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Commutator of Marcinkiewicz Integral Operators on Herz-Morrey-Hardy Spaces with Variable Exponents

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Abstract

In this paper, our aim is to prove the boundedness of commutators generated by the Marcinkiewicz integrals operator $[b, \mu_\Omega]$ and obtain the result with Lipschitz function and BMO function *f* on the Herz-Morrey-Hardy spaces with variable exponents $HMK_{p(\cdot)\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n)$.

Keywords

Marcinkiewicz Integral Operator, Herz-Morrey-Hardy Space, Commutator, Variable Exponent, Lipschitz Space

1. Introduction

Firstly in 1938, Marcinkiewicz [1] introduced the Marcinkiewicz integral. Next, the Marcinkiewicz integral operator has been studied extensively by many mathematicians in various fields. For example, Stain in [2] introduced the Marcinkiewicz integral operator related to the littlewood-Paley g function on $\mathbb{R}^n$ and proved that μ_Ω is of type (p, p) for $1 < p \le 2$ and of week type $(1,1)$. In [3], Ding, Fan and Pan improved the above result and obtained the $L^p(1 < p < \infty)$ and weighted $L^p(1 < p < \infty)$ boundedness of the Marcinkiewicz cussed the boundedness for the commutator generated by the Marcinkiintegral μ under some weak conditions. Torchinsky and Wang in [4] discussed integral μ_Ω and $BMO(\mathbb{R}^n)$ function on Lebesgue spaces $L^p(\mathbb{R}^n)$.

On the other hand, a class of functional spaces called Herz-Morrey-Hardy spaces with variable exponent has attracted great interest in recent years. We find that in successive studies in this field, in [5] [6] Xu, Yang introduced Herz-

Morrey-Hardy spaces with variable exponents and their some applications. He obtained that certain singular integral operators are bounded from Herz-Morrey-Hardy spaces with variable exponents into Herz-Morrey spaces with variable exponents as an application of the atomic characterization. Also, he established their molecular decomposition, and by using their atomic and molecular decompositions, he gave the boundedness of a convolution type singular integral on Herz-Morrey-Hardy spaces with variable exponents. Omer in [7] proved the boundedness of commutators generated by the Calderón-Zygmund and used properties of variable exponent, BMO(R^n) function and Lipschitz function to prove this boundedness. Also, Yang in [8] established some boundedness for $TD^{\gamma}-D^{\gamma}T$ and $\left(T^{*}-T^{\#}\right)D^{\gamma}$ on the homogeneous Morrey-Herz-type Hardy spaces with variable exponents and studied Boundedness of Calderón-Zygmund operator on these spaces.

Suppose $\mathbb{S}^{n-1}(n\geq 2)$ denotes the unit sphere in $\mathbb{R}^n$ equipped with the normalized measure $\mathrm{d}\sigma$. Let Ω be homogenous function of degree zero and satisfies

$$\int_{\mathbb{S}^{n-1}}\Omega(x')\mathrm{d}\sigma(x')=0, \tag{1.1}$$

where $x'=x/|x|$ for any $x\neq 0$.

Then the Marcinkiewicz integral operator μ_{Ω} is defined by

$$\mu_{\Omega}(f)(x)=\left(\int_0^{\infty}\left|F_{\Omega,t}(f)(x)\right|^2\frac{\mathrm{d}t}{t^3}\right)^{1/2}, \tag{1.2}$$

where

$$F_{\Omega,t}(f)(x)=\int_{|x-y|\leq 1}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)\mathrm{d}y. \tag{1.3}$$

Let $b\in Lip_{\gamma}(\mathbb{R}^n)$ and $b\in BMO$ be a locally integrable function on $\mathbb{R}^n$, the commutator generated by the Marcinkiewicz integral μ_{Ω} and b is defined by

$$[b,\mu_{\Omega}]=\left(\int_0^{\infty}\left|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]f(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{1/2}. \tag{1.4}$$

Motivated by [6] and [7], the aim of this paper is to study the boundedness for the commutator of Marcinkiewicz integral operator $[b,\mu_{\Omega}]$ on the Herz-Morrey-Hardy space with variable exponent where $\Omega\in L^s(S^{n-1})$ for $s\geq 1$, with BMO function and Lipschitz function, we will define The definitions of the Morrey-Herz spaces with variable exponents, the Morrey-Herz-Hardy spaces with variable exponents (which will be defined in the next section), and the preliminary lemmas are presented in Section 2. In Section 3, we will prove the boundedness of the commutator of Marcikiewicz integrals on Herz-Morrey-Hrdy spaces with variable exponent with $b\in Lip_{\gamma}(\mathbb{R}^n)$. Lastly, in Section 4 we will prove the boundedness of the commutator of Marcikiewicz integrals on Herz-Morrey-Hrdy spaces with variable exponent with function $b\in BMO(\mathbb{R}^n)$.

A given open set $\Omega \subset \mathbb{R}^n$ and a measurable function $p(\cdot):\Omega \to [1,\infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable function *f* on Ω such that for some $\lambda > 0$,

$$L^{p(\cdot)}(\Omega) = \left\{ f \text{ is measurable} : \int_\Omega \left(\frac{|f(x)|}{\eta} \right)^{p(x)} \mathrm{d}x < \infty \text{ for some constant } \eta > 0 \right\}, \tag{1.5}$$

the space $L^{p(\cdot)}_{Loc}(\Omega)$ is defined by

$$L^{p(\cdot)}_{Loc}(\Omega) = \left\{ f \text{ is measurable} : f \in L^{p(\cdot)}(K) \text{ for all compact } K \subset \Omega \right\}. \tag{1.6}$$

The Lebesgue spaces $L^{p(\cdot)}(\Omega)$ is Banach spaces with the norm defined by

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \eta > 0 : \int_\Omega \left(\frac{|f(x)|}{\eta} \right)^{p(x)} \mathrm{d}x \le 1 \right\}, \tag{1.7}$$

where $p_- = \operatorname{essinf}\{p(x): x \in \Omega\} > 1$, $p_+ = \operatorname{esssup}\{p(x): x \in \Omega\} < \infty$.

Denotes $p'(x) = p(x)/(p(x)-1)$. Let M be the Hardy-Littlewood maximal operator. We denote $\mathcal{B}(\Omega)$ to be the set of all functions $p(\cdot) \in \mathcal{P}(\Omega)$ satisfying the M is bounded on $L^{p(\cdot)}(\Omega)$.

Definition 1.1. [6]

Let $0 < q \le \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 \le \lambda < \infty$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The nonhomogeneous Morrey-Herz space $MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)$ and homogeneous Morrey-Herz space with variable exponents $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)$ are respectively defined by

$$MK^{\alpha(\cdot),q}_{p(\cdot),\lambda} := \left\{ f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} < \infty \right\}, \tag{1.8}$$

and

$$M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda} := \left\{ f \in L^{p(\cdot)}_{Loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} < \infty \right\}, \tag{1.9}$$

where

$$\|f\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} := \sup_{L \in \mathbb{N}_0} 2^{-L\lambda} \left(\sum_{k=0}^{L} \left\| 2^{k\alpha(\cdot)} f \tilde{\chi}_k \right\|^q_{L^{p(\cdot)}} \right)^{1/q}, \tag{1.10}$$

$$\|f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} := \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} \left\| 2^{k\alpha(\cdot)} f \chi_k \right\|^q_{L^{p(\cdot)}} \right)^{1/q}. \tag{1.11}$$

Definition 1.2. [9]

For all $0 < \gamma \le 1$, the Lipschitz space $Lip_\gamma(\mathbb{R}^n)$ is defined by

$$Lip_\gamma = \left\{ f : \|f\|_{Lip_\gamma} = \sup_{x,y \in \mathbb{R}^n; x \ne y} \frac{|f(x) - f(y)|}{|x-y|^\gamma} < \infty \right\}. \tag{1.12}$$

Definition 1.3. [5]

Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < q \le \infty$, $0 \le \lambda < \infty$ and $N > n+1$. The nonhomogeneous Herz-Morrey-Hardy space with variable exponent $HMK_{p(\cdot)\lambda}^{\alpha(\cdot),q}\left(\mathbb{R}^n\right)$ and homogeneous Herz-Morrey-Hardy space with variable exponents $HM\dot{K}_{p(\cdot)\lambda}^{\alpha(\cdot),q}\left(\mathbb{R}^n\right)$ are respectively defined by

$$HMK_{p(\cdot)\lambda}^{\alpha(\cdot),q}\left(\mathbb{R}^n\right):=\left\{f\in\mathcal{S}'\left(\mathbb{R}^n\right):\|f\|_{HMK_{p(\cdot)\lambda}^{\alpha(\cdot),q}}:=\|G_N f\|_{MK_{p(\cdot)\lambda}^{\alpha(\cdot),q}}<\infty\right\}, \tag{1.13}$$

$$HM\dot{K}_{p(\cdot)\lambda}^{\alpha(\cdot),q}\left(\mathbb{R}^n\right):=\left\{f\in\mathcal{S}'\left(\mathbb{R}^n\right):\|f\|_{HM\dot{K}_{p(\cdot)\lambda}^{\alpha(\cdot),q}}:=\|G_N f\|_{M\dot{K}_{p(\cdot)\lambda}^{\alpha(\cdot),q}}<\infty\right\}. \tag{1.14}$$

Definition 1.4. [10] (Hölder's inequality) Let $\alpha>1$ and $1/\alpha+1/\beta=1$. Then the discrete and integral forms of Hölder's inequality are given as

$$\int_a^b\left|f(x)g(x)\right|\mathrm{d}x\le\left(\int_a^b\left|f(x)\right|^\alpha\right)^{1/\alpha}\left(\int_a^b|g(x)|^\beta\right)^{1/\beta}, \tag{1.15}$$

for continuous function f and g on $[a,b]$.

Definition 1.5. [10] (Minkowski's inequality) Let $u>1$. Then the discrete and integral forms of Minkowski's inequality are given as

$$\left(\int_a^b\left|f(x)+g(x)\right|^u\mathrm{d}x\right)^{1/u}\le\left(\int_a^b\left|f(x)\right|^u\right)^{1/u}+\left(\int_a^b\left|g(x)\right|^u\right)^{1/u}, \tag{1.16}$$

for continuous function f and g on $[a,b]$. for more general functions can be obtained naturally. A further generalization is: If $u>1$, then

$$\left(\int\left(\int\left|f(x,y)\right|\mathrm{d}y\right)^u\mathrm{d}x\right)^{1/u}\le\int\left(\int\left|f(x)\right|^u\mathrm{d}x\right)^{1/u}\mathrm{d}y. \tag{1.17}$$

2. Preliminaries

In this section, we give some preliminaries which we used to prove theorems.

Lemma 2.1. [11] Let $p(\cdot)\in\mathcal{P}\left(\mathbb{R}^n\right)$. Then for any $f\in L^{p(\cdot)}$ and $g\in L^{p'(\cdot)}$, we have

$$\int_{\mathbb{R}^n}\left|f(x)g(x)\right|\mathrm{d}x\le C_p\|f\|_{L^{p(\cdot)}\left(\mathbb{R}^n\right)}\|g\|_{L^{p'(\cdot)}\left(\mathbb{R}^n\right)},$$

where $C_p=1+\frac{1}{p_-}-\frac{1}{p_+}$.

This inequality is called the generalized Hölder inequality with respect to the variable $L^{p(\cdot)}$ spaces.

Lemma 2.2. [12] Given $p(\cdot), p_1(\cdot), p_2(\cdot)\in\mathcal{P}\left(\mathbb{R}^n\right)$, for any $f\in L^{p_1(\cdot)}\left(\mathbb{R}^n\right), g\in L^{p_2(\cdot)}\left(\mathbb{R}^n\right)$, when $\frac{1}{p(\cdot)}=\frac{1}{p_2(\cdot)}+\frac{1}{p_1(\cdot)}$, we get

$$\left\|f(x)g(x)\right\|_{L^{p(\cdot)}\left(\mathbb{R}^n\right)}\le C\|f\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)}\|g\|_{L^{p_2}\left(\mathbb{R}^n\right)},$$

where $C_{p_1,p_2}=\left[1+\frac{1}{p_{1-}}-\frac{1}{p_{1+}}\right]^{\frac{1}{p_-}}$.

Proposition 2.3. [13] If $q(\cdot)\in\mathcal{P}\left(\mathbb{R}^n\right)$ satisfies

$$\left|q(x)-q(y)\right|\le\frac{-C}{\log\left(|x-y|\right)},\quad |x-y|\le 1/2,$$

$$\left|q(x)-q(y)\right| \le \frac{C}{\log\left(e+|x|\right)}, \quad |y| \ge |x|,$$

then $q(\cdot) \in \mathfrak{B}\left(\mathbb{R}^n\right)$.

Lemma 2.4. [14] Let k be a positive integer and B be a ball in $\mathbb{R}^n$. Then we have that for all $b \in BMO\left(\mathbb{R}^n\right)$ and $i, j \in \mathbb{Z}$ with $i < j$, we have

1) $C^{-1}\|b\|_*^k \le \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\|(b-b_B)\chi_B\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*^k$,

2) $\left\|\left(b-b_{B_i}\right)^k \chi_{B_j}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(j-i)^K \|b\|_*^k \left\|\chi_{B_j}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}$,

where $B_i = \left\{x \in \mathbb{R}^n : |x| \le 2^i\right\}$ and $B_j = \left\{x \in \mathbb{R}^n : |x| \le 2^j\right\}$.

Lemma 2.5. [15] Let $q(\cdot) \in \mathfrak{B}\left(\mathbb{R}^n\right)$, then there exist positive constants $C > 0$, such that for all balls $B \subset \mathbb{R}^n$ and all measurable subset $R \subset B$,

$$\frac{\|\chi_R\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|R|}{|B|}\right)^{\delta_1}, \quad \frac{\|\chi_R\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|R|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Lemma 2.6. [16] If $q(\cdot) \in \mathfrak{B}\left(\mathbb{R}^n\right)$, then there exists a constant $C > 0$ such that for any balls B in $\mathbb{R}^n$,

$$\frac{1}{|B|}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C.$$

Lemma 2.7. [6] Let $0 < q < \infty$, $p(\cdot) \in \mathfrak{B}\left(\mathbb{R}^n\right)$, $0 < \lambda < \infty$, and $\alpha(\cdot) \in L^\infty\left(\mathbb{R}^n\right)$ be log-Hölder continuous both at the origin and infinity,
$2\lambda \le \alpha(\cdot), n\delta_2 \le \alpha(0), \alpha_\infty < \infty$, δ_2 as in lemma 2.4. Then $f \in HM\dot{K}_{p(\cdot)\lambda}^{\alpha(\cdot),q}\left(\mathbb{R}^n\right)$ (or $HMK_{p(\cdot)\lambda}^{\alpha(\cdot),q}\left(\mathbb{R}^n\right)$) if and only if $f = \sum_{k=-\infty}^{\infty} \lambda_k f_k$ (or $f = \sum_{k=0}^{\infty} \lambda_k f_k$), in the sense of $f \in \mathcal{S}'(\mathbb{R})^n$, where each a_k is a central $(\alpha(\cdot), p(\cdot))$ atom with support contained in B_k and

$$\sup_{L \le 0, L \in Z} 2^{-L\lambda} \sum_{k=-\infty}^{L} |\lambda_k|^q < \infty \quad \text{or} \quad \left(\sup_{L \le 0, L \in Z} 2^{-L\lambda} \sum_{k=0}^{L} |\lambda_k|^q\right),$$

moreover

$$\|f\|_{HM\dot{K}_{p(\cdot)\lambda}^{\alpha(\cdot),q}} \approx \inf \sup_{L \le 0, L \in Z} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} |\lambda_k|^q\right)^{1/q}$$

or

$$\|f\|_{HMK_{p(\cdot)\lambda}^{\alpha(\cdot),q}} \approx \inf \sup_{L \le 0, L \in Z} 2^{-L\lambda} \left(\sum_{k=0}^{L} |\lambda_k|^q\right)^{1/q},$$

where infimum is taken over all above decomposition of f.

Lemma 2.8. [17] Let $q(\cdot) \in \mathcal{P}\left(\mathbb{R}^n\right), q \in (0, \infty]$ and $\lambda \in [0, \infty)$. If $\alpha(\cdot) \in L^\infty\left(\mathbb{R}^n\right) \cap \mathfrak{P}_0^{log}\left(\mathbb{R}^n\right) \cap \mathfrak{P}_\infty^{log}\left(\mathbb{R}^n\right)$, then

$$\|f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n)}^q = \max\left\{\sup_{L \le 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \|f\chi_k\|_{L^{p(\cdot)}}^q,\right.$$
$$\left.\sup_{L \le 0, L \in Z} 2^{-L\lambda q}\left(\sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \|f\chi_k\|_{L^{p(\cdot)}}^q + \sum_{k=0}^{L} 2^{kq\alpha(\infty)} \|f\chi_k\|_{L^{p(\cdot)}}^q\right)\right\}.$$

Lemma 2.9. [18] Let Ω satisfies L^r-Dini condition with $r \in [1, \infty)$. If there exist constants $C > 0$ and $R > 0$ such that $|y| < R/2$, then for every $x \in \mathbb{R}^n$, we have

$$\left(\int_{R<|R|<2R} \left| \frac{\Omega(x-y)}{|x-y|} - \frac{\Omega(x)}{|x|} \right|^r \mathrm{d}x \right)^{1/r} \le CR^{\left(\frac{n}{r}-n\right)} \left\{ \frac{|y|}{R} + \int_{|y|/2R<\delta<|y|/R} \frac{\omega_r(\delta)}{\delta} \mathrm{d}\delta \right\}.$$

Lemma 2.10. [15] Given *E*, let $q(\cdot) \in \mathcal{P}(E)$, $f : E \times E \to \mathbb{R}^n$ be a measurable function (with respect to product measure) such that for almost every $y \in E$, $f(., y) \in L^{q(\cdot)}(E)$. Then

$$\left\| \int_E f(., y) dy \right\|_{L^{q(\cdot)}(E)} \le C \int_E \| f(., y) \|_{L^{q(\cdot)}(E)} \mathrm{d}y.$$

Lemma 2.11. [19] If $a > 0, 1 \le s \le \infty, 0 \le d \le s$ and $-n + (n-1)d/s < v < \infty$, then

$$\left(\int_{|y|\le a|x|} |y|^v |\Omega(x-y)|^d \mathrm{d}y \right)^{1/d} \le C |x|^{(v+n)/d} \|\Omega\|_{L^s(S^{n-1})}.$$

Lemma 2.12. [19] Let $q(\cdot) \in \mathcal{P}$ satisfies Proposition 2.3. Then

$$\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{\frac{1}{q(x)}} & \text{if } |Q| \le 2^n \text{ and } x \in Q \\ |Q|^{\frac{1}{q(\infty)}} & \text{if } |Q| \ge 1 \end{cases}$$

for every cube (or ball) $Q \in \mathbb{R}^n$, where $p(\infty) = \lim_{x\to\infty} p(x)$.

3. Lipschitz Boundedness for the Commutator of Marcikiewicz Integrals Operator

In this section, we prove the boundedness of the commutator of Marcikiewicz integrals on Herz-Morrey-Hrdy spaces with variable exponent so when $b \in Lip_\gamma(\mathbb{R}^n)$ under some conditions.

Theorem 3.1.

Suppose that $b \in Lip_\gamma(\mathbb{R}^n)$ with $0 < \gamma \le 1$. If $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies proposition 2.3 with $q_1^+ < n/\gamma, 1/q_1(x) - 1/q_2(x) = \gamma/n$, $\Omega \in L^s(S^{n-1})(s > q_2^+)$ with $1 \le s' < q_1^-$ and satisfies

$$\int_0^1 \frac{\Omega_s(\delta)}{\delta^{1+\gamma}} \mathrm{d}\delta < \infty,$$

let $0 < p_1 \le q_2 < \infty$ and $n\delta_2 \le \alpha < n\delta_2 + \gamma$ or ($0 < \max(n\delta_2, \alpha_2) \le \alpha_1 < n\delta_2 + \gamma$). Then the commutator $[b, \mu_\Omega]$ is bounded from $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot)\lambda}(\mathbb{R}^n)$ (or $HMK^{\alpha(\cdot),q}_{p(\cdot)\lambda}(\mathbb{R}^n)$) to $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot)\lambda}(\mathbb{R}^n)$ (or $MK^{\alpha(\cdot),q}_{p(\cdot)\lambda}(\mathbb{R}^n)$).

To the proof the above theorem, we will recall the following lemma.

Lemma 3.1. [15]

Suppose that $b \in Lip_\gamma(\mathbb{R}^n)$ with $0 < \gamma \le 1$. If $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies Proposition 2.3 with $q_1^+ < n/\gamma, 1/q_1(x) - 1/q_2(x) = \gamma/n$ with $\Omega \in L^s(S^{n-1})(s > q_2^+)$. Then the commutator $[b, \mu_\Omega]$ is bounded from $L^{q_1(\cdot)}(\mathbb{R}^n)$ to $L^{q_1(\cdot)}(\mathbb{R}^n)$.

Next, we will give the Lipschitz estimate about the commutator $[b,\mu_\Omega]$ on Herz-Morrey-Hardy spaces with variable exponent.

Proof Theorem 3.1:

To prove this theorem, we only prove the homogeneous case. Let $f\in HMK^{\alpha(\cdot),q}_{p(\cdot)\lambda}(\mathbb{R}^n)$. By lemma 2.6 we have $f=\sum_{j=-\infty}^{\infty}\lambda_j f_j$ converged in $\mathcal{S}'(\mathbb{R})^n$, where each b_j is a central $(\alpha(\cdot),p(\cdot))$ atom with support contained in B_j and

$$\|f\|_{HMK^{\alpha(\cdot),q}_{p(\cdot)\lambda}}\approx\inf\sup_{L\le 0,L\in Z}2^{-L\lambda}\left(\sum_{j=-\infty}^{L}|\lambda_j|^q\right)^{1/q}.$$

Here we denote $\Delta=\sup_{L\le 0,L\in Z}2^{-L\lambda}\sum_{k=-\infty}^{L}|\lambda_k|^q$. By lemma 2.8 we have

$$\begin{aligned}\|[b,\mu_\Omega](f)\|^q_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)}=\max\Bigg\{&\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\|[b,\mu_\Omega](f)\chi_k\|^q_{L^{p(\cdot)}},\\&\sup_{L\le 0,L\in Z}2^{-L\lambda q}\left(\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\|[b,\mu_\Omega](f)\chi_k\|^q_{L^{p(\cdot)}}\right.\\&\left.+\sum_{k=0}^{L}2^{kq\alpha(\infty)}\|[b,\mu_\Omega](f)\chi_k\|^q_{L^{p(\cdot)}}\right)\Bigg\}.\end{aligned}$$

$$I=\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\|[b,\mu_\Omega](f)\chi_k\|^q_{L^{p(\cdot)}},$$

$$II=\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\|[b,\mu_\Omega](f)\chi_k\|^q_{L^{p(\cdot)}},$$

$$III=\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha(\infty)}\|[b,\mu_\Omega](f)\chi_k\|^q_{L^{p(\cdot)}}.$$

In beginning, we examine a function which we will use in proving

$$\begin{aligned}|[b,\mu_\Omega](b_j)(x)|\le&\left\{\int_0^{|x|}\left|\int_{|x-y|\le t}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}[b(x)-b(y)]b_j(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right\}^{1/2}\\&+\left\{\int_{|x|}^{\infty}\left|\int_{|x-y|\le t}\frac{|\Omega(x-py)|}{|x-y|^{n-1}}[b(x)-b(y)]b_j(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right\}^{1/2}\\&:\simeq\Upsilon_1+\Upsilon_2.\end{aligned}$$

When $x\in A_k$ and $|x-y|\le t$ with $t\le|x|$, it follows from $j\le k-2$ that $|x-y|\sim|x|$. We have

$$\left|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right|\le\frac{|y|}{|x-y|^3}.\tag{3.1}$$

Then by (3.1), the Minkowski's inequality, the generalized Hölder's inequality and the vanishing of the moment of b_j we have

$$\begin{aligned}\Upsilon_1&\le C\int_{\mathbb{R}^n}\left|\frac{|\Omega(x-y)|}{|x-y|^2}-\frac{|\Omega(x)|}{|x|^2}\right||b(x)-b(y)||b_j(y)|\left(\int_{|x-y|}^{|x|}\frac{\mathrm{d}t}{t^3}\right)^{1/2}\mathrm{d}y\\&\le C\int_{\mathbb{R}^n}\left|\frac{|\Omega(x-y)|}{|x-y|^2}-\frac{|\Omega(x)|}{|x|^2}\right||b(x)-b(y)||b_j(y)|\left|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right|^{1/2}\mathrm{d}y\end{aligned}$$

$$\le C\int_{\mathbb{R}^n}\left|\frac{|\Omega(x-y)|}{|x-y|^2}-\frac{|\Omega(x)|}{|x|^2}\right||b(x)-b(y)||b_j(y)|\frac{|y|^{1/2}}{|x-y|^{3/2}}\mathrm{d}y$$
$$\le C2^{(j-k)/2}\int_{B_j}\left|\frac{|\Omega(x-y)|}{|x-y|^2}-\frac{|\Omega(x)|}{|x|^2}\right||b(x)-b(y)||b_j(y)|\mathrm{d}y.$$

Similarly, we consider Υ_2. Noting that $|x-y|\sim|x|$. By the Minkowski's inequality, the generalized Hölder's inequality and the vanishing moments of b_j we have

$$\Upsilon_2\le C\int_{\mathbb{R}^n}\left|\frac{|\Omega(x-y)|}{|x-y|^{n-1}}-\frac{|\Omega(x)|}{|x|^{n-2}}\right||b(x)-b(y)||b_j(y)|\left(\int_{|x|}^{\infty}\frac{\mathrm{d}t}{t^3}\right)^{1/2}\mathrm{d}y$$
$$\le C\int_{B_j}\left|\frac{|\Omega(x-y)|}{|x-y|^2}-\frac{|\Omega(x)|}{|x|^2}\right||b(x)-b(y)||b_j(y)|\mathrm{d}y.$$

So we have

$$\left|[b,\mu_\Omega](b_j)(x)\right|\le C\int_{B_j}\left|\frac{|\Omega(x-y)|}{|x-y|^n}-\frac{|\Omega(x)|}{|x|^n}\right||b(x)-b(y)||b_j(y)|\mathrm{d}y.$$

From lemma 2.10 and the Minkowski's inequality we have

$$\left\|[b,\mu_\Omega](b_j)\chi_k\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\le C\int_{B_j}\left\|\left|\frac{|\Omega(x-y)|}{|x-y|^n}-\frac{|\Omega(x)|}{|x|^n}\right||b(x)-b(y)|\chi_k(\cdot)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}|b_j(y)|\mathrm{d}y$$
$$\le C\int_{B_j}\left\|\left|\frac{|\Omega(\cdot-y)|}{|\cdot-y|^n}-\frac{|\Omega(\cdot)|}{|\cdot|^n}\right||b(\cdot)-b(0)|\chi_k(\cdot)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}|b_j(y)|$$
$$+C\int_{B_j}\left\|\left|\frac{|\Omega(\cdot-y)|}{|\cdot-y|^n}-\frac{|\Omega(\cdot)|}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}|b(0)-b(y)||b_j(y)|$$
$$:=\Upsilon_1^*+\Upsilon_2^*.$$

For Υ_1^*, noting $s>p'$, we denote $\tilde{p}'(\cdot)>1$ and $\frac{1}{p(x)}=\frac{1}{\tilde{p}'(x)}+\frac{1}{s}$. By lemma 2.2 we have

$$\left\|\left|\frac{|\Omega(\cdot-y)|}{|\cdot-y|^n}-\frac{|\Omega(\cdot)|}{|\cdot|^n}\right||b(\cdot)-b(0)|\chi_k(\cdot)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\le\left\|\left|\frac{|\Omega(\cdot-y)|}{|\cdot-y|^n}-\frac{|\Omega(\cdot)|}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{s(\cdot)}(\mathbb{R}^n)}\left\||b(0)-b(y)||b_j(y)|\right\|_{L^{\tilde{p}'(\cdot)}(\mathbb{R}^n)}$$
$$\le C\|b\|_{Lip_\gamma}2^{k\gamma}\left\|\left|\frac{|\Omega(\cdot-y)|}{|\cdot-y|^n}-\frac{|\Omega(\cdot)|}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{s(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_K}\right\|_{L^{\tilde{p}'(\cdot)}(\mathbb{R}^n)}.$$

When $|B_k|\le 2^n$ and $x_k\in B_k$, by Lemma 2.12 we have

$$\left\|\chi_{B_K}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \simeq \left|B_k\right|^{\frac{1}{p'(x_k)}} \approx \left\|\chi_{B_K}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \left|B_k\right|^{\frac{-1}{s}-\frac{-\gamma}{n}}.$$

When $\left|B_k\right| \geq 1$ we have

$$\left\|\chi_{B_K}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \simeq \left|B_k\right|^{\frac{1}{p'(\infty)}} \approx \left\|\chi_{B_K}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \left|B_k\right|^{\frac{-1}{s}-\frac{-\gamma}{n}}.$$

So we obtain

$$\left\|\chi_{B_K}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \approx \left\|\chi_{B_K}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \left|B_k\right|^{\frac{-1}{s}-\frac{-\gamma}{n}}.$$

By lemma 2.9 we have

$$\begin{aligned}
&\left\|\left|\frac{\left|\Omega(\cdot-y)\right|}{\left|\cdot-y\right|^n}-\frac{\left|\Omega(\cdot)\right|}{\left|\cdot\right|^n}\right|\chi_k(\cdot)\right\|_{L^{s(\cdot)}(\mathbb{R}^n)} \\
&\leq 2^{(k-1)\left(\frac{n}{s}-n\right)}\left\{\frac{|y|}{2^k}+\int_{\frac{|y|}{2^k}}^{\frac{|y|}{2^{k-1}}}\frac{\omega_s(\delta)}{\delta}\mathrm{d}\delta\right\} \\
&\leq 2^{(k-1)\left(\frac{n}{s}-n\right)}\left\{2^{j-k+1}+2^{(j-k+1)\gamma}\int_0^1\frac{\omega_s(\delta)}{\delta}\mathrm{d}\delta\right\} \\
&\leq 2^{(k-1)\left(\frac{n}{s}-n\right)}2^{(j-k)\gamma}.
\end{aligned}$$

Now, by using the generalized Hölder's inequality we get:

$$\begin{aligned}
\Upsilon_1^* &\leq \int_{B_j}\left\|\left|\frac{\left|\Omega(\cdot-y)\right|}{\left|\cdot-y\right|^n}-\frac{\left|\Omega(\cdot)\right|}{\left|\cdot\right|^n}\right|\left|b(\cdot)-b(y)\right|\chi_k(\cdot)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left|b_j(y)\right|\mathrm{d}y \\
&\leq C\|b\|_{Lip_\gamma}\, 2^{-kn+(j-k)\gamma-k\gamma}\left\|\chi_{B_K}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left|B_k\right|^{\frac{-1}{s}-\frac{-\gamma}{n}}\int_{B_j}\left|b_j(y)\right|\mathrm{d}y \qquad (3.2)\\
&\leq C\|b\|_{Lip_\gamma}\, 2^{-kn+(j-k)\gamma-k\gamma}\left\|\chi_{B_K}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left\|b_j\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For Υ_2^* similar to the method of Υ_1^* we have

$$\begin{aligned}
&\left\|\left|\frac{\left|\Omega(\cdot-y)\right|}{\left|\cdot-y\right|^n}-\frac{\left|\Omega(\cdot)\right|}{\left|\cdot\right|^n}\right|\chi_k(\cdot)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq \left\|\left|\frac{\left|\Omega(\cdot-y)\right|}{\left|\cdot-y\right|^n}-\frac{\left|\Omega(\cdot)\right|}{\left|\cdot\right|^n}\right|\chi_k(\cdot)\right\|_{L^{s(\cdot)}(\mathbb{R}^n)}\left\|\chi_k(\cdot)\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\leq \left\|\left|\frac{\left|\Omega(\cdot-y)\right|}{\left|\cdot-y\right|^n}-\frac{\left|\Omega(\cdot)\right|}{\left|\cdot\right|^n}\right|\chi_k(\cdot)\right\|_{L^{s(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_k}(\cdot)\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\leq 2^{(k-1)\left(\frac{n}{s}-n\right)}2^{(j-k)\gamma}\left\|\chi_{B_K}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\leq 2^{-kn+(j-k)\gamma-k\gamma}\left\|\chi_{B_k}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Now, by using the generalized Hölder's inequality we get:

$$\begin{aligned}\Upsilon_2^* &\le \int_{B_j}\left\|\left|\frac{|\Omega(\cdot-y)|}{|\cdot-y|^n}-\frac{|\Omega(\cdot)|}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}|b(0)-b(y)||b_j(y)|\mathrm{d}y\\ &\le C\|b\|_{Lip_\gamma}2^{-kn+2(j-k)\gamma}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|b_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\ &\le C\|b\|_{Lip_\gamma}2^{-kn+(j-k)\gamma}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|b_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{3.3}$$

Now by (3.3), (3.4), and lemmas 2.5 and 2.6, we have

$$\begin{aligned}&\|[b,\mu_\Omega](b_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\|b\|_{Lip_\gamma}2^{-kn+(j-k)\gamma}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|b_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\ &\le C\|b\|_{Lip_\gamma}2^{(j-k)\gamma}\|b_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}\\ &\le C2^{-j\alpha+(j-k)(\gamma+n\delta_2)}\|b\|_{Lip_\gamma}.\end{aligned}$$

Firstly we estimate I. We need to show that there exists a positive constant C, such that $I\le C\Delta$, we consider

$$\begin{aligned}I &= \sup_{L\ge0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}|\lambda_j|\|[b,\mu_\Omega](f)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le \sup_{L\ge0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=K}^{\infty}|\lambda_j|\|[b,\mu_\Omega]\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\\ &\quad+\sup_{L\ge0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|[b,\mu_\Omega]\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\\ &:=I_1+I_2.\end{aligned}$$

By the $\left(L^{p(\cdot)}(\mathbb{R}^n),L^{q(\cdot)}(\mathbb{R}^n)\right)$, bounbedness of the commutator $[b,\mu_\Omega]$ on $L^{p(\cdot)}$ (see [15]), we have the following. Therefore, when $0<q\le1$

$$\begin{aligned}I_1 &= \sup_{L\ge0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=K}^{\infty}\|[b,\mu_\Omega]\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\\ &\le \sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=k}^{\infty}|\lambda_j|2^{-j\alpha_j}\right)^q\\ &\le \sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=k}^{-1}|\lambda_j|2^{-j\alpha(0)jq}+\sum_{j=0}^{\infty}|\lambda_j|2^{-j\alpha_\infty q}\right)\\ &\le \sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\sum_{j=k}^{-1}|\lambda_j|^q2^{\alpha(0)(k-j)q}\\ &\quad+\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\sum_{j=0}^{\infty}|\lambda_j|^q2^{-j\alpha_\infty q}\\ &\le \sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{-1}|\lambda_j|^q\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q}\\ &\quad+\sup_{L\le0,L\in\mathbb{Z}}\sum_{j=0}^{\infty}2^{-j\lambda q}|\lambda_j|^q2^{(\lambda-\alpha_\infty)jq}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\end{aligned}$$

$$\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{L}\left|\lambda_j\right|^q+\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=L}^{-1}\left|\lambda_j\right|^q\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q}$$

$$+\Delta\sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=0}^{\infty}2^{(\lambda-\alpha_\infty)jq}\sum_{k=-\infty}^{L}2^{(\alpha(0)k-L\lambda)q}$$

$$\le \Delta+\sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=L}^{-1}2^{-j\lambda q}\left|\lambda_j\right|^q 2^{(j-L)\lambda q}\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q}+\Delta \tag{3.4}$$

$$\le \Delta+\sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=L}^{-1}2^{-j\lambda q}\left|\lambda_j\right|^q 2^{(j-L)\lambda q}\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q}$$

$$\le \Delta.$$

When $0<q<\infty$, let $1/q+1/q'=1$ we have

$$I_1=\sup_{L\ge 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=K}^{\infty}\left|\lambda_j\right|\left\|\left[b,\mu_\Omega\right]\chi_k\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q$$

$$\le \sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|2^{-j\alpha}\right)^q$$

$$\le \sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{j=k}^{-1}\left|\lambda_j\right|2^{\alpha(0)(k-j)}\right)^q$$

$$+\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=0}^{\infty}\left|\lambda_j\right|2^{-j\alpha_\infty}\right)^q$$

$$\le \sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{j=k}^{-1}\left|\lambda_j\right|^q 2^{\alpha(0)(k-j)q/2}\right)\left(\sum_{j=k}^{-1}2^{\alpha(0)(k-j)q'/2}\right)^{q/q'}$$

$$+\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=0}^{\infty}\left|\lambda_j\right|^q 2^{-j\alpha_\infty q/2}\right)\left(\sum_{j=0}^{\infty}2^{-j\alpha_\infty q'/2}\right)^{q/q'}$$

$$\le \sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\sum_{j=k}^{-1}\left|\lambda_j\right|^q 2^{(j-k)q/2}$$

$$+\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\sum_{j=0}^{\infty}\left|\lambda_j\right|^q 2^{-j\alpha_\infty q/2}$$

$$\le \sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{-1}\left|\lambda_j\right|^q\sum_{k=-\infty}^{j}2^{\alpha(0)(j-k)q/2}$$

$$+\sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=0}^{\infty}2^{j\lambda q}\left|\lambda_j\right|^q 2^{(\lambda-\alpha_\infty/2)jq}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}$$

$$\le \sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{L}\left|\lambda_j\right|^q+\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=L}^{-1}\left|\lambda_j\right|^q\sum_{j=-\infty}^{j}2^{\alpha(0)(k-j)q/2}$$

$$+\Delta\sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=0}^{\infty}2^{(\lambda-\alpha_\infty/2)j/2q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)-L\lambda q}$$

$$\le \Delta+\sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=L}^{-1}2^{-j\lambda q}\left|\lambda_j\right|^q 2^{(j-L)\lambda q}\sum_{j=-\infty}^{j}2^{\alpha(0)(k-j)q/2}+\Delta$$

$$\le \Delta+\Delta\sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=L}^{-1}2^{(j-L)\lambda q}\sum_{j=-\infty}^{j}2^{\alpha(0)(k-j)q/2} \tag{3.5}$$

$$\le \Delta.$$

We estimate I_2 by lemma 2.1 when $0<q\le 1$ by $n\delta_2\le\alpha(0)<\gamma+n\delta_2$, we get

$$
\begin{aligned}
I_2 &= \sup_{L\ge 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \left\| [b,\mu_\Omega] \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
&\le C\|b\|_{Lip_\gamma}^q \sup_{L\ge 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} 2^{-j\alpha+(j-k)(\gamma+n\delta_2)} |\lambda_j| \right)^p \\
&\le C\|b\|_{Lip_\gamma}^q \sup_{L\ge 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^p 2^{(-j\alpha+(j-k)(\gamma+n\delta_2))q} \right) \qquad (3.6)\\
&\le C\|b\|_{Lip_\gamma}^q \sup_{L\ge 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^p \sum_{k=j+1}^{-1} 2^{q(j-k)[\gamma+n\delta_2-\alpha(0)]} \\
&\le C\|b\|_{Lip_\gamma}^q \Delta.
\end{aligned}
$$

When $0<q<\infty$, let $1/q+1/q'=1$. Since $n\delta_2 \le \alpha(0) < \gamma+n\delta_2$, by Hölder's inequality, we have

$$
\begin{aligned}
I_2 &= \sup_{L\ge 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \left\| [b,\mu_\Omega] \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
&\le C\|b\|_{Lip_\gamma}^q \sup_{L\ge 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} 2^{-j\alpha+(j-k)(\gamma+n\delta_2)} |\lambda_j| \right)^p \\
&\le C\|b\|_{Lip_\gamma}^q \sup_{L\ge 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^p 2^{(-j\alpha+(j-k)(\gamma+n\delta_2))q/2} \right) \\
&\quad \times \left(\sum_{j=-\infty}^{k-1} 2^{(-j\alpha+(j-k)(\gamma+n\delta_2))q'/2} \right)^{q/q'} \\
&\le C\|b\|_{Lip_\gamma}^q \sup_{L\ge 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^p 2^{(-j\alpha+(j-k)(\gamma+n\delta_2))q/2} \right) \\
&\le C\|b\|_{Lip_\gamma}^q \sup_{L\ge 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^p \sum_{k=j+1}^{-1} 2^{q/2(j-k)[\gamma+n\delta_2-\alpha(0)]} \qquad (3.7)\\
&\le C\|b\|_{Lip_\gamma}^q \Delta.
\end{aligned}
$$

Secondly we estimate II. We need to show that there exists a positive constant C, such that $II \le C\Delta$, we consider

$$
\begin{aligned}
II &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left\| [b,\mu_\Omega](f)\chi_k \right\|_{L^{p(\cdot)}}^q \\
&\le \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| \left\| [b,\mu_\Omega](b_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \left\| [b,\mu_\Omega](b_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&:= II_1 + II_2.
\end{aligned}
$$

When $0<q\le 1$, we get

$$
\begin{aligned}
II_1 &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| \left\| [b,\mu_\Omega](b_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&\le \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} 2^{-j\alpha_j} |\lambda_j| \right)^p
\end{aligned}
$$

$$\begin{aligned}
&\le \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{-1} \left|\lambda_j\right|^q 2^{-jq\alpha(0)} + \sum_{j=0}^{\infty} \left|\lambda_j\right|^q 2^{-jq\alpha_\infty} \right) \\
&\le \sum_{k=-\infty}^{-1} \sum_{j=k}^{-1} \left|\lambda_j\right|^q 2^{\alpha(0)(k-j)q} + \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \sum_{j=0}^{\infty} \left|\lambda_j\right|^q 2^{-j\alpha_\infty q} \\
&\le \sum_{k=-\infty}^{-1} \left|\lambda_j\right|^q \sum_{k=\infty}^{j} 2^{q\alpha(0)(k-j)} + \sum_{j=0}^{\infty} \left|\lambda_j\right|^q 2^{-jq\alpha_\infty} \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \\
&\le \sum_{k=-\infty}^{-1} \left|\lambda_j\right|^q + \sum_{j=0}^{\infty} 2^{-j\lambda q} \left|\lambda_j\right|^q 2^{-jq\alpha_\infty} \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \\
&\le \Delta + \Delta \sum_{i=-\infty}^{j} \left|\lambda_i\right|^q \sum_{j=0}^{\infty} 2^{(\lambda-\alpha_\infty)jq} \sum_{k=-\infty}^{j} 2^{kq\alpha(0)} \\
&\le \Delta.
\end{aligned} \tag{3.8}$$

When $0 < q < \infty$, let $1/q_1 + 1/q_1' = 1$ we have

$$\begin{aligned}
II_1 &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} \left|\lambda_j\right| \left\| \left[b, \mu_\Omega\right]\left(b_j\right) \chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&\le \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} \left|\lambda_j\right| 2^{-j\alpha} \right)^q \\
&\le \sum_{k=-\infty}^{-1} \left(\sum_{j=k}^{-1} \left|\lambda_j\right| 2^{\alpha(0)(j-k)} \right)^q + \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=0}^{\infty} \left|\lambda_j\right| 2^{-j\alpha_\infty} \right)^q \\
&\le \sum_{k=-\infty}^{-1} \left(\sum_{j=k}^{-1} \left|\lambda_j\right|^q 2^{q/2\alpha(0)(j-k)} \right) \left(\sum_{j=k}^{-1} 2^{\alpha(0)(j-k)q'/2} \right)^{q/q'} \\
&\quad + \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=0}^{\infty} \left|\lambda_j\right|^q 2^{-q/2 j\alpha_\infty} \right) \left(\sum_{j=0}^{\infty} 2^{-q'/2 j\alpha_\infty} \right)^{q/q'} \\
&\le \sum_{k=-\infty}^{-1} \left|\lambda_j\right|^q \sum_{k=-\infty}^{j} 2^{q/2\alpha(0)(j-k)} + \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \sum_{j=0}^{\infty} \left|\lambda_j\right|^q 2^{-q/2 j\alpha_\infty} \\
&\le \sum_{k=-\infty}^{-1} \left|\lambda_j\right|^q + \sum_{j=0}^{\infty} 2^{(\lambda-\alpha_\infty/2)jq} 2^{-\lambda jq} \sum_{i=-\infty}^{j} \left|\lambda_i\right|^q \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \\
&\le \Delta + \Delta \sum_{j=0}^{\infty} 2^{(\lambda-\alpha_\infty/2)jq} \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \\
&\le \Delta.
\end{aligned} \tag{3.9}$$

For II_2, when $0 < q \le 1$, by $n\delta_2 \le \alpha(0) < \gamma + n\delta_2$ we get

$$\begin{aligned}
II_2 &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=\infty}^{k-1} \left|\lambda_j\right| \left\| \left[b, \mu_\Omega\right]\left(b_j\right) \chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&\le \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(C \left\|b\right\|_{Lip_\gamma} \sum_{j=\infty}^{k-1} \left|\lambda_j\right| 2^{-j\alpha+(j-k)(\gamma+n\delta_2)} \right)^q \\
&\le C \left\|b\right\|_{Lip_\gamma}^q \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=\infty}^{k-1} \left|\lambda_j\right|^q 2^{\left[-j\alpha+(j-k)(\gamma+n\delta_2)\right]q} \right) \\
&\le C \left\|b\right\|_{Lip_\gamma}^q \sum_{j=-\infty}^{-1} \left|\lambda_j\right|^q \sum_{k=j+1}^{-1} 2^{\left[-j\alpha_\infty+(j-k)(\gamma+n\delta_2)\right]q} \\
&\le C \left\|b\right\|_{Lip_\gamma}^q \Delta.
\end{aligned} \tag{3.10}$$

When $1<q<\infty$, let $1/q+1/q'=1$. Since $n\delta_2 \le \alpha(0) < \gamma + n\delta_2$, by Hölder's inequality, we have

$$
\begin{aligned}
II_2 &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=\infty}^{k-1} |\lambda_j| \left\| [b,\mu_\Omega](b_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&\le \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(C\|b\|_{Lip_\gamma} \sum_{j=\infty}^{k-1} |\lambda_j| 2^{-j\alpha+(j-k)(\gamma+n\delta_2)} \right)^q \\
&\le C\|b\|_{Lip_\gamma}^q \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=\infty}^{k-1} |\lambda_j|^q 2^{[-j\alpha+(j-k)(\gamma+n\delta_2)]q/2} \right) \\
&\quad \times \left(\sum_{j=\infty}^{k-1} |\lambda_j|^q 2^{[-j\alpha+(j-k)(\gamma+n\delta_2)]q'/2} \right)^{q/q'} \\
&\le C\|b\|_{Lip_\gamma}^q \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=\infty}^{k-1} |\lambda_j|^q 2^{[-j\alpha+(j-k)(\gamma+n\delta_2)]q/2} \right) \\
&\le C\|b\|_{Lip_\gamma}^q \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=j+1}^{-1} 2^{(j-k)[\gamma+n\delta_2-\alpha(0)]q/2} \\
&\le C\|b\|_{Lip_\gamma}^q \Delta.
\end{aligned}
\tag{3.11}
$$

Thirdly, we estimate III, we need to show that there exists a positive constant C, such that $III \le C\Delta$

$$
\begin{aligned}
III &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha(\infty)} \left\| [b,\mu_\Omega](f)\chi_k \right\|_{L^{p(\cdot)}}^q \\
&\le \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha(\infty)} \left(\sum_{j=k}^{\infty} |\lambda_j| \left\| [b,\mu_\Omega](b_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha(\infty)} \left(\sum_{j=\infty}^{k-1} |\lambda_j| \left\| [b,\mu_\Omega] L^{p(\cdot)} \chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&:= III_1 + III_2.
\end{aligned}
$$

When $0<q\le 1$, by the boundedness of $[b,\mu_\Omega]$ in $L^{p(\cdot)}$ ([20]), we have

$$
\begin{aligned}
III_1 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{j=k}^{\infty} |\lambda_j| \left\| [b,\mu_\Omega](b_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&\le \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \sum_{j=k}^{\infty} |\lambda_j|^q \left\| [b,\mu_\Omega](b_j)\chi_k \right\|_{L^{p(\cdot)}}^q \\
&\le \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \sum_{j=k}^{\infty} |\lambda_j|^q 2^{-\alpha_j jq} \\
&\le \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \sum_{j=k}^{\infty} |\lambda_j|^q 2^{-\alpha_\infty jq} \\
&= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^{L} |\lambda_j|^q \sum_{k=0}^{j} 2^{\alpha_\infty (k-j)q} + \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{\infty} |\lambda_j|^q \sum_{k=0}^{L} 2^{\alpha_\infty (k-j)q} \\
&\le \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} |\lambda_j|^q + \sup_{L\le 0, L\in\mathbb{Z}} \sum_{j=L}^{\infty} 2^{(j-L)q\lambda} 2^{-j\lambda q} \sum_{i=-\infty}^{j} |\lambda_i|^q \sum_{k=0}^{L} 2^{\alpha_\infty (k-j)q} \\
&\le \Delta + \Delta \sup_{L\le 0, L\in\mathbb{Z}} \sum_{j=L}^{\infty} 2^{(j-L)q\lambda} 2^{-j\lambda q} 2^{\alpha_\infty (L-j)q} \\
&\le \Delta + \Delta \sup_{L\le 0, L\in\mathbb{Z}} \sum_{j=L}^{\infty} 2^{(j-L)q(\lambda-\alpha_\infty)} \\
&\le \Delta.
\end{aligned}
\tag{3.12}
$$

When $0<q\le\infty$, by $n\delta_2\le\alpha(0),\alpha_\infty<\gamma+n\delta_2$ and the boundedness of $[b,\mu_\Omega]$ in $L^{p(\cdot)}$ ([20]) and Hölder's inequality, we get

$$\begin{aligned}
III_1 &= \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha(\infty)}\left(\sum_{j=k}^{\infty}|\lambda_j|\left\|[b,\mu_\Omega](b_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{j=k}^{\infty}|\lambda_j|^q\left\|[b,\mu_\Omega](b_j)\chi_k\right\|_{L^{p(\cdot)}}^{q/2}\right)\\
&\quad\times\left(\sum_{j=k}^{\infty}\left\|[b,\mu_\Omega](b_j)\chi_k\right\|_{L^{p(\cdot)}}^{q'/2}\right)^{q/q'}\\
&\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{j=k}^{\infty}|\lambda_j|^q\|b_j\|_{L^{p(\cdot)}}^{q/2}\right)\left(\sum_{j=k}^{\infty}\|b_j\|_{L^{p(\cdot)}}^{q'/2}\right)^{q/q'}\\
&\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{j=k}^{\infty}|\lambda_j|^q|B_j|^{-\alpha_j q/(2n)}\right)\left(\sum_{j=k}^{\infty}|B_j|^{-\alpha_j q'/(2n)}\right)^{q/q'}\\
&\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{j=k}^{\infty}|\lambda_j|^q|B_j|^{-\alpha_j q/(2n)}\right)\\
&\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq/2}\left(\sum_{j=k}^{\infty}|\lambda_j|^q|B_j|^{-\alpha_j q/(2n)}\right)\\
&= \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L}|\lambda_j|^q\sum_{k=0}^{j}2^{(k-j)\alpha_\infty q/2}+\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=L}^{\infty}|\lambda_j|^q\sum_{k=0}^{L}2^{(k-j)\alpha_\infty q/2}\\
&\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L}|\lambda_j|^q+\sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{(j-L)\lambda q}2^{-j\lambda q}\sum_{i=-\infty}^{j}|\lambda_j|^q\sum_{k=0}^{L}2^{(k-j)\alpha_\infty q/2}\\
&\le \Delta+\Delta\sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{(j-L)\lambda q}2^{(L-j)\alpha_\infty q/2}\\
&\le \Delta+\Delta\sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{(j-L)q(\alpha-\alpha_\infty/2)}\\
&\le \Delta.
\end{aligned}\tag{3.13}$$

When $0<q\le 1$, by $n\delta_2\le\alpha(0),\alpha_\infty<\gamma+n\delta_2$ we get

$$\begin{aligned}
III_2 &= \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{j=\infty}^{k-1}|\lambda_j|\left\|[b,\mu_\Omega](b_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(C\|b\|_{Lip_\gamma}^q\sum_{j=\infty}^{k-1}|\lambda_j|^q 2^{[-j\alpha_j+(j-k)(\gamma+n\delta_2)]q}\right)\\
&= C\|b\|_{Lip_\gamma}^q\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{j=-\infty}^{-1}|\lambda_j|^q 2^{[-j\alpha(0)+(j-k)(\gamma+n\delta_2)]q}\right)\\
&\quad+C\|b\|_{Lip_\gamma}^q\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha(\infty)}\left(\sum_{j=0}^{k-1}|\lambda_j|^q 2^{[-j\alpha_\infty+(j-k)(\gamma+n\delta_2)]q}\right)\\
&\le C\|b\|_{Lip_\gamma}^q\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq[\alpha_\infty+\gamma+n\delta_2]}\sum_{j=-\infty}^{-1}|\lambda_j|^q 2^{[\gamma+n\delta_2+\alpha(0)]jq}\\
&\quad+C\|b\|_{Lip_\gamma}^q\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L}|\lambda_j|^q\sum_{k=j+1}^{\infty}2^{[\gamma+n\delta_2-\alpha_\infty](j-k)q}\\
&\le C\|b\|_{Lip_\gamma}^q\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{-1}|\lambda_j|^q+C\|b\|_{Lip_\gamma}^q\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L-1}|\lambda_j|^q\\
&\le C\|b\|_{Lip_\gamma}^q\Delta.
\end{aligned}\tag{3.14}$$

When $1<q<\infty$, let $1/q+1/q'=1$. Since $n\delta_2 \le \alpha(0), \alpha_\infty < \gamma + n\delta_2$, and by Hölder's inequality, we have

$$\begin{aligned}
III_2 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha(\infty)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \left\| [b,\mu_\Omega](b_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&\le \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha(\infty)} \left(\sum_{j=-\infty}^{-1} C\|b\|^q_{Lip_\gamma} |\lambda_j| 2^{[-j\alpha_j+(j-k)(\gamma+n\delta_2)]} \right)^q \\
&\le C\|b\|^q_{Lip_\gamma} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha(\infty)} \left(\sum_{j=-\infty}^{-1} |\lambda_j| 2^{[-j\alpha(0)+(j-k)(\gamma+n\delta_2)]} \right)^q \\
&\quad + C\|b\|^q_{Lip_\gamma} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha(\infty)} \left(\sum_{j=0}^{k-1} |\lambda_j| 2^{[-j\alpha_\infty+(j-k)(\gamma+n\delta_2)]} \right)^q \\
&\le C\|b\|^q_{Lip_\gamma} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq[\alpha_\infty-(\gamma+n\delta_2)]} \left(\sum_{j=-\infty}^{-1} |\lambda_j| 2^{[(\gamma+n\delta_2)-\alpha(0)]j} \right)^q \\
&\quad + C\|b\|^q_{Lip_\gamma} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} \left(\sum_{j=0}^{k-1} |\lambda_j| 2^{(j-k)[\gamma+n\delta_2-\alpha_\infty]} \right)^q \\
&\le \left(C\|b\|^q_{Lip_\gamma} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{[(\gamma+n\delta_2)-\alpha(0)]jq/2} \right) \\
&\quad \times \left(\sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{[(\gamma+n\delta_2)-\alpha(0)]jq'/2} \right)^{q/q'} \\
&\quad + C\|b\|^q_{Lip_\gamma} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} \left(\sum_{j=0}^{k-1} |\lambda_j|^q 2^{(j-k)[\gamma+n\delta_2-\alpha_\infty]q/2} \right) \\
&\quad \times \left(\sum_{j=0}^{k-1} |\lambda_j| 2^{(j-k)[\gamma+n\delta_2-\alpha_\infty]q'/2} \right)^{q/q'} \\
&\le C\|b\|^q_{Lip_\gamma} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{[(\gamma+n\delta_2)-\alpha(0)]jq/2} \\
&\quad + C\|b\|^q_{Lip_\gamma} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} \sum_{j=0}^{k-1} |\lambda_j|^q 2^{(j-k)[\gamma+n\delta_2-\alpha_\infty]q/2} \\
&\le C\|b\|^q_{Lip_\gamma} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q \\
&\quad + C\|b\|^q_{Lip_\gamma} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^{L-1} |\lambda_j|^q \sum_{k=j+1}^{L} 2^{(j-k)[\gamma+n\delta_2-\alpha_\infty]q/2} \\
&\le C\|b\|^q_{Lip_\gamma} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q + C\|b\|^q_{Lip_\gamma} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^{L-1} |\lambda_j|^q \\
&\le C\|b\|^q_{Lip_\gamma} \Delta.
\end{aligned} \tag{3.15}$$

Joint the estimates for I, II and III, we obtain

$$\left\| [b,\mu_\Omega](f) \right\|^q_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)} \le C\|b\|^q_{Lip_\gamma} \|f\|_{HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}}.$$

Then we complete the proof of Theorem 3.1.

4. BMO Boundedness for the Commutator of Marcikiewicz Integrals Operator

In this section, we prove the boundedness of the commutator of Marcikiewicz integrals on Herz-Morrey-Hrdy spaces with variable exponent with function $b \in BMO(\mathbb{R}^n)$.

Theorem 4.1.

Suppose that $b \in BMO(\mathbb{R}^n)$ with $0<\gamma\le 1$. If $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ satisfies proposition 2.3 and $\Omega \in L^s(S^{n-1})(s>q'^{-})$. Let $0<p_1\le p_2<\infty$ and $0<\lambda<\alpha<n\delta_2-\gamma-\frac{n}{s}$ (or $0<\lambda<\alpha_1\le\alpha_1<n\delta_2-\gamma-\frac{n}{s}$). Then $[b,\mu_\Omega]$ is bounded from $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)$ (or $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)$) to $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)$ (or $MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)$).

proof:

In a way similar to theorem (3.2) we only prove the homogeneous case. Let $b \in BMO(\mathbb{R}^n)$ and $f \in HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)$. Let us write

$$f(x)=\sum_{j=0}^{\infty} f(x)\chi_j(x)=\sum_{j=0}^{\infty} f_j(x).$$

Then we have

$$\begin{aligned}\left\|[b,\mu_\Omega](f)\right\|^q_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)} &= \max\Bigg\{\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left\|[b,\mu_\Omega](f)\chi_k\right\|^q_{L^{p(\cdot)}},\\ &\quad \sup_{L\le 0,L\in Z} 2^{-L\lambda q}\Bigg(\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left\|[b,\mu_\Omega](f)\chi_k\right\|^q_{L^{p(\cdot)}}\\ &\quad +\sum_{k=0}^{L}2^{kq\alpha(\infty)}\left\|[b,\mu_\Omega](f)\chi_k\right\|^q_{L^{p(\cdot)}}\Bigg)\Bigg\}\\ &:= \max\{H, HH+HHH\}.\end{aligned}$$

$$H=\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left\|[b,\mu_\Omega](f)\chi_k\right\|^q_{L^{p(\cdot)}},$$

$$HH=\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left\|[b,\mu_\Omega](f)\chi_k\right\|^q_{L^{p(\cdot)}},$$

$$HHH=\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha(\infty)}\left\|[b,\mu_\Omega](f)\chi_k\right\|^q_{L^{p(\cdot)}}.$$

From the Hölder's inequality, we have

$$\begin{aligned}&\left|[b,\mu_\Omega](b_j)\chi_k\right|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\int_{B_j}\frac{|\Omega(x-y)|}{|x-y|^n}|b(x)-b(y)||f_j(y)|\mathrm{d}y\\ &\le C2^{-kn}\int_{B_j}|\Omega(x-y)||b(x)-b(y)||f_j(y)|\mathrm{d}y\\ &\le C2^{-kn}\Big(|b(x)-b_{B_j}|\int_{B_j}|\Omega(x-y)||f_j(y)|\mathrm{d}y\\ &\quad+\int_{B_j}|\Omega(x-y)||b_{B_j}-b(y)||f_j(y)|\mathrm{d}y\Big)\\ &\le C2^{-kn}\Big(|b(x)-b_{B_j}|\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\mathrm{d}y\\ &\quad+\left\|\Omega(x-\cdot)(b_{B_j}-b(\cdot))\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\mathrm{d}y\Big).\end{aligned}$$

Noting $s > q'^{-}$, we denote $\tilde{q}'(\cdot) > 1$ and $\frac{1}{q'(x)} = \frac{1}{\tilde{q}'(x)} + \frac{1}{s}$. By lemmas 3.2, 3.10 we have

$$\begin{aligned}\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} &\le \left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{s(\cdot)}(\mathbb{R}^n)}\left\|\chi_j(\cdot)\right\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \\ &\le \left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{s(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}(\cdot)\right\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \\ &\le 2^{-j\gamma}\left(\int_{A_j}|y|^{s\gamma}|\Omega(x-y)|^s\,\mathrm{d}y\right)^{\frac{1}{s}}\left\|\chi_{B_j}(\cdot)\right\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \\ &\le C2^{-j\gamma}|2|^{k\left(\gamma+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\left\|\chi_{B_j}(\cdot)\right\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

By lemma (2.12), when $|B_j| \le 2^n, x_j \in B_j$ and when $|B_k| \ge 1$ respectively we have

$$\left\|\chi_{B_j}\right\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)} \approx |B_k|^{\frac{1}{\tilde{q}'(x_k)}} \approx \left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{\frac{-1}{s}},$$

and

$$\left\|\chi_{B_j}\right\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{\frac{1}{\tilde{q}'(\infty)}} \approx \left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{\frac{-1}{s}},$$

we obtain $\left\|\chi_{B_j}\right\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)} \approx \left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{\frac{-1}{s}}$.

So we have

$$\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C2^{(k-j)\left(\gamma+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\left\|\chi_{B_j}(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \tag{4.1}$$

Similarly by lemma 2.4 we have

$$\begin{aligned}&\left\|\Omega(x-\cdot)\left(b_{B_j}-b(\cdot)\right)\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\le \left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^s(\mathbb{R}^n)}\left\|\left(b_{B_j}-b(\cdot)\right)\chi_j(\cdot)\right\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)} \\ &\le C\|b\|_*\left\|\chi_{B_j}(\cdot)\right\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)}\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^s(\mathbb{R}^n)} \\ &\le C\|b\|_* 2^{(k-j)\left(\gamma+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{4.2}$$

Now, by (4.1), (4.2), lemmas 2.4, 2.5 and 2.3, we have

$$\begin{aligned}&\left\|[b,\mu_\Omega](f_j)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C2^{-kn}\Bigg(2^{(k-j)\left(\gamma+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left\|\left(b(\cdot)-b_{B_j}\right)\chi_k(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad+\|b\|_* 2^{(k-j)\left(\gamma+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left\|\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\,\mathrm{d}y\Bigg) \\ &\le C2^{-kn}\Bigg((k-j)\|b\|_* 2^{(k-j)\left(\gamma+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\end{aligned}$$

$$+\|b\|_* 2^{(k-j)\left(\gamma+\frac{n}{s}\right)} \|\Omega\|_{L^s(S^{n-1})} \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \mathrm{d}y \Bigg)$$

$$\begin{aligned}
&\le C(k-j)\|b\|_* 2^{-kn} 2^{(k-j)\left(\gamma+\frac{n}{s}\right)} \|\Omega\|_{L^s(S^{n-1})} \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C(k-j)\|b\|_* 2^{(k-j)\left(\gamma+\frac{n}{s}\right)} \|\Omega\|_{L^s(S^{n-1})} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \\
&\le C\|b\|_* (k-j) 2^{(k-j)\left(n\delta_2-\gamma-\frac{n}{s}\right)} \|\Omega\|_{L^s(S^{n-1})} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{4.3}$$

By the boundedness of μ_Ω in $L^{p(\cdot)}$ see [7], we have

$$\|(\mu_\Omega f_j)\chi_k\|_{L^{p(\cdot)}} \le \|f_j\|_{L^{p(\cdot)}} \le |B_j|^{-\alpha_j/n} = 2^{-j\alpha_j}.$$

So we have

$$\|[b,\mu_\Omega](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le \|b\|_* (k-j)\|\Omega\|_{L^s(S^{n-1})} 2^{(k-j)\left(n\delta_2-\gamma-\frac{n}{s}\right)-j\alpha_j}.$$

Firstly we estimate H. We need to show that there exists a positive constant C, such that $H \le C\Delta$ Consider

$$\begin{aligned}
H &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \|[b,\mu_\Omega](f)\chi_k\|_{L^{p(\cdot)}}^q \\
&\le \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| \|[b,\mu_\Omega](f_j)\chi_k\|_{L^{p(\cdot)}}^q\right)^q \\
&\quad + \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|[b,\mu_\Omega](f_j)\chi_k\|_{L^{p(\cdot)}}^q\right)^q \\
&:= H_1 + H_2.
\end{aligned}$$

By boundedness of $[b,\mu_\Omega]$ in $L^{p(\cdot)}$, see ([20]), when $0<q\le 1$ we have

$$\begin{aligned}
H_1 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| \|[b,\mu_\Omega](f_j)\chi_k\|_{L^{p(\cdot)}}^q\right)^q \\
&\le \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| 2^{-j\alpha_j}\right)^q \\
&\le \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=k}^{-1} |\lambda_j| 2^{-j\alpha(0)jq} + \sum_{j=0}^{\infty} |\lambda_j| 2^{-j\alpha_\infty q}\right) \\
&\le \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{j=k}^{-1} |\lambda_j|^q 2^{\alpha(0)(k-j)q} \\
&\quad + \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-j\alpha_\infty q} \\
&\le \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=-\infty}^{j} 2^{\alpha(0)(k-j)q} \\
&\quad + \sup_{L\le 0, L\in\mathbb{Z}} \sum_{j=0}^{\infty} 2^{-j\lambda q} |\lambda_j|^q 2^{(\lambda-\alpha_\infty)jq} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)}
\end{aligned}$$

$$\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{L}\left|\lambda_j\right|^q + \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=L}^{-1}\left|\lambda_j\right|^q\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q}$$

$$+\Delta \sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=0}^{\infty}2^{(\lambda-\alpha_\infty)jq}\sum_{k=-\infty}^{L}2^{(\alpha(0)k-L\lambda)q}$$

$$\le \Delta + \sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=L}^{-1}2^{-j\lambda q}\left|\lambda_j\right|^q 2^{(j-L)\lambda q}\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q}+\Delta \tag{4.4}$$

$$\le \Delta + \sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=L}^{-1}2^{-j\lambda q}\left|\lambda_j\right|^q 2^{(j-L)\lambda q}\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q}$$

$$\le \Delta.$$

When $1<q<\infty$ and $1/q+1/q'=1$, and let $\gamma+n\delta_2-\alpha>0$, we have

$$H_1 = \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|\left\|[b,\mu_\Omega](f_j)\chi_k\right\|_{L^{p(\cdot)}}^q\right)^q$$

$$\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|2^{-j\alpha}\right)^q$$

$$\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{j=k}^{-1}\left|\lambda_j\right|2^{\alpha(0)(k-j)}\right)^q$$

$$+\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=0}^{\infty}\left|\lambda_j\right|2^{-j\alpha_\infty}\right)^q$$

$$\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{j=k}^{-1}\left|\lambda_j\right|^q 2^{\alpha(0)(k-j)q/2}\right)\left(\sum_{j=k}^{-1}2^{\alpha(0)(k-j)q'/2}\right)^{q/q'}$$

$$+\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=0}^{\infty}\left|\lambda_j\right|^q 2^{-j\alpha_\infty q/2}\right)\left(\sum_{j=0}^{\infty}2^{-j\alpha_\infty q'/2}\right)^{q/q'}$$

$$\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\sum_{j=k}^{-1}\left|\lambda_j\right|^q 2^{(j-k)q/2}$$

$$+\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\sum_{j=0}^{\infty}\left|\lambda_j\right|^q 2^{-j\alpha_\infty q/2}$$

$$\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{-1}\left|\lambda_j\right|^q\sum_{k=-\infty}^{j}2^{\alpha(0)(j-k)q/2}$$

$$+\sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=0}^{\infty}2^{j\lambda q}\left|\lambda_j\right|^q 2^{(\lambda-\alpha_\infty/2)jq}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}$$

$$\le \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{L}\left|\lambda_j\right|^q + \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=L}^{-1}\left|\lambda_j\right|^q\sum_{j=-\infty}^{j}2^{\alpha(0)(k-j)q/2}$$

$$+\Delta \sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=0}^{\infty}2^{(\lambda-\alpha_\infty/2)j/2q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)-L\lambda q}$$

$$\le \Delta + \sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=L}^{-1}2^{-j\lambda q}\left|\lambda_j\right|^q 2^{(j-L)\lambda q}\sum_{j=-\infty}^{j}2^{\alpha(0)(k-j)q/2}+\Delta$$

$$\le \Delta + \Delta \sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=L}^{-1}2^{(j-L)\lambda q}\sum_{j=-\infty}^{j}2^{\alpha(0)(k-j)q/2} \tag{4.5}$$

$$\le \Delta.$$

Now we estimate H_2, when $0<q\le 1$, by $n\delta_2\le\alpha(0)<n\delta_2-\gamma-\frac{n}{s}$, we get

$$
\begin{aligned}
H_2 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \left\| [b,\mu_\Omega](f_j)\chi_k \right\|_{L^{p(\cdot)}}^q \right)^q \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q (k-j)^q 2^{\left[-j\alpha(0)+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q} \right) \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} |\lambda_j|^q \sum_{j=k}^{\infty} (k-j)^q 2^{q(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha(0)\right)} \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \Delta.
\end{aligned} \tag{4.6}
$$

when $1<q<\infty$, let $1/q+1/q'=1$. Since $n\delta_2<\alpha(0)\le\gamma+n\delta_2$, by Hölder's inequality we have

$$
\begin{aligned}
H_2 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \left\| [b,\mu_\Omega](f_j)\chi_k \right\|_{L^{p(\cdot)}}^q \right)^q \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| (k-j) 2^{\left[-j\alpha+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]} \right)^q \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=k}^{k-1} |\lambda_j|^q (k-j)^q 2^{\left[-j\alpha+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q/2} \right) \\
&\quad \times \left(\sum_{j=0}^{\infty} 2^{\left[-j\alpha+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q/2} \right)^{q/q'} \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=k}^{k-1} |\lambda_j|^q (k-j)^q 2^{\left[-j\alpha+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q/2} \right) \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} |\lambda_j|^q \sum_{k=j+1}^{-1} (k-j)^q 2^{\left[+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha\right)\right]q/2} \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \Delta.
\end{aligned} \tag{4.7}
$$

Secondly, we estimate HH. We need to show that there exists a positive constant *C*, such that $HH \le C\Delta$ Consider

$$
\begin{aligned}
HH &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left\| [b,\mu_\Omega](f)\chi_k \right\|_{L^{p(\cdot)}}^q \\
&\le \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} \left\| [b,\mu_\Omega](f_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} \left\| [b,\mu_\Omega](f_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&:= HH_1 + HH_2.
\end{aligned}
$$

When $0<q\le 1$, we get

$$HH_1 = \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} \left\| [b,\mu_\Omega](f_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q$$

$$\leq C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| (k-j) 2^{\left[-j\alpha+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]} \right)^q$$

$$\leq C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j|^q (k-j)^q 2^{\left[-j\alpha(0)+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q} \right.$$

$$\left. + \sum_{j=k}^{\infty} |\lambda_j|^q (k-j)^q 2^{\left[-j\alpha_\infty+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q} \right)$$

$$\leq C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} \sum_{j=k}^{-1} |\lambda_j|^q (k-j)^q 2^{q(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha(0)\right)}$$

$$+ C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \sum_{j=0}^{-\infty} |\lambda_j|^q (k-j) 2^{\left[-j\alpha_\infty+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q}$$

$$\leq C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} |\lambda_j|^q \sum_{j=-\infty}^{j} (k-j)^q 2^{q(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha(0)\right)}$$

$$+ C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \sum_{j=0}^{\infty} |\lambda_j|^q (k-j) 2^{\left[-j\alpha_\infty+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q}$$

$$\leq C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} |\lambda_j|^q$$

$$+ C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{j=0}^{\infty} (k-j) 2^{\left[(\lambda-\alpha_\infty)j+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q} 2^{j\lambda q} \sum_{i=\infty}^{j} |\lambda_i| \sum_{k=-\infty}^{j} 2^{kq\alpha(0)}$$

$$\leq C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \Delta \tag{4.8}$$

$$+ C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \Delta \sum_{j=0}^{\infty} (k-j) 2^{\left[(\lambda-\alpha_\infty)j+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q} \sum_{k=-\infty}^{j} 2^{kq\alpha(0)}$$

$$\leq C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \Delta.$$

Now when $1 < q < \infty$, let $1/q + 1/q' = 1$ we have

$$HH_1 = \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} \left\| [b,\mu_\Omega](f_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q$$

$$\leq C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| (k-j) 2^{\left[-j\alpha+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]} \right)^q$$

$$\leq C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} \left(\sum_{j=k}^{-1} |\lambda_j| (k-j) 2^{\left[(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)-\alpha(0)\right]} \right)^q$$

$$+ C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=0}^{\infty} |\lambda_j| (k-j) 2^{\left[-j\alpha_\infty+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]} \right)^q$$

$$\leq C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} \left(\sum_{j=k}^{-1} |\lambda_j|^q (k-j)^q 2^{\left[(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)-\alpha(0)\right]q/2} \right)$$

$$\times\left(\sum_{j=k}^{-1}(k-j)^{q}\,2^{\left[(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)-\alpha(0)\right]q'/2}\right)^{q/q'}$$

$$+C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left(\sum_{j=0}^{\infty}|\lambda_j|^q(k-j)^q\,2^{\left[-j\alpha_\infty+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q/2}\right)^{q}$$

$$\times\left(\sum_{j=0}^{\infty}(k-j)^{q}\,2^{\left[-j\alpha_\infty+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q'/2}\right)^{q/q'}$$

$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sum_{k=-\infty}^{-1}|\lambda_j|^q\sum_{k=-\infty}^{j}(k-j)^q\,2^{\left[(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)-\alpha(0)\right]q/2}$$

$$+C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\sum_{j=0}^{\infty}|\lambda_j|^q(k-j)^q\,2^{\left[-j\alpha_\infty+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q/2}$$

$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sum_{k=-\infty}^{-1}|\lambda_j|^q\sum_{j=k}^{-1}(k-j)^q\,2^{\left[(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)-\alpha(0)\right]q/2}$$

$$+C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sum_{j=0}^{\infty}|\lambda_j|^q\,2^{kq\left(\alpha(0)-\left(n\delta_2-\gamma-\frac{n}{s}\right)/2\right)}\sum_{k=-\infty}^{-1}(k-j)^q\,2^{\left(n\delta_2-\gamma-\frac{n}{s}-\alpha_\infty\right)jq/2}$$

$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sum_{k=-\infty}^{-1}|\lambda_j|^q$$

$$+C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sum_{j=0}^{\infty}(k-j)^q\,2^{\left[(\lambda-\alpha_\infty/2)j+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)q/2\right]}2^{-jq\lambda}\sum_{i=-\infty}^{j}|\lambda_j|^q\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}$$

$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\Delta$$

$$+C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\Delta\sum_{j=0}^{\infty}(k-j)^q\,2^{\left[(\lambda-\alpha_\infty/2)j+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)q/2\right]}\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}$$

$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\Delta. \tag{4.9}$$

For HH_2, when $0<q\le 1$, by $n\delta_2\le\alpha(0)<s+\delta+n\delta_2$ we get

$$HH_2=\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left(\sum_{j=-\infty}^{k-1}\left\|[b,\mu_\Omega](f_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q$$

$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|(k-j)\,2^{\left[-j\alpha(0)+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]}\right)^q$$

$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|^q(k-j)^q\,2^{q\left[-j\alpha(0)+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]}\right) \tag{4.10}$$

$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sum_{k=-\infty}^{-1}|\lambda_j|^q\sum_{k=j+1}^{-1}(k-j)^q\,2^{q(j-k))\left(n\delta_2-\gamma-\frac{n}{s}-\alpha(0)\right)}$$

$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\Delta.$$

Now $1<q<\infty$, let $1/q+1/q'=1$. Since $n\delta_2\le\alpha(0)<s+\delta+n\delta_2$, by Hölder's inequality we have

$$
\begin{aligned}
HH_2 &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} \left\| [b,\mu_\Omega](f_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| (k-j) 2^{-j\alpha(0)+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)} \right)^q \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q (k-j)^q 2^{\left[-j\alpha(0)+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q} \right) \\
&\times \left(\sum_{j=-\infty}^{k-1} (k-j) 2^{\left[-j\alpha(0)+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q'/2} \right)^{q/q'} \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q (k-j)^q 2^{\left[-j\alpha(0)+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q/2} \right) \\
&= C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=-\infty}^{-1} |\lambda|^q \sum_{k=j+1}^{-1} (k-j)^q 2^{(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha(0)\right)q/2} \qquad (4.11) \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \Delta.
\end{aligned}
$$

Thirdly, we estimate HHH, we need to show that there exists a positive constant C, such that $HHH \le C\Delta$

$$
\begin{aligned}
HHH &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left\| [b,\mu_\Omega](f)\chi_k \right\|_{L^{p(\cdot)}}^q \\
&\le \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{j=k}^{\infty} \left\| [b,\mu_\Omega](f_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{j=-\infty}^{k-1} \left\| [b,\mu_\Omega](f_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&:= HHH_1 + HHH_2.
\end{aligned}
$$

When $0 < q \le 1$ by boundedness of $[b,\mu_\Omega]$ in $L^{p(\cdot)}$

$$
\begin{aligned}
HHH_1 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{j=k}^{\infty} \left\| [b,\mu_\Omega](f_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&\le \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \sum_{j=k}^{\infty} \left\| [b,\mu_\Omega](f_j)\chi_k \right\|_{L^{p(\cdot)}}^q \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \sum_{j=k}^{\infty} |\lambda_j|^q (k-j)^q 2^{q\left[-j\alpha_j+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]} \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \sum_{j=k}^{\infty} |\lambda_j|^q (k-j)^q 2^{q\left[-j\alpha_\infty+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]} \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^{L} |\lambda_j|^q \sum_{k=0}^{j} (k-j)^q 2^{q(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha_\infty\right)} \\
&+ C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{\infty} |\lambda_j|^q \sum_{k=0}^{L} (k-j)^q 2^{q(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha_\infty\right)} \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^{L} |\lambda_j|^q
\end{aligned}
$$

$$+C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{(j-L)q\lambda}2^{-L\lambda q}\sum_{i=-\infty}^{j}|\lambda_i|^q\sum_{k=0}^{L}(k-j)^q\,2^{q(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha_\infty\right)}$$
$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\Delta+C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\Delta\sum_{k=0}^{L}(k-j)^q\,2^{q(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha_\infty\right)}. \tag{4.12}$$

Now when $0<q<\infty$, by boundedness of $[b,\mu_\Omega]$ in $L^{p(\cdot)}$, see ([20]) by Hölder's inequality we have

$$HHH_1=\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{j=k}^{\infty}\left\|[b,\mu_\Omega](f_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q$$
$$\le\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{j=k}^{\infty}\left\|[b,\mu_\Omega](f_j)\chi_k\right\|_{L^{p(\cdot)}}^{q/2}\right)\times\left(\sum_{j=k}^{\infty}\left\|[b,\mu_\Omega](f_j)\chi_k\right\|_{L^{p(\cdot)}}^{q'/2}\right)^{q/q'}$$
$$\le\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{j=k}^{\infty}|\lambda_j|^q\|b_j\|_{L^{p(\cdot)}}^{q/2}\right)\times\left(\sum_{j=k}^{\infty}\|b_j\|_{L^{p(\cdot)}}^{q'/2}\right)^{q/q'}$$
$$\le\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{j=k}^{\infty}|\lambda_j|^q|B_j|^{-\alpha_j q/(2n)}\right)\times\left(\sum_{j=k}^{\infty}|B_j|^{-\alpha_j q'/(2n)}\right)^{q/q'}$$
$$\le\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq/2}\left(\sum_{j=k}^{\infty}|\lambda_j|^q|B_j|^{-\alpha_j q/(2n)}\right)$$
$$=\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=0}^{L}|\lambda_j|^q\sum_{k=0}^{j}2^{\alpha_\infty(k-j)q/2}+\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=L}^{\infty}|\lambda_j|^q\sum_{k=0}^{L}2^{\alpha_\infty(k-j)q/2}$$
$$\le\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=L}^{\infty}|\lambda_j|^q+\sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{(j-L)q\lambda}2^{-j\lambda q}\sum_{i=-\infty}^{j}|\lambda_i|^q\sum_{k=0}^{L}2^{\alpha_\infty(k-j)q/2} \tag{4.13}$$
$$\le\Delta+\Delta\sup_{L\le 0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{(j-L)q\lambda}2^{\alpha_\infty(k-j)q/2}$$
$$\le\Delta.$$

We have $0<q\le 1$, by $n\delta_2\le\alpha(0),\alpha_\infty<s+\delta+n\delta_2$ we get

$$HHH_2=\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{j=-\infty}^{k-1}\left\|[b,\mu_\Omega](f_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q$$
$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha_\infty}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|^q(k-j)^q\,2^{\left[-j\alpha_j+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q}\right)$$
$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha_\infty}\left(\sum_{j=-\infty}^{-1}|\lambda_j|^q(k-j)^q\,2^{\left[-j\alpha(0)+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q}\right)$$
$$+C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{j=0}^{k-1}|\lambda_j|^q(k-j)^q\,2^{\left[-j\alpha_\infty+(j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)\right]q}\right)$$
$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq[\alpha_\infty+\gamma+n\delta_2]}\sum_{j=-\infty}^{-1}|\lambda_j|^q(k-j)^q\,2^{\left[n\delta_2-\gamma-\frac{n}{s}-\alpha(0)\right]jq}$$
$$+C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}|\lambda_j|^q\sum_{j=0}^{k-1}(k-j)^q\,2^{(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha_\infty\right)q} \tag{4.14}$$
$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{-1}|\lambda_j|^q+C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L-1}|\lambda_j|^q$$
$$\le C\|b\|_*^q\|\Omega\|_{L^s(S^{n-1})}^q\Delta.$$

Now when $1<q<\infty$, let $1/q+1/q'=1$. Since $n\delta_2 \le \alpha(0), \alpha(\infty) < s+\delta+n\delta_2$, by Hölder's inequality, we have

$$
\begin{aligned}
HHH_2 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{j=-\infty}^{k-1} \left\| [b,\mu_\Omega](f_j)\chi_k \right\|_{L^{p(\cdot)}} \right)^q \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| (k-j) 2^{-j\alpha_j + (j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)} \right)^q \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{j=-\infty}^{-1} |\lambda_j| (k-j) 2^{-j\alpha(0) + (j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)} \right)^q \\
&\quad + C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{j=0}^{k-1} |\lambda_j| (k-j) 2^{-j\alpha_\infty + (j-k)\left(n\delta_2-\gamma-\frac{n}{s}\right)} \right)^q \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=0}^{L} 2^{kq\left[\alpha_\infty - \left(n\delta_2-\gamma-\frac{n}{s}\right)\right]} \left(\sum_{j=-\infty}^{-1} |\lambda_j| (k-j) 2^{j\left(n\delta_2-\gamma-\frac{n}{s}-\alpha(0)\right)} \right)^q \\
&\quad + C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=0}^{L} \left(\sum_{j=0}^{k-1} |\lambda_j| (k-j) 2^{(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha_\infty\right)} \right)^q \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \left(\sum_{j=-\infty}^{-1} |\lambda_j|^q (k-j)^q 2^{jq/2\left(n\delta_2-\gamma-\frac{n}{s}-\alpha(0)\right)} \right) \\
&\quad \times \left(\sum_{j=-\infty}^{-1} 2^{q'/2 j\left(n\delta_2-\gamma-\frac{n}{s}-\alpha(0)\right)} \right)^{q/q'} \\
&\quad + C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=0}^{L} \left(\sum_{j=0}^{k-1} |\lambda_j|^q (k-j)^q 2^{q/2(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha_\infty\right)} \right) \\
&\quad \times \left(\sum_{j=0}^{k-1} 2^{q'/2(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha_\infty\right)} \right)^{q/q'} \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{j=-\infty}^{-1} |\lambda_j|^q (k-j)^q 2^{jq/2\left(n\delta_2-\gamma-\frac{n}{s}-\alpha(0)\right)} \\
&\quad + C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=0}^{L} \sum_{j=0}^{k-1} |\lambda_j|^q (k-j)^q 2^{q/2(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha_\infty\right)} \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{j=-\infty}^{-1} |\lambda_j|^q \\
&\quad + C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=0}^{L-1} |\lambda_j|^q \sum_{k=j+1}^{L} (k-j)^q 2^{q/2(j-k)\left(n\delta_2-\gamma-\frac{n}{s}-\alpha_\infty\right)} \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{j=-\infty}^{-1} |\lambda_j|^q + C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \sum_{k=0}^{L-1} |\lambda_j|^q \\
&\le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^q \Delta.
\end{aligned}
\tag{4.15}
$$

Joint the estimates for H, HH and HHH, we obtain

$$\left\| [b,\mu_\Omega](f) \right\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n)}^q \le C\|b\|_*^q \|\Omega\|_{L^s(S^{n-1})}^p \|f\|_{HMK_{p(\cdot)\lambda}^{\alpha(\cdot),q}}.$$

Then we complete the proof of Theorem 4.1.

5. Conclusion

The study concluded that we can proof of boundedness for commutator of Marcinkiewicz integrals on Herz-Morrey-Hrdy spaces with variable exponent, which we use The main tools are properties of variable exponent in theorem 3.1 when $b \in Lip_{\gamma}\left(\mathbb{R}^n\right)$, in theorem 4.1 when $b \in BMO\left(\mathbb{R}^n\right)$. We can obtain a solution for proof that commutator of Marcinkiewicz integrals are boundedness.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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