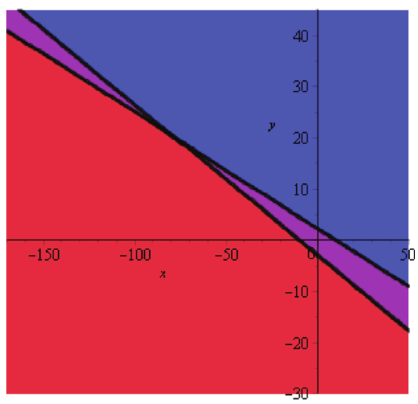
where $a_2, a_3, a_4, \Omega_5, \Omega_6, \Omega_7$ are arbitrary values.

Case 5.

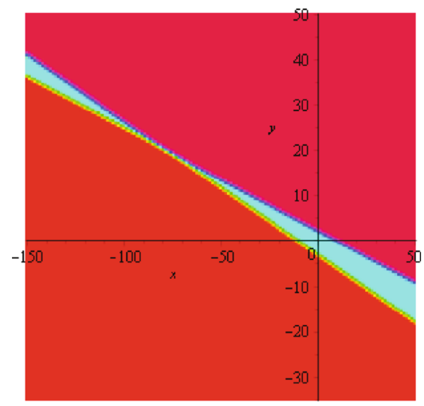
$$\Psi_5 = \Psi_0 + 2\,\frac{a_4 \sinh\left(\Omega_7 x + y\Omega_8\right)\Omega_7}{a_1 e^{\Omega_3 t} + a_2 e^{-\Omega_3 t} + a_3 \cos\left(\Omega_6 t\right) + a_4 \cosh\left(\Omega_7 x + y\Omega_8\right)}, \tag{19}$$



(a)



(b)



(c)

Figure 2: Plot evolution of periodic waves (14) utilizing values $a_1 = .5, a_2 = 1.5, a_3 = 2, a_4 = 1, \Omega_1 = 1.9, \Omega_5 = .2, \Omega_9 = 2, \delta_1 = 1, \delta_2 = 1.3, \delta_3 = 2, \delta_4 = 1.3, \delta_5 = 2, \Psi_0 = 1, t = 10$ and (a) 3D plot, (b) density plot, and (c) contour plot.

where $a_2, a_3, a_4, \Omega_3, \Omega_6, \Omega_7$ are arbitrary values.

Case 6.

$$\Psi_6 = \Psi_0 + 2\,\frac{a_1\Omega_1 e^{\Omega_1 x+\Omega_2 y} - a_2\Omega_1 e^{-\Omega_1 x-\Omega_2 y}}{a_1 e^{\Omega_1 x+\Omega_2 y} + a_2 e^{-\Omega_1 x-\Omega_2 y} + a_3\cos(\Omega_6 t) + a_4\cosh(\Omega_9 t)}, \tag{20}$$

where $a_2, a_3, a_4, \Omega_1, \Omega_2, \Omega_6, \Omega_9$ are arbitrary values.

Case 7.

$$\Psi_7 = \Psi_0 - 2\,\frac{a_3\sin(\Omega_4 x + \Omega_5 y)\Omega_4}{a_1 e^{\Omega_3 t} + a_2 e^{-\Omega_3 t} + a_3\cos(\Omega_4 x + \Omega_5 y) + a_4\cosh(\Omega_9 t)}, \tag{21}$$

where $a_2, a_3, a_4, \Omega_3, \Omega_4, \Omega_5, \Omega_9$ are arbitrary values.

Case 8.

$$\Psi_8 = \Psi_0 + 2\,\frac{a_1\Omega_7 e^{\Omega_7 x+y\Omega_8} - a_2\Omega_7 e^{-\Omega_7 x-y\Omega_8} + a_4\sinh(\Omega_7 x + y\Omega_8)\Omega_7}{a_1 e^{\Omega_7 x+y\Omega_8} + a_2 e^{-\Omega_7 x-y\Omega_8} + a_3\cos(\Omega_6 t) + a_4\cosh(\Omega_7 x + y\Omega_8)}, \tag{22}$$

where $a_2, a_3, a_4, \Omega_6, \Omega_7, \Omega_8$ are arbitrary values.

Case 9.

$$\Psi_9 = \Psi_0 + 2\,\frac{1/2a_1\sqrt{-\delta_4}e^{G_1} - 1/2a_2\sqrt{-\delta_4}e^{-G_1} - a_3\sin(G_2)\Omega_4 + 1/2a_4\sinh(G_1)\sqrt{-\delta_4}}{a_1e^{G_1} + a_2e^{-G_1} + a_3\cos(G_2) + a_4\cosh(G_1)}, \tag{23}$$

$$\begin{aligned} G_1 &= \frac{8}{3}\frac{\Omega_4\Xi_6\Omega_5\sqrt{-\delta_4}t}{\delta_4\left(4\Omega_4^2-\delta_4\right)^2} + \frac{1}{2}\sqrt{-\delta_4}x, \\ G_2 &= \frac{4}{3}\frac{\Omega_5\Xi_6 t}{\left(4\Omega_4^2-\delta_4\right)^2} + \Omega_4 x + \Omega_5 y, \\ \Xi_6 &= 4\Omega_4^2\delta_3 + 4\Omega_4\Omega_5\delta_5 - \delta_3\delta_4, \end{aligned} \tag{24}$$

where $a_2, a_3, a_4, \Omega_6, \Omega_7, \Omega_8$ are arbitrary values. Also, we need to satisfy condition $4\Omega_4^2 - \delta_4 \neq 0, \delta_4 < 0$. By assigning particular values of the parameters, we can easily observe the wave motion as periodic waves in parallel to the y axis in Figure 4.

Case 10.

$$\begin{aligned} \Psi_{10} &= \Psi_0 + 2\,\frac{a_1\Omega_7e^{G_1} - a_2\Omega_7e^{-G_1} - a_3\sin(G_2)\Omega_4 + a_4\sinh(G_1)\Omega_7}{a_1e^{G_1} + a_2e^{-G_1} + a_3\cos(G_2) + a_4\cosh(G_1)}, \\ \Xi_7 &= 2\Omega_4^2 - 2\Omega_7^2 - \delta_4, \\ G_1 &= \frac{4}{3}\frac{\delta_3^2\Omega_4^2\Omega_7 t}{\delta_5(\Xi_7)^2} + \Omega_7 x + \frac{1}{2}\frac{\Omega_7\delta_3\left(4\Omega_7^2+\delta_4\right)y}{\delta_5(\Xi_7)}, \\ G_2 &= -\frac{4}{3}\frac{\Omega_4\Omega_7^2\delta_3^2 t}{(\Xi_7)^2\delta_5} + \Omega_4 x - \frac{1}{2}\frac{\Omega_4\delta_3\left(4\Omega_4^2-\delta_4\right)y}{(\Xi_7)\delta_5}, \end{aligned} \tag{25}$$

where $a_2, a_3, a_4, \Omega_4, \Omega_7$ are arbitrary values. Also, we need to satisfy the condition $(2\Omega_4^2 - 2\Omega_7^2 - \delta_4)\delta_5 \neq 0$. By assigning particular values of the parameters, we can easily observe the wave motion as periodic waves in the line of $x-y$ axis in which they intersect at one point in Figure 5.

Case 11.

$$\begin{aligned} \Psi_{11} &= \Psi_0 + 2\,\frac{a_1\Omega_1e^{G_1} - a_2\Omega_1e^{-G_1} - a_3\sin(G_2)\Omega_4 + a_4\sinh(G_1)\Omega_1}{a_1e^{G_1} + a_2e^{-G_1} + a_3\cos(G_2) + a_4\cosh(G_1)}, \\ \Xi_8 &= \frac{\left(3\Omega_1\delta_3 + 2\Omega_2\delta_5\right)^2}{\left(\Omega_1^2 - 3\Omega_4^2 + \delta_4\right)^2}, \end{aligned} \tag{26}$$

$$\begin{aligned} G_1 &= \frac{1}{3}\frac{\Omega_4^2\Xi_8 t}{\Omega_1\delta_5} + \Omega_1 x + \Omega_2 y, \\ G_2 &= -\frac{1}{3}\frac{\Omega_4\Xi_8 t}{\delta_5} + \Omega_4 x + \frac{\Omega_4\left(3\Omega_1^3\delta_3 + 3\Omega_1^2\Omega_2\delta_5 + 3\Omega_1\Omega_4^2\delta_3 - \Omega_2\Omega_4^2\delta_5 + \Omega_2\delta_4\delta_5\right)y}{\Omega_1\delta_5\left(\Omega_1^2 - 3\Omega_4^2 + \delta_4\right)}, \end{aligned} \tag{27}$$

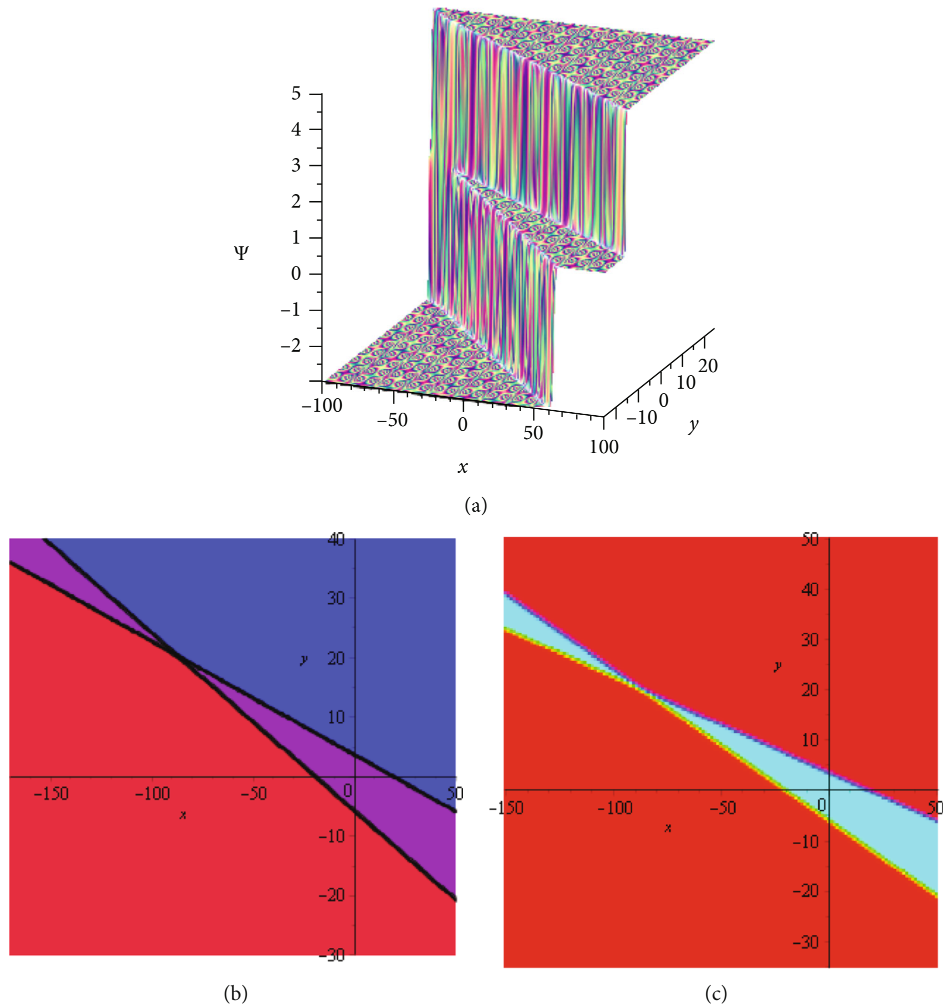


FIGURE 3: Plot evolution of periodic waves (16) utilizing values $a_1 = .5, a_2 = 1.5, a_3 = 2, a_4 = 1, \Omega_2 = 1.9, \Omega_6 = .2, \Omega_7 = 2, \delta_1 = 1, \delta_2 = 1.3, \delta_3 = 2, \delta_4 = 1.3, \delta_5 = 2, \Psi_0 = 1, t = 10$ and (a) 3D plot, (b) density plot, and (c) contour plot.

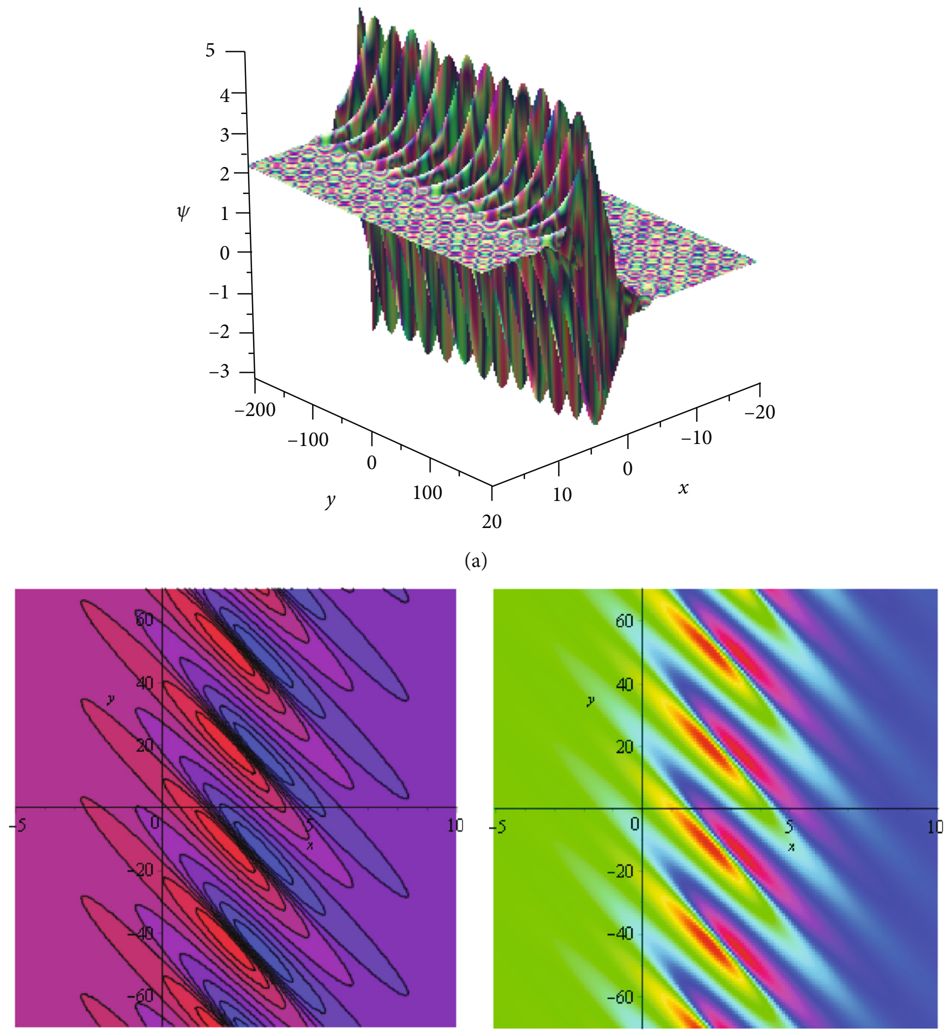
where $a_2, a_3, a_4, \Omega_1, \Omega_2, \Omega_4$ are arbitrary values. Also, we need to satisfy the condition $\Omega_1\delta_5(\Omega_1{}^2 - 3\,\Omega_4{}^2 + \delta_4) \neq 0$. By assigning particular values of the parameters, we can easily observe the wave motion as periodic waves in the line of $x - y$ axis in which they intersect at one point in Figure 6.

Case 12.

$$\Psi_{12} = \Psi_0 + 2\,\frac{a_1\sqrt{3\,\Omega_4{}^2 - \delta_4}e^{G_1} - a_2\sqrt{3\,\Omega_4{}^2 - \delta_4}e^{-G_1} - a_3\sin(G_2)\Omega_4 + a_4\sinh(G_1)\sqrt{3\,\Omega_4{}^2 - \delta_4}}{a_1e^{G_1} + a_2e^{-G_1} + a_3\cos(G_2) + a_4\cosh(G_1)}, \quad (28)$$

$$G_1 = \frac{\sqrt{3\,\Omega_4{}^2 - \delta_4}\,\Xi_9 t}{\delta_5} + \sqrt{3\,\Omega_4{}^2 - \delta_4}\,x - \frac{3}{2}\frac{\delta_3\sqrt{3\,\Omega_4{}^2 - \delta_4}\,y}{\delta_5},$$
$$G_2 = -\frac{\left(3\,\Omega_4{}^2 - \delta_4\right)\Xi_9 t}{\Omega_4\delta_5} + \Omega_4 x + \Omega_5 y, \quad (29)$$

$$\Xi_9 = \frac{1}{12}\frac{(3\,\Omega_4\delta_3 + 2\,\Omega_5\delta_5)^2}{\left(16\,\Omega_4{}^4 - 8\,\Omega_4{}^2\delta_4 + \delta_4{}^2\right)}, \quad (30)$$



(a) (b) (c)

Figure 4: Plot evolution of periodic waves (23) utilizing values $a_1 = .5, a_2 = 1.5, a_3 = 2, a_4 = 1, \Omega_4 = 1.9, \Omega_5 = .2, \Omega_7 = 2, \delta_1 = 1, \delta_2 = 1.3, \delta_3 = 2, \delta_4 = -1.3, \delta_5 = 2, \Psi_0 = 1, t = 10$ and (a) 3D plot, (b) density plot, and (c) contour plot.

where $a_2, a_3, a_4, \Omega_1, \Omega_2, \Omega_4$ are arbitrary values. Also, we need to satisfy condition $3\,\Omega_4{}^2 - \delta_4 > 0$. By assigning particular values of the parameters, we can easily observe the wave motion as periodic waves in the line of $x - y$ axis in which they intersect at one point in Figure 7.

Case 13.

$$
\begin{aligned}
\Psi_{13} &= \Psi_0 + 2\,\frac{a_1\Omega_1 e^{G_1} - a_2\Omega_1 e^{-G_1} - a_3 \sin(G_2)\Omega_4 + a_4 \sinh(G_1)\Omega_1}{a_1 e^{G_1} + a_2 e^{-G_1} + a_3 \cos(G_2) + a_4 \cosh(G_1)},\\
\Xi_{10} &= \frac{1}{3}\frac{\left(3\,\Omega_1\delta_3 + 2\,\Omega_2\delta_5\right)^2}{\left(\Omega_1{}^2 - 3\,\Omega_4{}^2 + \delta_4\right)^2},\\
G_1 &= \frac{\Omega_4{}^2\Xi_{10}t}{\Omega_1\delta_5} + \Omega_1 x + \Omega_2 y,\\
G_2 &= -\frac{\Omega_4\Xi_{10}t}{\delta_5} + \Omega_4 x + \frac{\Omega_4\left(3\,\Omega_1{}^3\delta_3 + 3\,\Omega_1{}^2\Omega_2\delta_5 + 3\,\Omega_1\Omega_4{}^2\delta_3 - \Omega_2\Omega_4{}^2\delta_5 + \Omega_2\delta_4\delta_5\right)y}{\Omega_1\delta_5\left(\Omega_1{}^2 - 3\,\Omega_4{}^2 + \delta_4\right)},
\end{aligned}
\tag{31}
$$

where $a_2, a_3, a_4, \Omega_1, \Omega_2, \Omega_4$ are arbitrary values. Also, it needs to satisfy $3\,\Omega_4{}^2 - \delta_4 > 0$.

3. New Cross-Kink Wave Solutions for Generalized HSI Eq

Based on the Hirota operator [42] for Equation (1), we get

$$f = a_1 H_1 + a_2 H_2 + a_3 H_3 + a_4 H_4, \tag{32}$$

$$\begin{aligned} H_1 &= \exp\left(\Omega_1 x + \Omega_2 y + \Omega_3 t\right), \\ H_2 &= \exp\left(-\Omega_1 x - \Omega_2 y - \Omega_3 t\right), \\ H_3 &= \sin\left(\Omega_4 x + \Omega_5 y + \Omega_6 t\right), \\ H_4 &= \sinh\left(\Omega_7 x + \Omega_8 y + \Omega_9 t\right), \\ H_5 &= \cos\left(\Omega_4 x + \Omega_5 y + \Omega_6 t\right), \end{aligned} \tag{33}$$

$$H_6 = \cosh\left(\Omega_7 x + \Omega_8 y + \Omega_9 t\right), \tag{34}$$

$$u = 2\frac{\partial}{\partial x}\ln(f) = 2\,\frac{a_1\Omega_1 H_1 - a_2\Omega_1 H_2 + a_3 H_5 \Omega_4 + a_4 H_6 \Omega_7}{f}, \tag{35}$$

where $\Omega_i, i = 1, \cdots, 9, a_i, j = 1, \cdots, 4$, are free parameters. By inserting (35) into Equation (3), one concludes

$$\begin{cases} 2\,a_4{}^2\left(4\,\Omega_7{}^3\Omega_9 + \Omega_7{}^2\delta_2 + \Omega_7\Omega_8\delta_3 + \Omega_7\Omega_9\delta_4 + \Omega_8{}^2\delta_5 + \Omega_8\Omega_9\delta_1\right) = 0, \\ 2\,a_3a_4\left(\Omega_4{}^3\Omega_6 - 3\,\Omega_4{}^2\Omega_7\Omega_9 - 3\,\Omega_4\Omega_6\Omega_7{}^2 + \Omega_7{}^3\Omega_9 - \Omega_4{}^2\delta_2 - \Omega_4\Omega_5\delta_3 - \Omega_4\Omega_6\delta_4 - \Omega_5{}^2\delta_5 - \Omega_5\Omega_6\delta_1 + \Omega_7{}^2\delta_2 + \Omega_7\Omega_8\delta_3 + \Omega_7\Omega_9\delta_4 + \Omega_8{}^2\delta_5 + \Omega_8\Omega_9\delta_1\right) = 0, \\ 2\,a_1a_4\left(\Omega_1{}^3\Omega_3 + 3\,\Omega_1{}^2\Omega_7\Omega_9 + 3\,\Omega_1\Omega_3\Omega_7{}^2 + \Omega_7{}^3\Omega_9 + \Omega_1{}^2\delta_2 + \Omega_1\Omega_2\delta_3 + \Omega_1\Omega_3\delta_4 + \Omega_2{}^2\delta_5 + \Omega_2\Omega_3\delta_1 + \Omega_7{}^2\delta_2 + \Omega_7\Omega_8\delta_3 + \Omega_7\Omega_9\delta_4 + \Omega_8{}^2\delta_5 + \Omega_8\Omega_9\delta_1\right) = 0, \\ -2\,a_3a_4\left(\Omega_4{}^3\Omega_9 + 3\,\Omega_4{}^2\Omega_6\Omega_7 - 3\,\Omega_4\Omega_7{}^2\Omega_9 - \Omega_6\Omega_7{}^3 - 2\,\Omega_4\Omega_7\delta_2 - \Omega_4\Omega_8\delta_3 - \Omega_4\Omega_9\delta_4 - \Omega_5\Omega_7\delta_3 - 2\,\Omega_5\Omega_8\delta_5 - \Omega_5\Omega_9\delta_1 - \Omega_6\Omega_7\delta_4 - \Omega_6\Omega_8\delta_1\right) = 0 \\ -2\,a_1a_4\left(\Omega_1{}^3\Omega_9 + 3\,\Omega_1{}^2\Omega_3\Omega_7 + 3\,\Omega_1\Omega_7{}^2\Omega_9 + \Omega_3\Omega_7{}^3 + 2\,\Omega_1\Omega_7\delta_2 + \Omega_1\Omega_8\delta_3 + \Omega_1\Omega_9\delta_4 + \Omega_2\Omega_7\delta_3 + 2\,\Omega_2\Omega_8\delta_5 + \Omega_2\Omega_9\delta_1 + \Omega_3\Omega_7\delta_4 + \Omega_3\Omega_8\delta_1\right) = 0, \\ 2\,a_3{}^2\left(4\,\Omega_4{}^3\Omega_6 - \Omega_4{}^2\delta_2 - \Omega_4\Omega_5\delta_3 - \Omega_4\Omega_6\delta_4 - \Omega_5{}^2\delta_5 - \Omega_5\Omega_6\delta_1\right) = 0, \\ 2\,a_1a_3\left(\Omega_1{}^3\Omega_3 - 3\,\Omega_1{}^2\Omega_4\Omega_6 - 3\,\Omega_1\Omega_3\Omega_4{}^2 + \Omega_4{}^3\Omega_6 + \Omega_1{}^2\delta_2 + \Omega_1\Omega_2\delta_3 + \Omega_1\Omega_3\delta_4 + \Omega_2{}^2\delta_5 + \Omega_2\Omega_3\delta_1 - \Omega_4{}^2\delta_2 - \Omega_4\Omega_5\delta_3 - \Omega_4\Omega_6\delta_4 - \Omega_5{}^2\delta_5 - \Omega_5\Omega_6\delta_1\right) = 0, \\ 2\,a_1a_3\left(\Omega_1{}^3\Omega_6 + 3\,\Omega_1{}^2\Omega_3\Omega_4 - 3\,\Omega_1\Omega_4{}^2\Omega_6 - \Omega_3\Omega_4{}^3 + 2\,\Omega_1\Omega_4\delta_2 + \Omega_1\Omega_5\delta_3 + \Omega_1\Omega_6\delta_4 + \Omega_2\Omega_4\delta_3 + 2\,\Omega_2\Omega_5\delta_5 + \Omega_2\Omega_6\delta_1 + \Omega_3\Omega_4\delta_4 + \Omega_3\Omega_5\delta_1\right) = 0, \\ 8\,a_1a_2\left(4\,\Omega_1{}^3\Omega_3 + \Omega_1{}^2\delta_2 + \Omega_1\Omega_2\delta_3 + \Omega_1\Omega_3\delta_4 + \Omega_2{}^2\delta_5 + \Omega_2\Omega_3\delta_1\right) = 0. \end{cases} \tag{36}$$

Solving the above equations, we get the following.

Case 1.

$$\Psi_1 = \Psi_0 - \frac{\left[2\,a_3\cos\left(\Xi_1\right)\Omega_4\right]}{\left[a_1 e^{-\Omega_2\delta_5 t/\delta_1 + \Omega_2 y} + a_2 e^{\Omega_2\delta_5 t/\delta_1 - \Omega_2 y} + a_3\sin\left(\Xi_1\right) + a_4\sinh\left(\Omega_9 t - \Omega_9\delta_1 y/\delta_5\right)\right]}, \tag{37}$$

$$\Xi_1 = \Omega_4 x - \frac{\Omega_4\left(\Omega_4{}^2\delta_5 + \delta_1\delta_3 - \delta_4\delta_5\right)y}{\delta_1\delta_5}, \tag{38}$$

where $a_2, a_3, a_4, \Omega_2, \Omega_4, \Omega_9$ are arbitrary values. Also, we need to satisfy the condition $\Omega_7 \neq 0, \delta_1\delta_5 \neq 0$. By assigning particular values of the parameters, we can easily observe the wave motion as cross-kink waves in parallel to the x axis in Figure 8.

Case 2.

$$\Psi_2 = \Psi_0 + \frac{\left[2\,a_1\Omega_1 e^{\Xi_2} - 2\,a_2\Omega_1 e^{-\Xi_2}\right]}{\left[a_1 e^{\Xi_2} + a_2 e^{-\Xi_2} + a_3\sin\left(-\Omega_5\delta_5 t/\delta_1 + \Omega_5 y\right) + a_4\sinh\left(\Omega_9 t - \Omega_9\delta_1 y/\delta_5\right)\right]}, \tag{39}$$

$$\Xi_2 = \Omega_1 x + \frac{\Omega_1\left(\Omega_1{}^2\delta_5 - \delta_1\delta_3 + \delta_4\delta_5\right)y}{\delta_1\delta_5}, \tag{40}$$

where $a_2, a_3, a_4, \Omega_1, \Omega_5, \Omega_9$ are arbitrary values. Also, we need to satisfy the condition $\Omega_1 \neq 0, \delta_1\delta_5 \neq 0$. By assigning particular values of the parameters, we can easily observe the wave motion as cross-kink waves in the line of $x - y$ axis in which they intersect at one point in Figure 9.

Case 3.

$$\Psi_3 = \Psi_0 + \frac{[2 a_4 \cosh(\Xi_3)\Omega_7]}{\left[a_1 e^{-\Omega_2\delta_5 t/\delta_1 + \Omega_2 y} + a_2 e^{\Omega_2\delta_5 t/\delta_1 - \Omega_2 y} + a_3 \sin(\Omega_6 t - \Omega_6\delta_1 y/\delta_5) + a_4 \sinh(\Xi_3)\right]}, \tag{41}$$

$$\Xi_3 = \Omega_7 x + \frac{\Omega_7\left(\Omega_7^2\delta_5 - \delta_1\delta_3 + \delta_4\delta_5\right)y}{\delta_1\delta_5}, \tag{42}$$

where $a_2, a_3, a_4, \Omega_2, \Omega_6, \Omega_7$ are arbitrary values. Also, we need to satisfy the condition $\Omega_7 \neq 0, \delta_1\delta_5 \neq 0$. By assigning particular values of the parameters, we can easily observe the wave motion as cross-kink waves in the line of $x - y$ axis in which they intersect at one point in Figure 10.

Case 4.

$$\begin{aligned}
\Psi_4 &= \Psi_0 + \frac{\left[2 a_1\Omega_7 e^{\Xi_4} - 2 a_2\Omega_7 e^{-\Xi_4} + 2 a_4 \cosh(\Xi_5)\Omega_7\right]}{\left[a_1 e^{\Xi_4} + a_2 e^{-\Xi_4} + a_3 \sin(\Omega_6 t - \Omega_6\delta_1 y/\delta_5) + a_4 \sinh(\Xi_5)\right]}, \\
\Xi_4 &= \Omega_7 x + \frac{\Omega_7\left(\Omega_7^2\delta_5 - \delta_1\delta_3 + \delta_4\delta_5\right)y}{\delta_1\delta_5}, \\
\Xi_5 &= \Omega_7 x + \frac{\Omega_7\left(\Omega_7^2\delta_5 - \delta_1\delta_3 + \delta_4\delta_5\right)y}{\delta_1\delta_5},
\end{aligned} \tag{43}$$

where $a_2, a_3, a_4, \Omega_5, \Omega_6, \Omega_7$ are arbitrary values.

Case 5.

$$\Psi_5 = \Psi_0 + 2\,\frac{a_4 \cosh(\Omega_7 x + y\Omega_8)\Omega_7}{a_1 e^{\Omega_3 t} + a_2 e^{-\Omega_3 t} + a_3 \sin(\Omega_6 t) + a_4 \sinh(\Omega_7 x + y\Omega_8)}, \tag{44}$$

where $a_2, a_3, a_4, \Omega_3, \Omega_6, \Omega_7$ are arbitrary values.

Case 6.

$$\Psi_6 = \Psi_0 + 2\,\frac{a_1\Omega_1 e^{\Omega_1 x + \Omega_2 y} - a_2\Omega_1 e^{-\Omega_1 x - \Omega_2 y}}{a_1 e^{\Omega_1 x + \Omega_2 y} + a_2 e^{-\Omega_1 x - \Omega_2 y} + a_3 \sin(\Omega_6 t) + a_4 \sinh(\Omega_9 t)}, \tag{45}$$

where $a_2, a_3, a_4, \Omega_1, \Omega_2, \Omega_6, \Omega_9$ are arbitrary values.

Case 7.

$$\Psi_7 = \Psi_0 - 2\,\frac{a_3 \cos(\Omega_4 x + \Omega_5 y)\Omega_4}{a_1 e^{\Omega_3 t} + a_2 e^{-\Omega_3 t} + a_3 \sin(\Omega_4 x + \Omega_5 y) + a_4 \sinh(\Omega_9 t)}, \tag{46}$$

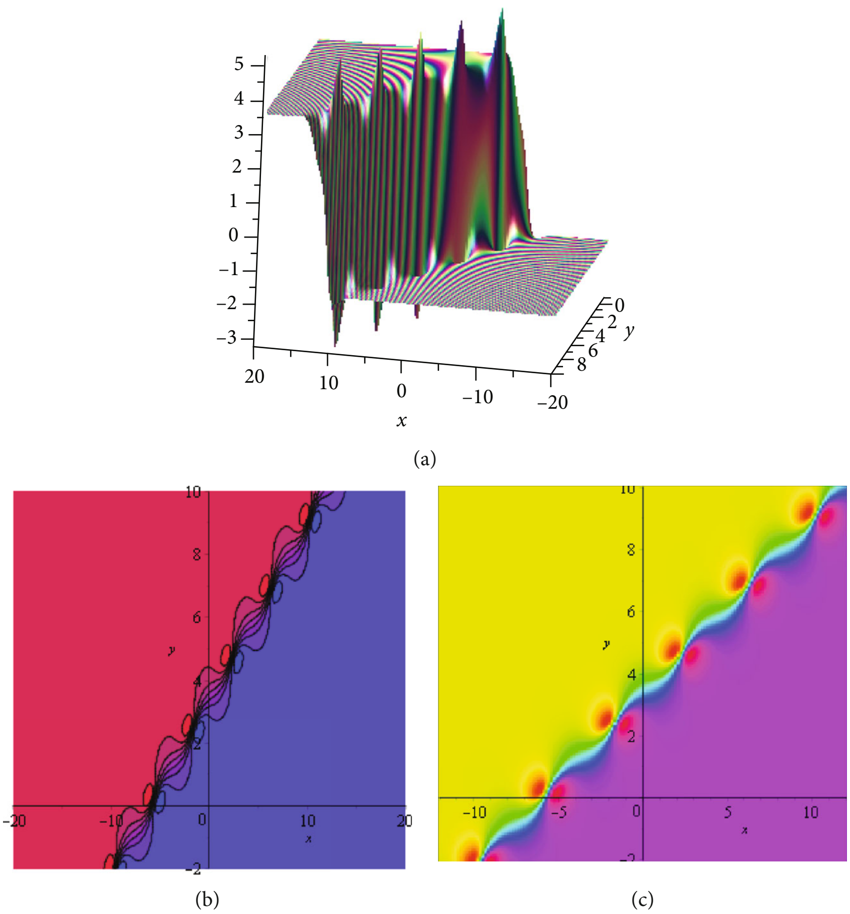


FIGURE 5: Plot evolution of periodic waves (23) utilizing values $a_1 = .5, a_2 = 1.5, a_3 = 2, a_4 = 1, \Omega_4 = 1.1, \Omega_7 = 1.2, \delta_1 = 1, \delta_2 = 1.3, \delta_3 = 2, \delta_4 = 1.3, \delta_5 = 2, \Psi_0 = 1, t = 10$ and (a) 3D plot, (b) density plot, and (c) contour plot.

where $a_2, a_3, a_4, \Omega_3, \Omega_4, \Omega_5, \Omega_9$ are arbitrary values.

Case 8.

$$\Psi_8 = \Psi_0 + 2\,\frac{a_1\Omega_7 e^{\Omega_7 x + y\Omega_8} - a_2\Omega_7 e^{-\Omega_7 x - y\Omega_8} + a_4 \cosh\left(\Omega_7 x + y\Omega_8\right)\Omega_7}{a_1 e^{\Omega_7 x + y\Omega_8} + a_2 e^{-\Omega_7 x - y\Omega_8} + a_3 \sin\left(\Omega_6 t\right) + a_4 \sinh\left(\Omega_7 x + y\Omega_8\right)}, \tag{47}$$

where $a_2, a_3, a_4, \Omega_6, \Omega_7, \Omega_8$ are arbitrary values.

Case 9.

$$\Psi_9 = \Psi_0 + 2\,\frac{1/2a_1\sqrt{-\delta_4}e^{G_1} - 1/2a_2\sqrt{-\delta_4}e^{-G_1} - a_3\cos\left(G_2\right)\Omega_4 + 1/2a_4\cosh\left(G_1\right)\sqrt{-\delta_4}}{a_1 e^{G_1} + a_2 e^{-G_1} + a_3\sin\left(G_2\right) + a_4\sinh\left(G_1\right)}, \tag{48}$$

$$\begin{aligned} G_1 &= \frac{8}{3}\frac{\Omega_4 \Xi_6 \Omega_5 \sqrt{-\delta_4}t}{\delta_4} + \frac{1}{2}\sqrt{-\delta_4}x, \\ G_2 &= \frac{4}{3}\frac{\Omega_5 \Xi_6 t}{1} + \Omega_4 x + \Omega_5 y, \\ \Xi_6 &= \frac{\left(4\Omega_4{}^2\delta_3 + 4\Omega_4\Omega_5\delta_5 - \delta_3\delta_4\right)}{\left(4\Omega_4{}^2 - \delta_4\right)^2}, \end{aligned} \tag{49}$$

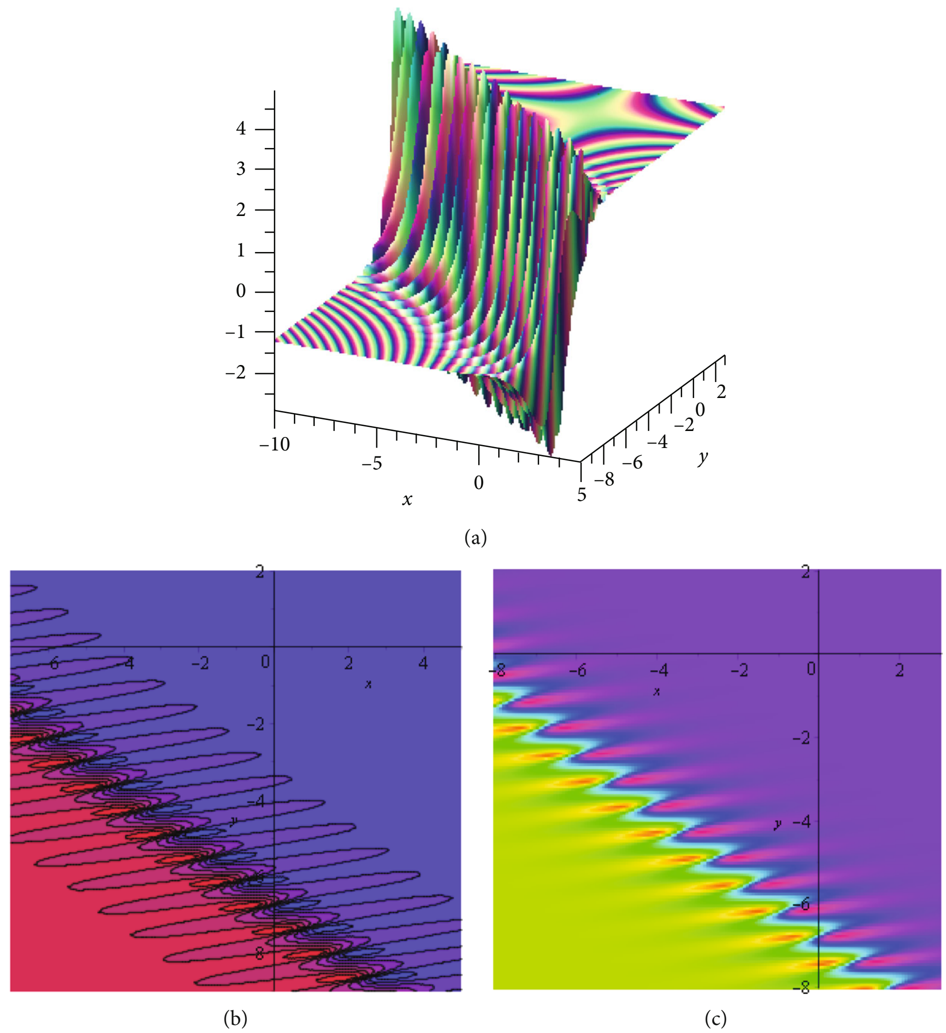
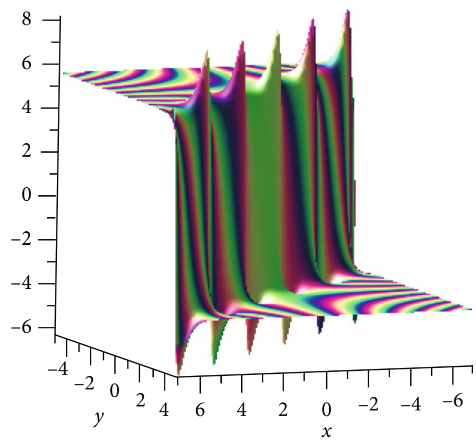


FIGURE 6: Plot evolution of periodic waves (26) utilizing values $a_1 = .5, a_2 = 1.5, a_3 = 2, a_4 = 1, \Omega_1 = 1.1, \Omega_2 = 1.5, \Omega_4 = 1.2, \delta_1 = 1, \delta_2 = 1.3, \delta_3 = 2, \delta_4 = 1.3, \delta_5 = 2, \Psi_0 = 1, t = 10$ and (a) 3D plot, (b) density plot, and (c) contour plot.

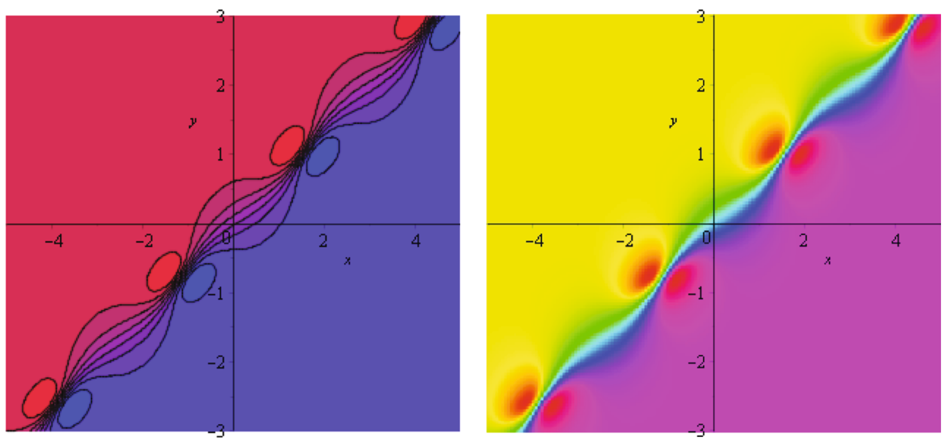
where $a_2, a_3, a_4, \Omega_6, \Omega_7, \Omega_8$ are arbitrary values. Also, we need to satisfy the condition $4\Omega_4^2 - \delta_4 \neq 0, \delta_4 < 0$. By assigning particular values of the parameters, we can easily observe the wave motion as cross-kink waves in the line of $x - y$ axis in which they intersect at one point in Figure 11.

Case 10.

$$\Psi_{10} = \Psi_0 + 2\,\frac{a_1\Omega_7 e^{G_1} - a_2\Omega_7 e^{-G_1} - a_3 \cos(G_2)\Omega_4 + a_4 \cosh(G_1)\Omega_7}{a_1 e^{G_1} + a_2 e^{-G_1} + a_3 \sin(G_2) + a_4 \sinh(G_1)},$$
$$\Xi_7 = 2\,\Omega_4{}^2 - 2\,\Omega_7{}^2 - \delta_4,$$
$$G_1 = \frac{4}{3}\frac{\delta_3{}^2\Omega_4{}^2\Omega_7 t}{\delta_5(\Xi_7)^2} + \Omega_7 x + \frac{1}{2}\frac{\Omega_7\delta_3\left(4\,\Omega_7{}^2 + \delta_4\right)y}{\delta_5(\Xi_7)}, \tag{50}$$
$$G_2 = -\frac{4}{3}\frac{\Omega_4\Omega_7{}^2\delta_3{}^2 t}{(\Xi_7)^2\delta_5} + \Omega_4 x - \frac{1}{2}\frac{\Omega_4\delta_3\left(4\,\Omega_4{}^2 - \delta_4\right)y}{(\Xi_7)\delta_5},$$



(a)



(b)

(c)

FIGURE 7: Plot evolution of periodic waves (28) utilizing values $a_1 = .5, a_2 = 1.5, a_3 = 2, a_4 = 1, \Omega_4 = 1.5, \Omega_5 = 1.2, \delta_1 = 1, \delta_2 = 1.3, \delta_3 = 2, \delta_4 = 1.3, \delta_5 = 2, \Psi_0 = 1, t = 10$ and (a) 3D plot, (b) density plot, and (c) contour plot.

where $a_2, a_3, a_4, \Omega_4, \Omega_7$ are arbitrary values. Also, we need to satisfy the condition $(2\,\Omega_4{}^2 - 2\,\Omega_7{}^2 - \delta_4)\delta_5 \neq 0$. By assigning particular values of the parameters, we can easily observe the wave motion as cross-kink waves in the line of $x - y$ axis in which they intersect at one point in Figure 12.

Case 11.

$$\Psi_{11} = \Psi_0 + 2\,\frac{a_1\Omega_1 e^{G_1} - a_2\Omega_1 e^{-G_1} - a_3\cos(G_2)\Omega_4 + a_4\cosh(G_1)\Omega_1}{a_1 e^{G_1} + a_2 e^{-G_1} + a_3\sin(G_2) + a_4\sinh(G_1)},$$
$$\Xi_8 = \frac{1}{3}\frac{(3\,\Omega_1\delta_3 + 2\,\Omega_2\delta_5)^2}{(\Omega_1{}^2 - 3\,\Omega_4{}^2 + \delta_4)^2}, \tag{51}$$

$$G_1 = \frac{\Omega_4{}^2\Xi_8 t}{\Omega_1\delta_5} + \Omega_1 x + \Omega_2 y,$$
$$G_2 = -\frac{\Omega_4\Xi_8 t}{\delta_5} + \Omega_4 x + \frac{\Omega_4\left(3\,\Omega_1{}^3\delta_3 + 3\,\Omega_1{}^2\Omega_2\delta_5 + 3\,\Omega_1\Omega_4{}^2\delta_3 - \Omega_2\Omega_4{}^2\delta_5 + \Omega_2\delta_4\delta_5\right)y}{\Omega_1\delta_5\left(\Omega_1{}^2 - 3\,\Omega_4{}^2 + \delta_4\right)}, \tag{52}$$

where $a_2, a_3, a_4, \Omega_1, \Omega_2, \Omega_4$ are arbitrary values. Also, we need to satisfy the condition $\Omega_1\delta_5(\Omega_1{}^2 - 3\,\Omega_4{}^2 + \delta_4) \neq 0$. By assigning particular values of the parameters, we can easily observe the wave motion as cross-kink waves in the line of $x - y$ axis in which they intersect at one point in Figure 13.

Case 12.

$$\Psi_{12}=\Psi_0+2\frac{a_1\sqrt{3\Omega_4^2-\delta_4}e^{G_1}-a_2\sqrt{3\Omega_4^2-\delta_4}e^{-G_1}-a_3\cos(G_2)\Omega_4+a_4\cosh(G_1)\sqrt{3\Omega_4^2-\delta_4}}{a_1e^{G_1}+a_2e^{-G_1}+a_3\sin(G_2)+a_4\sinh(G_1)}, \quad (53)$$

$$G_1=\frac{\sqrt{3\Omega_4^2-\delta_4}\Xi_9 t}{\delta_5}+\sqrt{3\Omega_4^2-\delta_4}x-\frac{3}{2}\frac{\delta_3\sqrt{3\Omega_4^2-\delta_4}y}{\delta_5},$$
$$G_2=-\frac{(3\Omega_4^2-\delta_4)\Xi_9 t}{\Omega_4\delta_5}+\Omega_4x+\Omega_5y, \quad (54)$$

$$\Xi_9=\frac{1}{12}\frac{(3\Omega_4\delta_3+2\Omega_5\delta_5)^2}{(16\Omega_4^4-8\Omega_4^2\delta_4+\delta_4^2)}, \quad (55)$$

where $a_2, a_3, a_4, \Omega_1, \Omega_2, \Omega_4$ are arbitrary values. Also, we need to satisfy the condition $3\Omega_4^2-\delta_4>0$. By assigning particular values of the parameters, we can easily observe the wave motion as cross-kink waves in the line of $x-y$ axis in which they intersect at one point in Figure 14.

Case 13.

$$\Psi_{13}=\Psi_0+2\frac{a_1\Omega_1e^{G_1}-a_2\Omega_1e^{-G_1}-a_3\cos(G_2)\Omega_4+a_4\cosh(G_1)\Omega_1}{a_1e^{G_1}+a_2e^{-G_1}+a_3\sin(G_2)+a_4\sinh(G_1)},$$
$$\Xi_{10}=\frac{1}{3}\frac{(3\Omega_1\delta_3+2\Omega_2\delta_5)^2}{(\Omega_1^2-3\Omega_4^2+\delta_4)^2},$$
$$G_1=\frac{\Omega_4^2\Xi_{10}t}{\Omega_1\delta_5}+\Omega_1x+\Omega_2y,$$
$$G_2=-\frac{\Omega_4\Xi_{10}t}{\delta_5}+\Omega_4x+\frac{\Omega_4(3\Omega_1^3\delta_3+3\Omega_1^2\Omega_2\delta_5+3\Omega_1\Omega_4^2\delta_3-\Omega_2\Omega_4^2\delta_5+\Omega_2\delta_4\delta_5)y}{\Omega_1\delta_5(\Omega_1^2-3\Omega_4^2+\delta_4)}, \quad (56)$$

where $a_2, a_3, a_4, \Omega_1, \Omega_2, \Omega_4$ are arbitrary values. Also, we need to satisfy the condition $3\Omega_4^2-\delta_4>0$.

4. Application of SIVP for Equation (1)

By utilizing $\xi=k(x+ay-ct)$ in Equation (1), one becomes

$$-ck^2\Psi''''-6kc\Psi'\Psi''+(a^2\delta_5-c\delta_4+a\delta_3+\delta_2-c\delta_1)\Psi''=0. \quad (57)$$

By points of Refs. [16, 17, 43] and by multiplying Equation (57) with Ψ', we get

$$J=\int_{-\infty}^{\infty}\left(-2kc(\Psi')^3+\frac{1}{2}ck^2(\Psi'')^2+\frac{1}{2}(a^2\delta_5-c\delta_4+a\delta_3+\delta_2-c\delta_1)(\Psi')^2-ck^2\Psi'\Psi'''\right)d\xi. \quad (58)$$

Case 1. With selection the below solution function

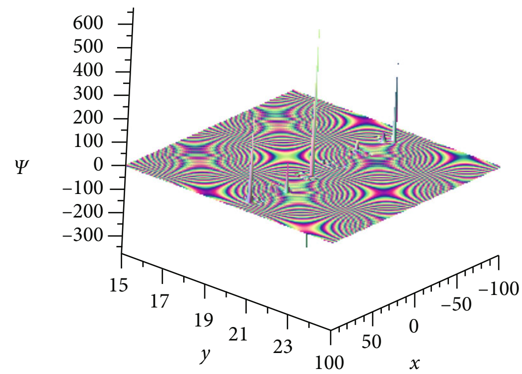
$$u(\xi)=A\operatorname{sech}(B\xi), \quad (59)$$

then the stationary integral changes to

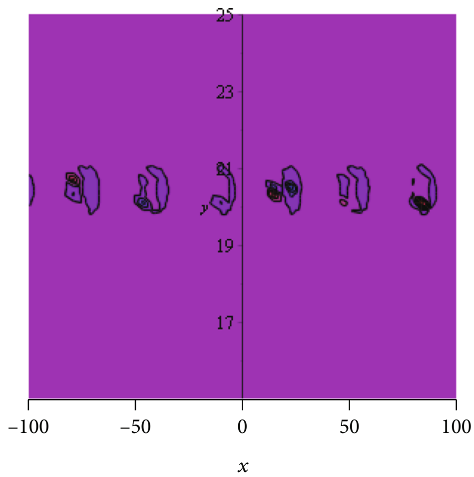
$$J=-\frac{1}{30}A^2B(-21B^2ck^2-12kABc-5S),$$
$$S=a^2\delta_5-c\delta_4+a\delta_3+\delta_2-c\delta_1. \quad (60)$$

With the help of below

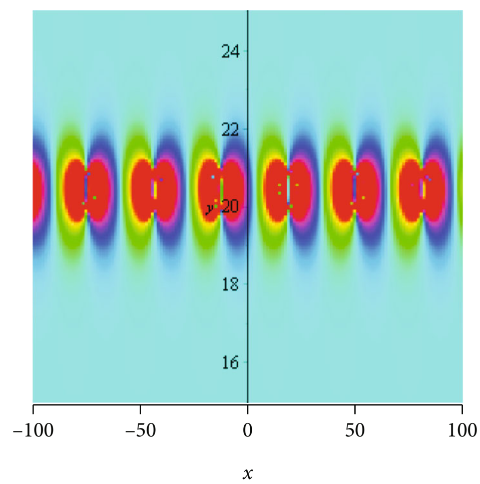
$$\frac{\partial J}{\partial A}=0,$$
$$\frac{\partial J}{\partial B}=0, \quad (61)$$



(a)



(b)



(c)

FIGURE 8: Plot evolution of cross-kink waves (37) utilizing values $a_1 = .5, a_2 = 1.5, a_3 = 2, a_4 = 1, \Omega_2 = 1.9, \Omega_4 = .2, \Omega_9 = 2, \delta_1 = 1, \delta_2 = 1.3, \delta_3 = 2, \delta_4 = 1.3, \delta_5 = 2, \Psi_0 = 1, t = 10$ and (a) 3D plot, (b) density plot, and (c) contour plot.

two nonlinear algebraic systems will be concluded as

$$\begin{aligned} -\frac{1}{15}AB\left(-21\,B^2ck^2 - 12\,kABc - 5\,S\right) + \frac{2}{5}A^2B^2kc = 0, \\ \frac{1}{30}A^2\left(-21\,B^2ck^2 - 12\,kABc - 5\,S\right) \\ -\frac{1}{30}A^2B\left(-42\,Bck^2 - 12\,Ack\right) = 0. \end{aligned} \tag{62}$$

By solving the above cases, one gets

$$\begin{aligned} A &= \pm\frac{S\sqrt{21}}{3\sqrt{cS}}, \\ B &= \pm\frac{\sqrt{21cS}}{21ck}. \end{aligned} \tag{63}$$

The domain of definition is

$$c\left(a^2\delta_5 - c\delta_4 + a\delta_3 + \delta_2 - c\delta_1\right) > 0. \tag{64}$$

Lastly, the solitary wave solution gets

$$\begin{aligned} \Psi(x, y, t) = \pm\frac{S\sqrt{21}}{3\sqrt{cS}}\ \mathrm{sech}\left[\pm\frac{\sqrt{21cS}}{21c}(x + ay - ct)\right], \\ S = a^2\delta_5 - c\delta_4 + a\delta_3 + \delta_2 - c\delta_1. \end{aligned} \tag{65}$$

Case 2. By selecting the below solution function

$$u(\xi) = A\ \sec\mathrm{h}^2(B\xi), \tag{66}$$

then the stationary integral changes to

$$J = \frac{\left(240\,B^2ck^2 + 70\,kABc + 28\,S\right)A^2B}{105}. \tag{67}$$

With the help of below,

$$\begin{aligned} \frac{\partial J}{\partial A} = \frac{2}{3}A^2B^2kc + \frac{\left(480\,B^2ck^2 + 140\,kABc + 56\,S\right)AB}{105} = 0, \\ \frac{\partial J}{\partial B} = \frac{\left(480\,Bck^2 + 70\,Ack\right)A^2B}{105} \\ + \frac{\left(240\,B^2ck^2 + 70\,kABc + 28\,S\right)A^2}{105} = 0. \end{aligned} \tag{68}$$

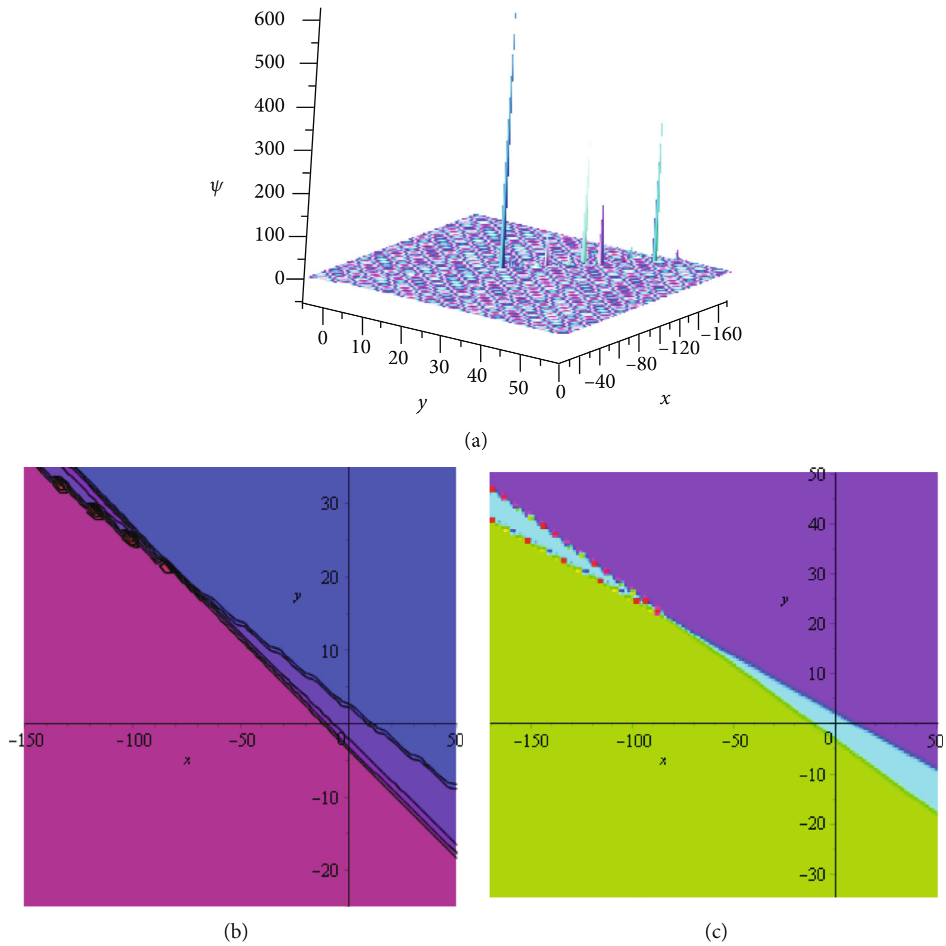


Figure 9: Plot evolution of cross-kink waves (39) utilizing values $a_1 = .5, a_2 = 1.5, a_3 = 2, a_4 = 1, \Omega_1 = 1.9, \Omega_5 = .2, \Omega_9 = 2, \delta_1 = 1, \delta_2 = 1.3, \delta_3 = 2, \delta_4 = 1.3, \delta_5 = 2, \Psi_0 = 1, t = 10$ and (a) 3D plot, (b) density plot, and (c) contour plot.

By solving the above cases, one gets

$$A = \mp \frac{16S\sqrt{21}}{35\sqrt{cS}}, \quad B = \pm \frac{\sqrt{21cS}}{30ck}. \tag{69}$$

The domain of definition is

$$c\left(a^2\delta_5 - c\delta_4 + a\delta_3 + \delta_2 - c\delta_1\right) > 0. \tag{70}$$

Lastly, the bright wave solution one becomes

$$\Psi(x, y, t) = \mp \frac{16S\sqrt{21}}{3\sqrt{cS}} \operatorname{sec\,h}^2\left[\pm \frac{\sqrt{21cS}}{30c}(x + ay - ct)\right], \quad S = a^2\delta_5 - c\delta_4 + a\delta_3 + \delta_2 - c\delta_1. \tag{71}$$

Case 3. Suppose the dark soliton wave solution as below forms

$$u(\xi) = A \tanh^2(B\xi), \tag{72}$$

then the stationary integral changes to

$$J = -\frac{2A^2B\left(-120B^2ck^2 + 35kABc - 14S\right)}{105}. \tag{73}$$

With the help of below,

$$\begin{aligned} \frac{\partial J}{\partial A} &= -\frac{4AB\left(-120B^2ck^2 + 35kABc - 14S\right)}{105} - \frac{2}{3}A^2B^2kc = 0, \\ \frac{\partial J}{\partial B} &= -\frac{2A^2\left(-120B^2ck^2 + 35kABc - 14S\right)}{105} - \frac{2A^2B\left(-240Bck^2 + 35Ack\right)}{105} = 0. \end{aligned} \tag{74}$$

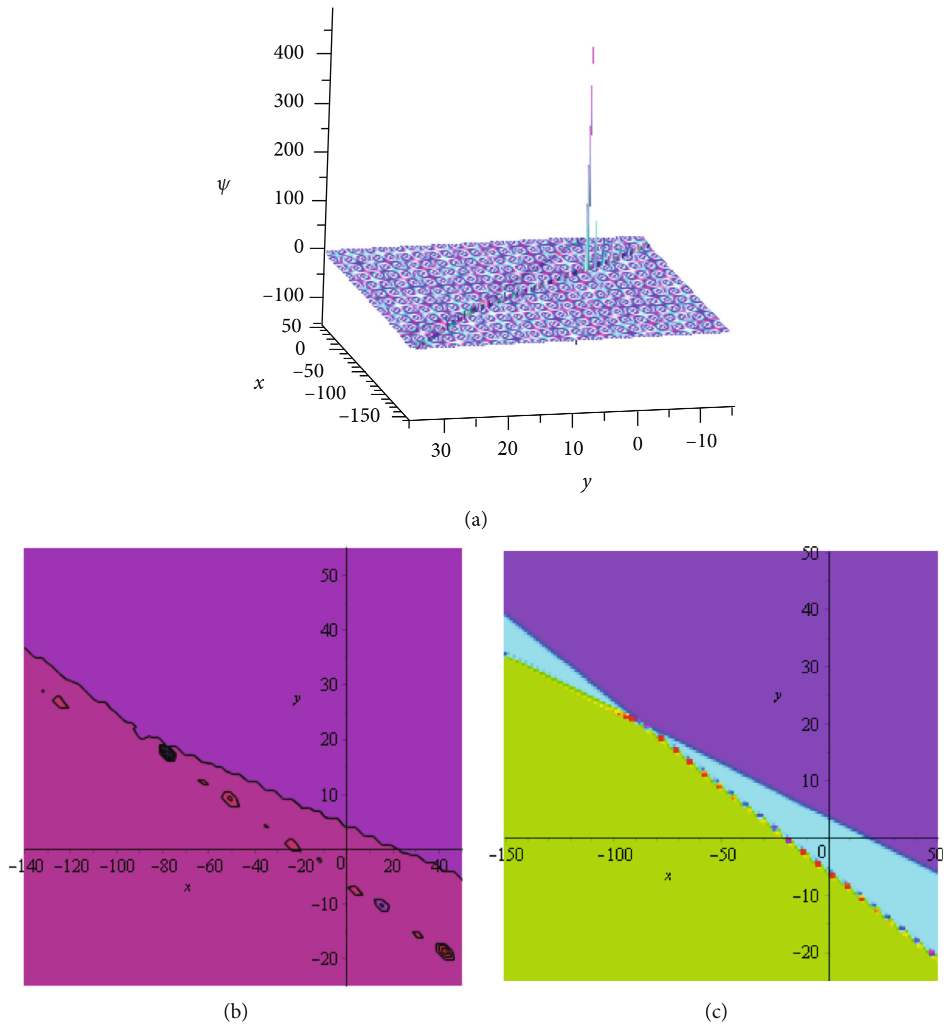


Figure 10: Plot evolution of cross-kink waves (41) utilizing values $a_1 = .5, a_2 = 1.5, a_3 = 2, a_4 = 1, \Omega_2 = 1.9, \Omega_6 = .2, \Omega_7 = 2, \delta_1 = 1, \delta_2 = 1.3$ $, \delta_3 = 2, \delta_4 = 1.3, \delta_5 = 2, \Psi_0 = 1, t = 10$ and (a) 3D plot, (b) density plot, and (c) contour plot.

By solving the above cases, one gets

$$A = \mp \frac{16S\sqrt{21}}{35\sqrt{cS}}, \quad B = \pm \frac{\sqrt{21cS}}{30ck}. \tag{75}$$

The domain of definition is

$$c\left(a^2\delta_5 - c\delta_4 + a\delta_3 + \delta_2 - c\delta_1\right) > 0. \tag{76}$$

Then, the dark wave solution will be obtained as

$$\Psi(x, y, t) = \mp \frac{16S\sqrt{21}}{3\sqrt{cS}} \tan\mathrm{h}^2\left[\pm \frac{\sqrt{21cS}}{30c}(x + ay - ct)\right], \quad S = a^2\delta_5 - c\delta_4 + a\delta_3 + \delta_2 - c\delta_1. \tag{77}$$

5. The Improved Rational tan $(\chi(\xi)/2)$-Expansion Method

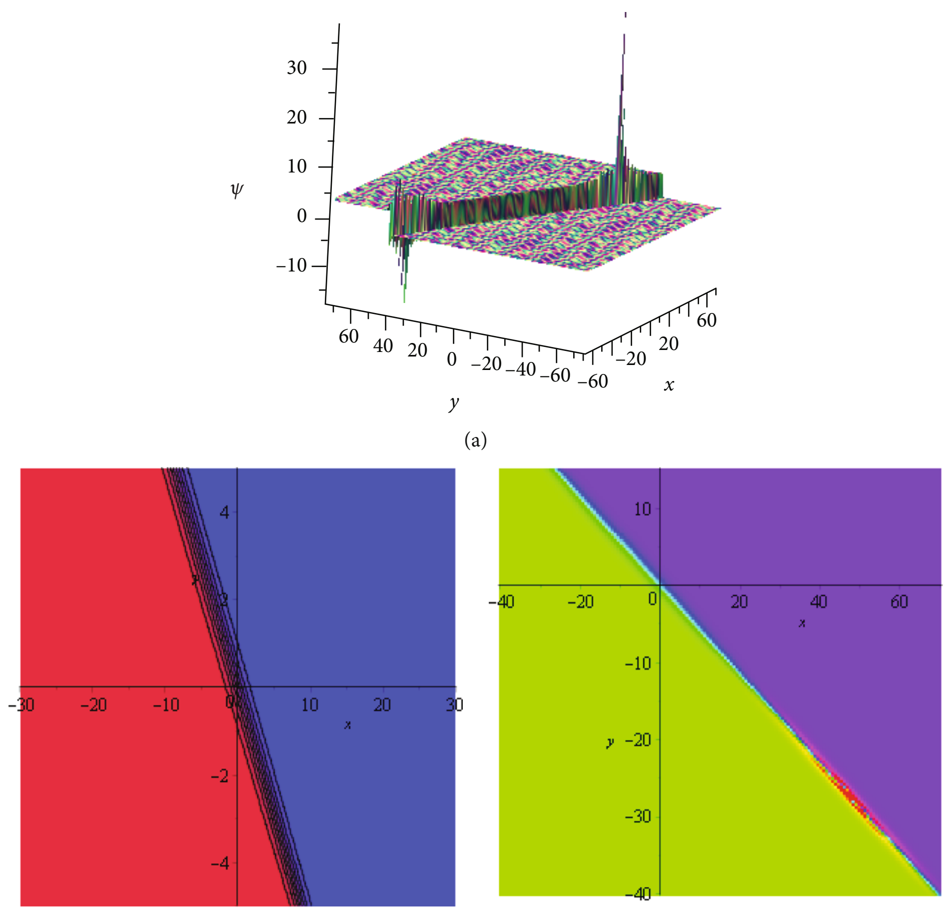
The application of the improved tan $(\chi(\xi)/2)$-expansion technique will be studied where for the first time is given here. We first discuss the mathematical analysis of nonlinear partial differential equations (NPDEs). Hence, we consider the NPDEs in the following way.

Step 1. Assume a nonlinear partial differential equation is given in the general form as follows:

$$\mathscr{L}\left(\chi, \chi_x, \chi_y, \chi_t, \chi_{xx}, \chi_{yy}, \chi_{tt}, \cdots\right) = 0. \tag{78}$$

After simple algebraic operations, this equation is transformed into an ordinary differential equation (ODE) with the below transformation:

$$\chi(x, y, t) = \chi(\xi), \quad \xi = k_1 x + k_2 y - ct, \tag{79}$$



(b) (c)

Figure 11: Plot evolution of cross-kink waves (48) utilizing values $a_1 = .5$, $a_2 = 1.5$, $a_3 = 2$, $a_4 = 1$, $\Omega_4 = 1.9$, $\Omega_5 = .2$, $\Omega_7 = 2$, $\delta_1 = 1$, $\delta_2 = 1.3$, $\delta_3 = 2$, $\delta_4 = -1.3$, $\delta_5 = 2$, $\Psi_0 = 1$, $t = 10$ and (a) 3D plot, (b) density plot, and (c) contour plot.

as below nonlinear ODE

$$\mathcal{S}\left(\chi, k_1\chi', k_2\chi', -c\chi', k_1^2\chi'', k_2^2\chi'', c^2\chi'', \cdots\right) = 0. \quad (80)$$

Step 2. Then, assume that the searched wave solutions of Equation (80) have the following representation:

$$\Psi(\xi) = \sum_{j=0}^{\sigma} \zeta_j \tan^j\left(\frac{\chi(\xi)}{2}\right) + \sum_{j=1}^{\sigma} \theta_j \cot^j\left(\frac{\chi(\xi)}{2}\right), \quad (81)$$

where $\zeta_j (0 \le j \le \sigma), \theta_j (0 \le j \le \sigma)$ are constants to be determined, such that $\zeta_\sigma, \theta_\sigma \neq 0$, and $\chi = \chi(\xi)$ is the solution of the following first order differential equation:

$$\chi' = \alpha_1 \sin(\chi) + \alpha_2 \cos(\chi) + \alpha_3. \quad (82)$$

The particular solutions of Equation (82) will be read as the following.

Family 1. If $\alpha_1^2 + \alpha_2^2 - \alpha_3^2 < 0$ and $\alpha_2 - \alpha_3 \neq 0$, afterwards $\mathfrak{Z}(\xi)/2 = \tan^{-1}[\alpha_1/\alpha_2 - \alpha_3 - \sqrt{\alpha_3^2 - \alpha_1^2 - \alpha_2^2}/\alpha_2 - \alpha_3 \tan(\sqrt{\alpha_3^2 - \alpha_1^2 - \alpha_2^2}/\ 2(\xi + \phi_0))]$.

Family 2. If $\alpha_1^2 + \alpha_2^2 - \alpha_3^2 > 0$ and $\alpha_2 - \alpha_3 \neq 0$, afterwards $\mathfrak{Z}(\xi)/2 = \tan^{-1}[\alpha_1/\alpha_2 - \alpha_3 + \sqrt{\alpha_1^2 + \alpha_2^2 - \alpha_3^2}/\alpha_2 - \alpha_3 \tanh(\sqrt{\alpha_1^2 + \alpha_2^2 - \alpha_3^2}/2(\xi + \phi_0))]$.

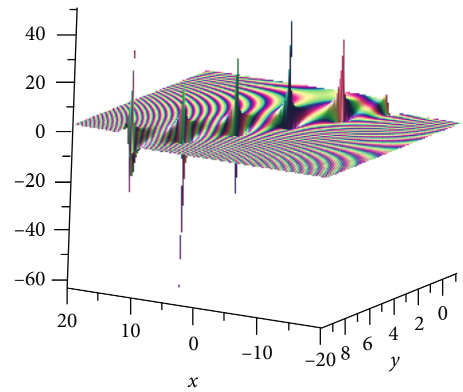
Family 3. If $\alpha_1^2 + \alpha_2^2 - \alpha_3^2 > 0$, $\alpha_2 \neq 0$ and $\alpha_3 = 0$, afterwards $\mathfrak{Z}(\xi)/2 = \tan^{-1}[\alpha_1/\alpha_2 + \sqrt{\alpha_1^2 + \alpha_2^2}/\alpha_2 \tanh(\sqrt{\alpha_1^2 + \alpha_2^2}/2(\xi + \phi_0))]$.

Family 4. If $\alpha_1^2 + \alpha_2^2 - \alpha_3^2 < 0$, $\alpha_3 \neq 0$ and $\alpha_2 = 0$, afterwards $\mathfrak{Z}(\xi)/2 = \tan^{-1}[-\alpha_1/\alpha_3 + \sqrt{\alpha_3^2 - \alpha_1^2}/\alpha_3 \tan(\sqrt{\alpha_3^2 - \alpha_1^2}/2(\xi + \phi_0))]$.

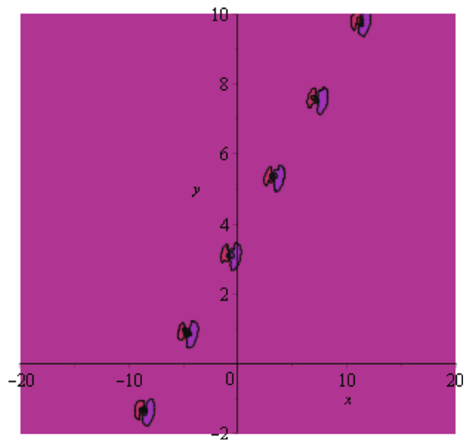
Family 5. If $\alpha_2 - \alpha_3 \neq 0$ and $\alpha_1 = 0$, afterwards $\mathfrak{Z}(\xi)/2 = \tan^{-1}[\sqrt{\alpha_2 + \alpha_3/\alpha_2 - \alpha_3} \tanh(\sqrt{\alpha_2^2 - \alpha_3^2}/2(\xi + \phi_0))]$.

Family 6. If $\alpha_1 = 0$ and $\alpha_3 = 0$, afterwards $\mathfrak{Z}(\xi) = \tan^{-1}[e^{2\alpha_2(\xi+\phi_0)} - 1/e^{2\alpha_2(\xi+\phi_0)} + 1, 2e^{\alpha_2(\xi+\phi_0)}/e^{2\alpha_2(\xi+\phi_0)} + 1]$.

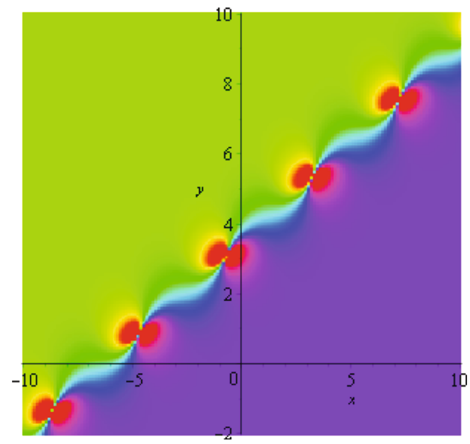
Family 7. If $\alpha_2 = 0$ and $\alpha_3 = 0$, afterwards $\mathfrak{Z}(\xi) = \tan^{-1}[2 e^{\alpha_1(\xi+\phi_0)}/e^{2\alpha_1(\xi+\phi_0)} + 1, e^{2\alpha_1(\rho+\phi_0)} - 1/e^{2\alpha_1(\xi+\phi_0)} + 1]$.



(a)



(b)



(c)

Figure 12: Plot evolution of cross-kink waves (48) utilizing values $a_1 = .5, a_2 = 1.5, a_3 = 2, a_4 = 1, \Omega_4 = 1.1, \Omega_7 = 1.2, \delta_1 = 1, \delta_2 = 1.3, \delta_3 = 2$ $, \delta_4 = 1.3, \delta_5 = 2, \Psi_0 = 1, t = 10$ and (a) 3D plot, (b) density plot, and (c) contour plot.

Family 8. If $\alpha_1^2 + \alpha_2^2 = \alpha_3^2$, afterwards $\mathfrak{Z}(\xi)/2 = \tan^{-1}[\alpha_1(\xi + \phi_0) + 2/(\alpha_2 - \alpha_3)(\xi + \phi_0)]$.

Family 9. If $\alpha_1 = \alpha_2 = \alpha_3 = k\alpha_1$, afterwards $\mathfrak{Z}(\xi)/2 = \tan^{-1}[e^{k\alpha_1(\xi+\phi_0)} - 1]$.

Family 10. If $\alpha_1 = \alpha_3 = k\alpha_1$ and $\alpha_2 = -k\alpha_1$, afterwards $\mathfrak{Z}(\xi)/2 = -\tan^{-1}[e^{k\alpha_1(\xi+\phi_0)}/-1 + e^{k\alpha_1(\xi+\phi_0)}]$.

Family 11. If $\alpha_3 = \alpha_1$, then $\mathfrak{Z}(\xi)/2 = -\tan^{-1}[(\alpha_1 + \alpha_2) e^{\alpha_2(\xi+\phi_0)} - 1/(\alpha_1 - \alpha_2)e^{\alpha_2(\xi+\phi_0)} - 1]$.

Family 12. If $\alpha_1 = \alpha_3$, afterwards $\mathfrak{Z}(\xi)/2 = \tan^{-1}[(\alpha_2 + \alpha_3) e^{\alpha_2(\xi+\phi_0)} + 1/(\alpha_2 - \alpha_3)e^{\alpha_2(\xi+\phi_0)} - 1]$.

Family 13. If $\alpha_3 = -\alpha_1$, afterwards $\mathfrak{Z}(\xi)/2 = \tan^{-1}[e^{\alpha_2(\xi+\phi_0)} + \alpha_2 - \alpha_1/e^{\alpha_2(\xi+\phi_0)} - \alpha_2 - \alpha_1]$.

Family 14. If $\alpha_2 = -\alpha_3$, afterwards $\mathfrak{Z}(\xi)/2 = \tan^{-1}[\alpha_1 e^{\alpha_1(\xi+\phi_0)} /1 - \alpha_3 e^{\alpha_1(\xi+\phi_0)}]$.

Family 15. If $\alpha_2 = 0$ and $\alpha_1 = \alpha_3$, afterwards $\mathfrak{Z}(\xi)/2 = -\tan^{-1}[\alpha_3(\xi + \phi_0) + 2/\alpha_3(\xi + \phi_0)]$.

Family 16. If $\alpha_1 = 0$ and $\alpha_2 = \alpha_3$, afterwards $\mathfrak{Z}(\xi)/2 = \tan^{-1}[\alpha_3(\xi + \phi_0)]$.

Family 17. If $\alpha_1 = 0$ and $\alpha_2 = -\alpha_3$, afterwards $\mathfrak{Z}(\xi)/2 = -\tan^{-1}[1/\alpha_3(\xi + \phi_0)]$.

Family 18. If $\alpha_1 = 0$ and $\alpha_2 = 0$, afterwards $\mathfrak{Z}(\xi) = \alpha_3(\xi + \phi_0) + C$.

Family 19. If $\alpha_2 = \alpha_3$ afterwards $\mathfrak{Z}(\xi)/2 = \tan^{-1}[e^{\alpha_1(\xi+\phi_0)} - \alpha_3 /\alpha_1]$, where ϕ_0 is an integration constant. Also, $\zeta_k, \theta_k (k = 1, 2, \cdots, \sigma), \alpha_1, \alpha_2$, and α_3 are constants to be determined later.

Step 3. To determine the positive integer η, we usually balance linear terms of the highest order in the resulting equation with the highest order nonlinear terms appearing in Equation (80).

Step 4. We collect all the terms with the same order of tan $(\chi(\xi)/2)^k$ $(k = 0, 1, 2, \cdots)$ together. Equating each coefficient of the polynomials of i to zero yields the set of algebraic equations for $\zeta_k, \theta_k (k = 1, 2, \cdots, \sigma), \alpha_1, \alpha_2$, and α_3 with the aid of the Maple.

Step 5. Solving the algebraic equations in Step 4, then substituting $\zeta_k, \theta_k (k = 1, 2, \cdots, \sigma)$, α_1, α_2, and α_3 in (81).

5.1. Application of Improved $\tan(\chi(\xi))$ *Method on the Generalized HSI Eq.* By utilizing the following transformation,

$$\xi = k_1 x + k_2 y - ct. \tag{83}$$

Based on the section before, the needed detail can be got in the below equation:

$$A_1 \frac{d^4}{d\xi^4}\Psi(\xi) + 2A_2\left(\frac{d}{d\xi}\Psi(\xi)\right)\frac{d^2}{d\xi^2}\Psi(\xi) + A_3\frac{d^2}{d\xi^2}\Psi(\xi) = 0, \tag{84}$$

with

$$\begin{aligned} A_1 &= -ck_1^{\,3}, \\ A_2 &= -3\,ck_1^{\,2}, \\ A_3 &= -c\delta_1 k_2 - ck_1\delta_4 + \delta_2 k_1^{\,2} + \delta_3 k_1 k_2 + \delta_5 k_2^{\,2}, \end{aligned} \tag{85}$$

where k_1, k_2, c are unspecified constants. The balance number will be obtained $\theta = 1$ by using the balance principle. Then, the exact solution is given as

$$\Psi(\xi) = \zeta_0 + \zeta_1 \tan\left(\frac{\chi(\xi)}{2}\right) + \theta_1 \cot\left(\frac{\chi(\xi)}{2}\right). \tag{86}$$

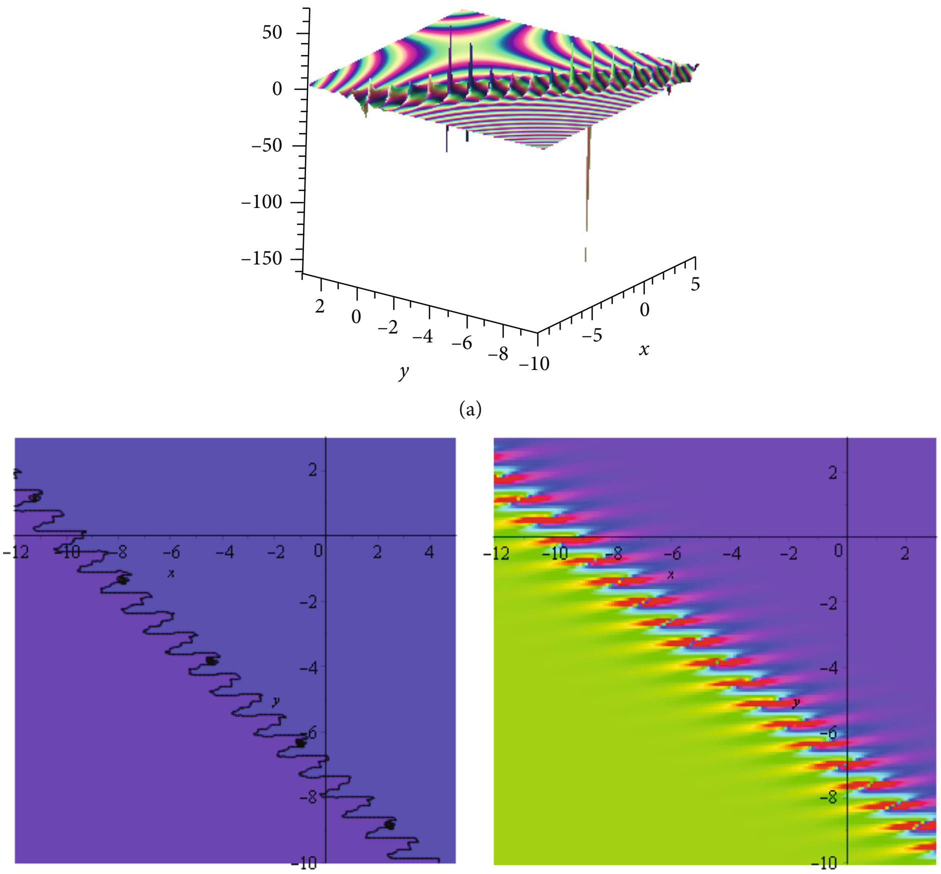
Firstly, we substitute the expressions of $\chi(\xi)$ in (86) into (84) and collect all terms with the same order of $\tan^k(\chi(\xi)/2)$. Then, by equating the coefficient of each polynomial to zero containing a system of eleven nonlinear equations and by solving the nonlinear system, the specified coefficients will be got as the below cases.

Case 1.

$$\begin{aligned} c &= c, \\ \zeta_0 &= \zeta_0, \\ \alpha_1 &= 0, \\ \alpha_2 &= \alpha_2, \\ \alpha_3 &= \alpha_3, \\ \zeta_1 &= 0, \\ \theta_1 &= k_1(\alpha_2 + \alpha_3), \\ k_2 &= -\frac{k_1\left(\alpha_2^{\,2}k_1^{\,2} - \alpha_3^{\,2}k_1^{\,2} + \delta_4\right)}{\delta_1}, \\ k_1 &= \pm\frac{1}{2}\,\frac{\sqrt{2}\sqrt{\delta_5(\alpha_2^{\,2} - \alpha_3^{\,2})\left(\delta_1\delta_3 - 2\,\delta_4\delta_5 + \sqrt{-4\,\delta_1^{\,2}\delta_2\delta_5 + \delta_1^{\,2}\delta_3^{\,2}}\right)}}{\delta_5(\alpha_2^{\,2} - \alpha_3^{\,2})}. \end{aligned} \tag{87}$$

According to Family 5, (86) becomes

$$\begin{aligned} \Psi_1(\xi) &= \zeta_0 + k_1(\alpha_2 + \alpha_3)\cot\left(\frac{\chi(\xi)}{2}\right), \\ \tan\left(\frac{1}{2}\chi(\xi)\right) &= \sqrt{\frac{\alpha_2 + \alpha_3}{\alpha_2 - \alpha_3}}\tanh\left(\frac{\sqrt{\alpha_2^2 - \alpha_3^2}}{2}(\xi + \phi_0)\right), \\ \xi &= \pm\frac{1}{2}\,\frac{\sqrt{2\delta_5(\alpha_2^{\,2} - \alpha_3^{\,2})\left(\delta_1\delta_3 - 2\,\delta_4\delta_5 + \sqrt{-4\,\delta_1^{\,2}\delta_2\delta_5 + \delta_1^{\,2}\delta_3^{\,2}}\right)}}{\delta_5(\alpha_2^{\,2} - \alpha_3^{\,2})}x - \frac{k_1\left(k_1^{\,2}\alpha_2^{\,2} - k_1^{\,2}\alpha_3^{\,2} + \delta_4\right)}{\delta_1}y - ct, \end{aligned} \tag{88}$$



(a) (b) (c)

Figure 13: Plot evolution of cross-kink waves (51) utilizing values $a_1 = .5, a_2 = 1.5, a_3 = 2, a_4 = 1, \Omega_1 = 1.1, \Omega_2 = 1.5, \Omega_4 = 1.2, \delta_1 = 1, \delta_2 = 1.3, \delta_3 = 2, \delta_4 = 1.3, \delta_5 = 2, \Psi_0 = 1, t = 10$ and (a) 3D plot, (b) density plot, and (c) contour plot.

so that $\delta_5(\alpha_2{}^2 - \alpha_3{}^2)(\delta_1\delta_3 - 2\,\delta_4\delta_5 + \sqrt{-4\,\delta_1{}^2\delta_2\delta_5 + \delta_1{}^2\delta_3{}^2}) > 0$ and $\delta_1{}^2\delta_3{}^2 - 4\,\delta_1{}^2\delta_2\delta_5 > 0$.

According to Family 6, (86) gets

$$\Psi_2(\xi) = \zeta_0 \pm \frac{1}{2}\frac{\sqrt{2\delta_5\left(\delta_1\delta_3 - 2\,\delta_4\delta_5 + \sqrt{\delta_1{}^2\delta_3{}^2 - 4\,\delta_1{}^2\delta_2\delta_5}\right)}}{\delta_5} \cdot \cot\left(\frac{\chi(\xi)}{2}\right),$$
$$\tan\left(\frac{1}{2}\chi(\xi)\right) = \left[\frac{e^{2\alpha_2(\xi+\phi_0)} - 1}{e^{2\alpha_2(\xi+\phi_0)} + 1}, \frac{2e^{\alpha_2(\xi+\phi_0)}}{e^{2\alpha_2(\xi+\phi_0)} + 1}\right],$$
$$\xi = k_1x - \frac{k_1\left(k_1{}^2\alpha_2{}^2 + \delta_4\right)}{\delta_1}y - ct,$$
$$k_1 = \pm\frac{1}{2}\frac{\sqrt{2}\sqrt{\delta_5\left(\delta_1\delta_3 - 2\,\delta_4\delta_5 + \sqrt{-4\,\delta_1{}^2\delta_2\delta_5 + \delta_1{}^2\delta_3{}^2}\right)}}{\delta_5\alpha_2}, \tag{89}$$

so that $\delta_5(\delta_1\delta_3 - 2\,\delta_4\delta_5 + \sqrt{-4\,\delta_1{}^2\delta_2\delta_5 + \delta_1{}^2\delta_3{}^2}) > 0$ and $\delta_1{}^2\delta_3{}^2 - 4\,\delta_1{}^2\delta_2\delta_5 > 0$.

Case 2.

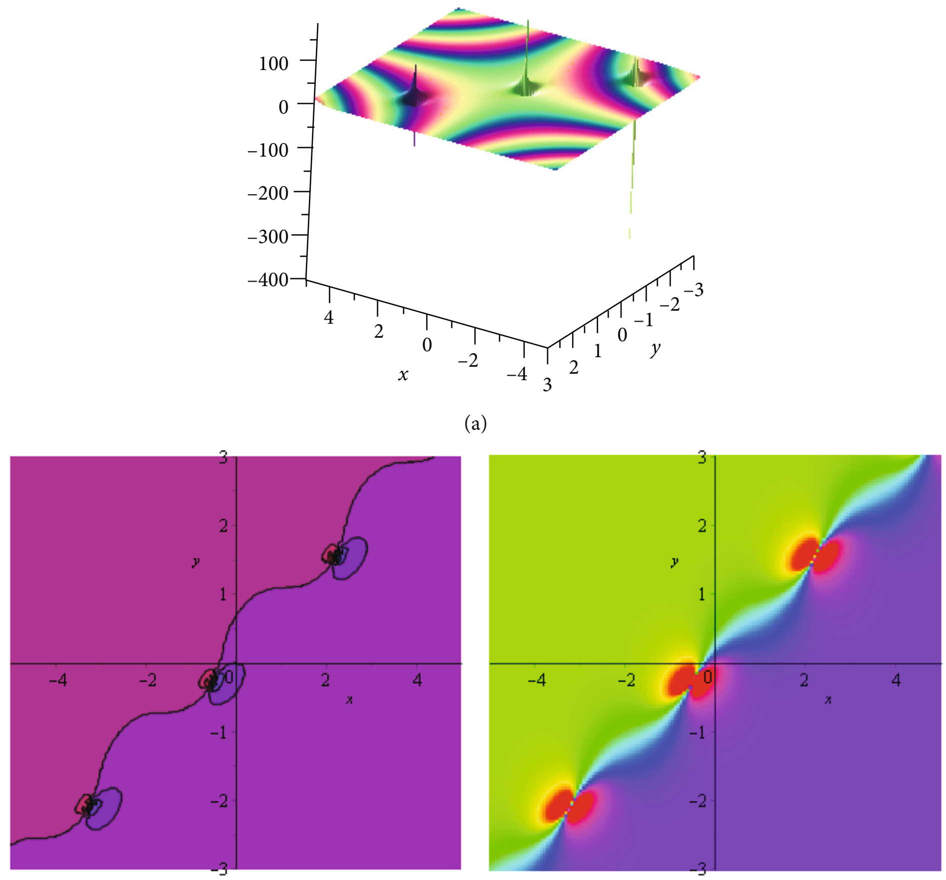
$$\begin{aligned} c &= c,\\ \zeta_0 &= \zeta_0,\\ \zeta_1 &= 0,\\ \alpha_1 &= 0,\\ \alpha_2 &= -\alpha_3,\\ \alpha_3 &= \alpha_3,\\ k_1 &= k_1,\\ k_2 &= k_2,\\ \theta_1 &= \theta_1. \end{aligned} \tag{90}$$

According to Family 17, (86) becomes

$$\Psi_1(\xi) = \zeta_0 - \alpha_3\theta_1(k_1x + k_2y - ct + \phi_0). \tag{91}$$

Case 3.

$$c = \frac{\delta_2k_1{}^2 + \delta_3k_1k_2 + \delta_5k_2{}^2}{k_1{}^3\left(\alpha_1^2 + \alpha_2^2 - \alpha_3^2\right) + \delta_1k_2 + \delta_4k_1},$$



(a)

(b) (c)

Figure 14: Plot evolution of cross-kink waves (53) utilizing values $a_1 = .5, a_2 = 1.5, a_3 = 2, a_4 = 1, \Omega_1 = 1.1, \Omega_2 = 1.5, \Omega_4 = 1.2, \delta_1 = 1, \delta_2 = 1.3, \delta_3 = 2, \delta_4 = 1.3, \delta_5 = 2, \Psi_0 = 1, t = 10$ and (a) 3D plot, (b) density plot, and (c) contour plot.

$$\begin{aligned} \zeta_0 &= \zeta_0, \\ \zeta_1 &= 0, \\ \alpha_1 &= \alpha_1, \\ \alpha_2 &= \alpha_2, \\ \alpha_3 &= \alpha_3, \\ k_1 &= k_1, \\ k_2 &= k_2, \\ \theta_1 &= k_1(\alpha_2 + \alpha_3). \end{aligned} \tag{92}$$

According to Family 1, (86) becomes

$$\Psi_1(\xi) = \zeta_0 + k_1(\alpha_2 + \alpha_3)\cot\left(\frac{\chi(\xi)}{2}\right),$$
$$\tan\left(\frac{1}{2}\chi(\xi)\right) = \frac{\alpha_1}{\alpha_2 - \alpha_3} - \frac{\sqrt{\alpha_3^2 - \alpha_1^2 - \alpha_2^2}}{\alpha_2 - \alpha_3}\tan \cdot \left(\frac{\sqrt{\alpha_3^2 - \alpha_1^2 - \alpha_2^2}}{2}(\xi + \phi_0)\right), \tag{93}$$

so that $\xi = k_1 x + k_2 y - (\delta_2 {k_1}^2 + \delta_3 k_1 k_2 + \delta_5 {k_2}^2 / {k_1}^3(\alpha_1^2 + \alpha_2^2 - \alpha_3^2) + \delta_1 k_2 + \delta_4 k_1)t$, $\alpha_1^2 + \alpha_2^2 - \alpha_3^2 < 0$, and $\alpha_2 - \alpha_3 \neq 0$. According to Family 2, (86) gets

$$\Psi_2(\xi) = \zeta_0 + k_1(\alpha_2 + \alpha_3)\cot\left(\frac{\chi(\xi)}{2}\right),$$
$$\tan\left(\frac{1}{2}\chi(\xi)\right) = \frac{\alpha_1}{\alpha_2 - \alpha_3} + \frac{\sqrt{\alpha_1^2 + \alpha_2^2 - \alpha_3^2}}{\alpha_2 - \alpha_3}\tanh \cdot \left(\frac{\sqrt{\alpha_1^2 + \alpha_2^2 - \alpha_3^2}}{2}(\xi + \phi_0)\right), \tag{94}$$

so that $\xi = k_1 x + k_2 y - (\delta_2 {k_1}^2 + \delta_3 k_1 k_2 + \delta_5 {k_2}^2 / {k_1}^3(\alpha_1^2 + \alpha_2^2 - \alpha_3^2) + \delta_1 k_2 + \delta_4 k_1)t$, $\alpha_1^2 + \alpha_2^2 - \alpha_3^2 > 0$, and $\alpha_2 - \alpha_3 \neq 0$. According to Family 6, (86) gets

$$\Psi_3(\xi) = \zeta_0 + k_1\alpha_2\cot\left(\frac{\chi(\xi)}{2}\right),$$
$$\tan\left(\frac{1}{2}\chi(\xi)\right) = \left[\frac{e^{2\alpha_2(\xi+\phi_0)} - 1}{e^{2\alpha_2(\xi+\phi_0)} + 1}, \frac{2e^{\alpha_2(\xi+\phi_0)}}{e^{2\alpha_2(\xi+\phi_0)} + 1}\right], \tag{95}$$

so that $\xi = k_1 x + k_2 y - (\delta_2 k_1{}^2 + \delta_3 k_1 k_2 + \delta_5 k_2{}^2 / k_1{}^3 \alpha_2{}^2 + \delta_1 k_2 + \delta_4 k_1) t$. According to Family 8, (86) gets

$$\Psi_4(\xi) == \zeta_0 + k_1(\alpha_2 + \alpha_3) \cot\left(\frac{\chi(\xi)}{2}\right), \quad \tan\left(\frac{1}{2}\chi(\xi)\right) = \frac{\alpha_1(\xi + \phi_0) + 2}{(\alpha_2 - \alpha_3)(\xi + \phi_0)}, \tag{96}$$

so that $\xi = k_1 x + k_2 y - (\delta_2 k_1{}^2 + \delta_3 k_1 k_2 + \delta_5 k_2{}^2 / \delta_1 k_2 + \delta_4 k_1) t$ and $\alpha_1^2 + \alpha_2^2 - \alpha_3^2 = 0$.

Case 4.

$$\begin{aligned} c &= c, \\ \zeta_0 &= \zeta_0, \\ \alpha_1 &= 0, \\ \alpha_2 &= \alpha_2, \\ \alpha_3 &= \alpha_3, \\ \zeta_1 &= k_1(\alpha_2 - \alpha_3), \\ \theta_1 &= 0, \\ k_2 &= -\frac{k_1\left(\alpha_2{}^2 k_1{}^2 - \alpha_3{}^2 k_1{}^2 + \delta_4\right)}{\delta_1}, \\ k_1 &= \pm\frac{1}{2}\frac{\sqrt{2}\sqrt{\delta_5(\alpha_2{}^2 - \alpha_3{}^2)\left(\delta_1\delta_3 - 2\delta_4\delta_5 + \sqrt{-4\delta_1{}^2\delta_2\delta_5 + \delta_1{}^2\delta_3{}^2}\right)}}{\delta_5(\alpha_2{}^2 - \alpha_3{}^2)}. \end{aligned} \tag{97}$$

According to Family 5, (86) becomes

$$\begin{aligned} \Psi_1(\xi) &= \zeta_0 + k_1(\alpha_2 - \alpha_3)\tan\left(\frac{\chi(\xi)}{2}\right), \\ \tan\left(\frac{1}{2}\chi(\xi)\right) &= \sqrt{\frac{\alpha_2 + \alpha_3}{\alpha_2 - \alpha_3}}\tanh\left(\frac{\sqrt{\alpha_2^2 - \alpha_3^2}}{2}(\xi + \phi_0)\right), \\ \xi &= \pm\frac{1}{2}\frac{\sqrt{2\delta_5(\alpha_2{}^2 - \alpha_3{}^2)\left(\delta_1\delta_3 - 2\delta_4\delta_5 + \sqrt{-4\delta_1{}^2\delta_2\delta_5 + \delta_1{}^2\delta_3{}^2}\right)}}{\delta_5(\alpha_2{}^2 - \alpha_3{}^2)} \\ &\quad \cdot x - \frac{k_1\left(k_1{}^2\alpha_2{}^2 - k_1{}^2\alpha_3{}^2 + \delta_4\right)}{\delta_1}y - ct, \end{aligned} \tag{98}$$

so that $\delta_5(\alpha_2{}^2 - \alpha_3{}^2)(\delta_1\delta_3 - 2\delta_4\delta_5 + \sqrt{-4\delta_1{}^2\delta_2\delta_5 + \delta_1{}^2\delta_3{}^2}) > 0$ and $\delta_1{}^2\delta_3{}^2 - 4\delta_1{}^2\delta_2\delta_5 > 0$.

According to Family 6, (86) gets

$$\begin{aligned} \Psi_2(\xi) &= \zeta_0 \pm \frac{1}{2}\frac{\sqrt{2\delta_5\left(\delta_1\delta_3 - 2\delta_4\delta_5 + \sqrt{\delta_1{}^2\delta_3{}^2 - 4\delta_1{}^2\delta_2\delta_5}\right)}}{\delta_5} \\ &\quad \cdot \tan\left(\frac{\chi(\xi)}{2}\right), \\ \tan\left(\frac{1}{2}\chi(\xi)\right) &= \left[\frac{e^{2\alpha_2(\xi+\phi_0)} - 1}{e^{2\alpha_2(\xi+\phi_0)} + 1}, \frac{2e^{\alpha_2(\xi+\phi_0)}}{e^{2\alpha_2(\xi+\phi_0)} + 1}\right], \\ \xi &= k_1 x - \frac{k_1\left(k_1{}^2\alpha_2{}^2 + \delta_4\right)}{\delta_1}y - ct, \\ k_1 &= \pm\frac{1}{2}\frac{\sqrt{2}\sqrt{\delta_5\left(\delta_1\delta_3 - 2\delta_4\delta_5 + \sqrt{-4\delta_1{}^2\delta_2\delta_5 + \delta_1{}^2\delta_3{}^2}\right)}}{\delta_5\alpha_2}, \end{aligned} \tag{99}$$

so that $\delta_5(\delta_1\delta_3 - 2\delta_4\delta_5 + \sqrt{-4\delta_1{}^2\delta_2\delta_5 + \delta_1{}^2\delta_3{}^2}) > 0$ and $\delta_1{}^2\delta_3{}^2 - 4\delta_1{}^2\delta_2\delta_5 > 0$.

Case 5.

$$\begin{aligned} c &= c, \\ \zeta_0 &= \zeta_0, \\ \zeta_1 &= -2\alpha_3, \\ \alpha_1 &= 0, \\ \alpha_2 &= -\alpha_3, \\ \alpha_3 &= \frac{c\delta_1 k_2 + c\delta_4 k_1 - \delta_2 k_1{}^2 - \delta_3 k_1 k_2 - \delta_5 k_2{}^2}{6ck_1{}^2\theta_1}, \\ k_1 &= k_1, \\ k_2 &= k_2, \\ \theta_1 &= \theta_1. \end{aligned} \tag{100}$$

According to Family 17, (86) becomes

$$\begin{aligned} \Psi_1(x, y, t) &= \zeta_0 + \frac{2}{k_1 x + k_2 y - ct + \phi_0} \\ &\quad - \frac{c\delta_1 k_2 + c\delta_4 k_1 - \delta_2 k_1{}^2 - \delta_3 k_1 k_2 - \delta_5 k_2{}^2}{6ck_1{}^2} \\ &\quad \cdot (k_1 x + k_2 y - ct + \phi_0). \end{aligned} \tag{101}$$

Case 6.

$$\begin{aligned} c &= c, \\ \zeta_0 &= \zeta_0, \\ \zeta_1 &= -\frac{c\delta_1 k_2 + c\delta_4 k_1 - \delta_2 k_1{}^2 - \delta_3 k_1 k_2 - \delta_5 k_2{}^2}{6c\alpha_3 k_1{}^2}, \\ \alpha_1 &= 0, \\ \alpha_2 &= \alpha_3, \\ \alpha_3 &= \alpha_3, \\ k_1 &= k_1, \\ k_2 &= k_2, \\ \theta_1 &= 2\alpha_3 k_1. \end{aligned} \tag{102}$$

According to Family 19, (86) becomes

$$\Psi_1(x,y,t)=\zeta_0-\frac{c\delta_1k_2+c\delta_4k_1-\delta_2k_1{}^2-\delta_3k_1k_2-\delta_5k_2{}^2}{6c\alpha_3k_1{}^2}\cdot\frac{e^{\alpha_1(k_1x+k_2y-ct+\phi_0)}-\alpha_3}{\alpha_1}+\frac{2\alpha_3k_1\alpha_1}{e^{\alpha_1(k_1x+k_2y-ct+\phi_0)}-\alpha_3}. \tag{103}$$

Case 7.

$$\begin{aligned} c&=\frac{\delta_2k_1{}^2+\delta_3k_1k_2+\delta_5k_2{}^2}{4k_1{}^3(\alpha_2{}^2-\alpha_3{}^2)+\delta_1k_2+\delta_4k_1},\\ \zeta_0&=\zeta_0,\\ \alpha_1&=0,\\ \alpha_2&=\alpha_2,\\ \alpha_3&=\alpha_3,\\ \zeta_1&=k_1(\alpha_2-\alpha_3),\\ \theta_1&=k_1(\alpha_2+\alpha_3). \end{aligned} \tag{104}$$

According to Family 5, (86) becomes

$$\begin{aligned} \Psi_1(\xi)&=\zeta_0+k_1(\alpha_2-\alpha_3)\tan\left(\frac{\chi(\xi)}{2}\right)+k_1(\alpha_2+\alpha_3)\cot\left(\frac{\chi(\xi)}{2}\right),\\ \tan\left(\frac{1}{2}\chi(\xi)\right)&=\sqrt{\frac{\alpha_2+\alpha_3}{\alpha_2-\alpha_3}}\tanh\left(\frac{\sqrt{\alpha_2^2-\alpha_3^2}}{2}(\xi+\phi_0)\right),\\ \xi&=k_1x+k_2y-\frac{\delta_2k_1{}^2+\delta_3k_1k_2+\delta_5k_2{}^2}{4k_1{}^3(\alpha_2{}^2-\alpha_3{}^2)+\delta_1k_2+\delta_4k_1}t, \end{aligned} \tag{105}$$

so that $4k_1{}^3(\alpha_2{}^2-\alpha_3{}^2)+\delta_1k_2+\delta_4k_1\neq0$.

According to Family 6, (86) gets

$$\begin{aligned} \Psi_2(\xi)&=\zeta_0+k_1\alpha_2\tan\left(\frac{\chi(\xi)}{2}\right)+k_1\alpha_2\cot\left(\frac{\chi(\xi)}{2}\right),\\ \tan\left(\frac{1}{2}\chi(\xi)\right)&=\left[\frac{e^{2\alpha_2(\xi+\phi_0)}-1}{e^{2\alpha_2(\xi+\phi_0)}+1},\frac{2e^{\alpha_2(\xi+\phi_0)}}{e^{2\alpha_2(\xi+\phi_0)}+1}\right],\\ \xi&=k_1x+k_2y-\frac{\delta_2k_1{}^2+\delta_3k_1k_2+\delta_5k_2{}^2}{4k_1{}^3\alpha_2{}^2+\delta_1k_2+\delta_4k_1}t. \end{aligned} \tag{106}$$

Case 8.

$$\begin{aligned} c&=c,\\ \zeta_0&=\zeta_0,\\ \alpha_1&=0,\\ \alpha_2&=\alpha_2,\\ \alpha_3&=\alpha_3,\\ \zeta_1&=k_1(\alpha_2-\alpha_3),\\ \theta_1&=k_1(\alpha_2+\alpha_3),\\ k_2&=-\frac{k_1\left(4\alpha_2{}^2k_1{}^2-4\alpha_3{}^2k_1{}^2+\delta_4\right)}{\delta_1},\\ k_1&=\pm\frac{1}{4}\frac{\sqrt{2}\sqrt{\delta_5(\alpha_2{}^2-\alpha_3{}^2)\left(\delta_1\delta_3-2\delta_4\delta_5+\sqrt{-4\delta_1{}^2\delta_2\delta_5+\delta_1{}^2\delta_3{}^2}\right)}}{\delta_5(\alpha_2{}^2-\alpha_3{}^2)}. \end{aligned} \tag{107}$$

According to Family 5, (86) becomes

$$\begin{aligned} \Psi_1(\xi)&=\zeta_0+k_1(\alpha_2-\alpha_3)\tan\left(\frac{\chi(\xi)}{2}\right)+k_1(\alpha_2+\alpha_3)\cot\left(\frac{\chi(\xi)}{2}\right),\\ \tan\left(\frac{1}{2}\chi(\xi)\right)&=\sqrt{\frac{\alpha_2+\alpha_3}{\alpha_2-\alpha_3}}\tanh\left(\frac{\sqrt{\alpha_2^2-\alpha_3^2}}{2}(\xi+\phi_0)\right),\\ \xi&=\pm\frac{1}{4}\frac{\sqrt{2\delta_5(\alpha_2{}^2-\alpha_3{}^2)\left(\delta_1\delta_3-2\delta_4\delta_5+\sqrt{-4\delta_1{}^2\delta_2\delta_5+\delta_1{}^2\delta_3{}^2}\right)}}{\delta_5(\alpha_2{}^2-\alpha_3{}^2)}\cdot x-\frac{k_1\left(4k_1{}^2\alpha_2{}^2-4k_1{}^2\alpha_3{}^2+\delta_4\right)}{\delta_1}y-ct, \end{aligned} \tag{108}$$

so that $\delta_5(\alpha_2{}^2-\alpha_3{}^2)(\delta_1\delta_3-2\delta_4\delta_5+\sqrt{-4\delta_1{}^2\delta_2\delta_5+\delta_1{}^2\delta_3{}^2})>0$ and $\delta_1{}^2\delta_3{}^2-4\delta_1{}^2\delta_2\delta_5>0$.

According to Family 6, (86) gets

$$\begin{aligned} \Psi_2(\xi)&=\zeta_0+k_1\alpha_2\tan\left(\frac{\chi(\xi)}{2}\right)+k_1\alpha_2\cot\left(\frac{\chi(\xi)}{2}\right),\\ \tan\left(\frac{1}{2}\chi(\xi)\right)&=\left[\frac{e^{2\alpha_2(\xi+\phi_0)}-1}{e^{2\alpha_2(\xi+\phi_0)}+1},\frac{2e^{\alpha_2(\xi+\phi_0)}}{e^{2\alpha_2(\xi+\phi_0)}+1}\right],\\ \xi&=k_1x-\frac{k_1\left(4k_1{}^2\alpha_2{}^2+\delta_4\right)}{\delta_1}y-ct,\\ k_1&=\pm\frac{1}{4}\frac{\sqrt{2}\sqrt{\delta_5\left(\delta_1\delta_3-2\delta_4\delta_5+\sqrt{-4\delta_1{}^2\delta_2\delta_5+\delta_1{}^2\delta_3{}^2}\right)}}{\delta_5\alpha_2}, \end{aligned} \tag{109}$$

so that $\delta_5(\delta_1\delta_3-2\delta_4\delta_5+\sqrt{-4\delta_1{}^2\delta_2\delta_5+\delta_1{}^2\delta_3{}^2})>0$ and $\delta_1{}^2\delta_3{}^2-4\delta_1{}^2\delta_2\delta_5>0$.

6. Conclusion

In this study, the periodic, cross-kink, solitary, bright, and dark wave solutions of the $(2+1)$-dimensional generalized Hirota-Satsuma-Ito equation have been achieved. From the bilinear form of this equation, one test function or ansatz has been chosen. Through Maple, the evolution phenomenon of these waves is seen in Figures 1–14, respectively. Mainly, by choosing specific parameter constraints, all cases of 2D and 3D in solitons can be captured from the periodic and cross-kink wave solutions. Also, the improved tan $(\chi(\xi))$ method on the generalized nonlinear wave equation studied and four sets of solutions were obtained. The obtained solutions are extended with numerical simulation to analyze graphically, which results into multiwave and cross-kink wave solutions. Moreover, we studied the solitary, bright, and dark soliton wave solutions of the generalized HSI equation by help of SIVP in the previous section. Finally, literature is full of nonlinear evolution that rich soliton structures are still to be constructed while applying these methods. Further investigations deserve to be made in order to ameliorate the improved tan $(\chi(\xi))$ scheme, so that it may be possible to provide all the different solutions to a nonlinear system. These questions will constitute future works.

Data Availability

The datasets supporting the conclusions of this article are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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