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Research Article

Control of the Cauchy System for an Elliptic Operator: The Controllability Method

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In this paper, we are dealing with the ill-posed Cauchy problem for an elliptic operator. This is a follow-up to a previous paper on the same subject. Indeed, in an earlier publication, we introduced a regularization method, called the controllability method, which allowed us to propose, on the one hand, a characterization of the existence of a regular solution to the ill-posed Cauchy problem. On the other hand, we have also succeeded in proposing, via a strong singular optimality system, a characterization of the optimal solution to the considered control problem, and this, without resorting to the Slater-type assumption, an assumption to which many analyses had to resort. On occasion, we have dealt with the control problem, with state boundary observation, the problem initially analyzed by J. L. Lions. The proposed point of view, consisting of the interpretation of the Cauchy system as a system of two inverse problems, then called naturally for conjectures in favor of which the present manuscript wants to constitute an argument. Indeed, we conjectured, in view of the first results obtained, that the proposed method could be improved from the point of view of the initial interpretation that we had made of the problem. In this sense, we analyze here two other variants (observation of the flow, then distributed observation) of the problem, the results of which confirm the intuition announced in the previous publication mentioned above. Those results, it seems to us, are of significant relevance in the analysis of the controllability method previously introduced.

1. Introduction

Let Ω be a regular bounded open subset of $\mathbb{R}^n$, of boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are disjointed, regular, and with superficial positive measures.

In Ω, we consider the state z and the control $v = (v_0, v_1)$ linked by

$$\begin{aligned} &\Delta z = 0 \quad \text{in}\, \Omega, \\ &z = v_0, \quad \frac{\partial z}{\partial \nu} = v_1 \quad \text{on}\, \Gamma_0. \end{aligned} \tag{1}$$

Problem (1) is ill-posed in Hadamard's sense. This means that, for $v = (v_0, v_1)$ given in $(L^2(\Gamma_0))^2$, the problem (1) does not always admit a solution, and there may be an instability of it when it exists. We refer to (1) as the ill-posed elliptic Cauchy problem.

We, therefore, consider a priori the pairs (v, z) such as

$$v = (v_0, v_1) \in (L^2(\Gamma_0))^2 \text{ and } z \in L^2(\Omega), \tag{2}$$

where (v, z) is solution of (1). It is said that such pairs (v, z) constitute the control-state pairs set.

Remark 1. Note that, when it exists, the solution of the ill-posed Cauchy problem (1) is unique.

Let $\mathscr{U}_{ad}^0$ and $\mathscr{U}_{ad}^1$ be two nonempty convex closed subsets of $L^2(\Gamma_0)$. We set

$$\mathcal{U}_{ad} = \mathcal{U}^0_{ad} \times \mathcal{U}^1_{ad}, \tag{3}$$

and

$$\mathcal{A} = \{(v, z) \in \mathcal{U}_{ad} \times L^2(\Omega) \text{ satisfying } (1) \text{ with } v = (v_0, v_1)\}. \tag{4}$$

A control-state pair (v, z) will be said admissible if $(v, z) \in \mathcal{A}$. We will refer to $\mathcal{A}$ as the set of admissible control-state pairs.

It is then a question of knowing how to characterize, via a strong singular optimality system, the optimal pair, known as the optimal control-state pair, the solution to the control problem

$$\inf\{J(v, z); (v, z) \in \mathcal{A}\}, \tag{5}$$

where the functional J can be, for example,

$$J(v, z) = \frac{1}{2}\left\|\frac{\partial z}{\partial \nu} - z_d\right\|^2_{L^2(\Gamma_1)} + \frac{N_0}{2}\|v_0\|^2_{L^2(\Gamma_0)} + \frac{N_1}{2}\|v_1\|^2_{L^2(\Gamma_0)}, \; z_d \in L^2(\Gamma_1), \tag{6}$$

or

$$J(v, z) = \frac{1}{2}\|z - z_d\|^2_{L^2(\Omega)} + \frac{N_0}{2}\|v_0\|^2_{L^2(\Gamma_0)} + \frac{N_1}{2}\|v_1\|^2_{L^2(\Gamma_0)}, \; z_d \in L^2(\Omega). \tag{7}$$

The following remark is then in order.

Remark 2. If $z \in L^2(\Omega)$ with $\Delta z = 0$, we have that $z \in H^{-1/2}(\Gamma)$ and $\frac{\partial z}{\partial \nu} \in H^{-3/2}(\Gamma)$ (see [1]).

So, the cost Function (6) must therefore be considered on the sets of admissible control-state pairs such as, in addition,

$$\left.\frac{\partial z}{\partial \nu}\right|_{\Gamma_1} \in L^2(\Gamma_1); \tag{8}$$

and it is, therefore, such sets that must be assumed, not empty, for the problem to make sense. So we necessarily have, for the problems (1), (6), (5) and (1), (7), (5), that $z \in H^{3/2}(\Omega)$.

The original problem analyzed by Lions [1] considered the cost function

$$J(v, z) = \frac{1}{2}\|z - z_d\|^2_{L^2(\Gamma_1)} + \frac{N_0}{2}\|v_0\|^2_{L^2(\Gamma_0)} + \frac{N_1}{2}\|v_1\|^2_{L^2(\Gamma_0)}, \; z_d \in L^2(\Gamma_1). \tag{9}$$

In order to obtain a singular optimality system where state and control are independent, Lions [1] uses the penalization method that makes it possible to obtain convergence results in particular cases

(1) $\mathcal{U}^0_{ad} = L^2(\Gamma_0)$, $\mathcal{U}^1_{ad} \subset L^2(\Gamma_0)$.
(2) $\mathcal{U}^0_{ad} \subset L^2(\Gamma_0)$, $\mathcal{U}^1_{ad} = L^2(\Gamma_0)$.

In the first case, a strong singular optimality system is directly obtained. But in the second case, we obtain a weak singular optimality system, whose strong formulation requires the additional assumption of Slater type that

$$\text{The interior of } \mathcal{U}^0_{ad} \text{ is non empty in } L^2(\Gamma_0). \tag{10}$$

However, Lions [1] conjectures that one should be able to solve the problem with only the usual assumptions of nonvacuity, convexity, and closure of the sets of admissible controls $\mathcal{U}^0_{ad}$ and $\mathcal{U}^1_{ad}$. Conjecture for which this paper is intended to constitute an argument.

Indeed, in a previous publication (cf. [2]), where we analyzed the problem initially posed by Lions [1] (the one considering the cost Function (9)), we managed to verify the conjecture of Lions [1]. Introducing, to do this, a regularization method called the controllability method, which consists of the interpretation of the Cauchy problem as a system of inverse problems. We show that, when it exists, the solution of the Cauchy problem (1) is a common solution of a system of two (well-posed) mixed problems resulting from the interpretation that we make of the problem, managing in passing to characterize the existence of a regular solution to the Cauchy problems itself. The initial control problem is then approached by a sequence of (classical) control problems posed on the mixed problems obtained. The novelty with the proposed method is that it allows, as announced, to know how to overcome the Slater-type assumption in the characterization of the optimal control-state pair; the interpretation that we make of the problem being sufficient to obtain directly the strong convergence of the process. And it is there, in the interpretation that we make of the problem as a system of inverse problems, all the originality of the proposed method, this point of view not having, to the best of our knowledge, been approached in work prior to Guel and Nakoulima [2], at least as far as the control of the Cauchy problem is concerned.

The results obtained in favor of these first reflections (cf. [2]) called for natural conjectures as to the pioneering analysis of Lions [1] concerning the appearance of the Slater-type assumption in one of the cases treated rather than in the other. But also as to ways of improving the method we are proposing. Indeed, we conjectured that sooner than considering both systems resulting from the interpretation of the problem as an inverse problem, we could be satisfied with only one of these states, according to the following specifications:

(i) For the boundary observation problem: the nature of the observation would dictate the adequate system to be considered.
(ii) For the one with distributed observation: one or the other of the systems should suffice; the choice then

being naturally guided by the ease of the difficulty in observing/controlling one or the other of the Cauchy data.

The results we present here are intended as an argument in favor of these conjectures. They certainly do not finish clarifying the point of view but are already reassured of the intuition inspired by the first.

Before going any further in this presentation, note that many authors have studied, mostly in the case of distributed observation, the control of the ill-posed Cauchy problem. Indeed, following the work of Lions [1], the first to be interested in the problem was Nakoulima [3], who obtained, for the cost Function (7), results already confirming the conjecture of Lions [1], without to end up addressing the problem. The results in question, using a regularization-penalization method, managed to do without the Slater-type assumption, but only for one class of constraints, namely in the case

$$\mathscr{U}_{ad}^{0} = \mathscr{U}_{ad}^{1} = (L^2(\Gamma_0))^{+}. \quad (11)$$

The control spaces are then considered being of the empty interior; the conjecture of Lions [1] is well confirmed by these results. Nevertheless, the problem remains globally open because only a particular class of constraints was considered.

A little later, Nakoulima and Mophou [4] looked again at the question, proposing this time (still for the problem with distributed observation (7)) a method of regularization, without penalization, called elliptic–elliptic regularization, interpreting the singular system as the limit of a family of well-posed problems. However, these results resort again to the Slater-type assumption, still leaving unanswered the conjecture of Lions [1].

Still with regard to the distributed observation problem, one of the latest results dates back to the work of Berhail and Omrane [5]. The latter then proposed the notion of no/least regrets controls, through which they succeed in characterizing the optimal solution through a strong singular optimality system, and this without recourse to the Slater-type assumption. But the authors then only consider the unconstrained case

$$\mathscr{U}_{ad}^{0} = \mathscr{U}_{ad}^{1} = L^2(\Gamma_0). \quad (12)$$

This is an opportunity to note that in this particular unconstrained case, we know how to do well, and this via various methods, the difficulty remaining in the general case with constraints.

To finish drawing up the state of the art concerning the problem in the spotlight, we can cite, in the cases of evolution, the work of Kernevez [6], Barry et al. [7], and Barry and Ndiaye [8]. Noting that in these last two references, the authors adapt to the cases of parabolic evolution, then hyperbolic, the penalization method introduced by Lions [1] in the stationary case.

So that, before [2], the problem of Lions [1] remained.

The paper is organized as follows: Section 2 is devoted to interpreting the initial problem as an inverse problem. We then take the liberty of ignoring certain calculation details, already well explained by Guel and Nakoulima [2]. In Sections 3 and 4, we return to the main object of the present paper, analyzing the control problems with boundary observation of the flow (Section 3), then with distributed observation (Section 4), starting by regularizing it via the controllability results previously obtained (Sections 3.1 and 4.1). After establishing the convergence of the process in Sections 3.2 and 4.2, then the approached optimality systems in Sections 3.3 and 4.3, we end in Sections 3.4 and 4.4 with the singular optimality systems for the initial problems.

2. Controllability for the Ill-Posed Elliptic Cauchy Problem

In this section, we introduce a point of view that seems to us new concerning the ill-posed Cauchy problem. It consists of interpreting the problem as an inverse problem and, therefore, a controllability problem.

We establish that, when it exists, the solution of the ill-posed Cauchy problem is a common solution of a system of two inverse problems. We then succeed in establishing a necessary and sufficient condition for the existence, not only of a solution but of a regular solution to the problem.

More precisely, we consider the systems

$$\begin{aligned} \Delta y_1 &= 0 \quad \text{in } \Omega, \\ y_1 &= v_0 \quad \text{on } \Gamma_0, \end{aligned} \quad (13)$$

$$\begin{aligned} \Delta y_2 &= 0 \quad \text{in } \Omega, \\ \frac{\partial y_2}{\partial \nu} &= v_1 \quad \text{on } \Gamma_0, \end{aligned} \quad (14)$$

and more

$$\frac{\partial y_1}{\partial \nu} = v_1 \text{ and } y_2 = v_0 \text{ on } \Gamma_0. \quad (15)$$

Remark 3. If the systems (13)–(15) admit a solution, then this latter verifies

$$y_1 = z = y_2, \quad (16)$$

where $(v = (v_0, v_1), z)$ constitutes a control-state pair for the Cauchy problem.

We can then interpret (13)–(15) as a system of inverse problems, that to say, for which we have a datum and an observation on the border Γ_0, but no information on the border Γ_1.

Then, we consider the following inverse problem: given $(v_0, v_1) \in (L^2(\Gamma_0))^2$, find $(w_1, w_2) \in (L^2(\Gamma_1))^2$ such that, if y_1 and y_2 are respective solutions of

$$\begin{aligned} &\Delta y_1 = 0 \quad \text{in } \Omega, \\ &y_1 = v_0 \quad \text{on } \Gamma_0, \quad \frac{\partial y_1}{\partial \nu} = w_1 \quad \text{on } \Gamma_1, \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\Delta y_2 = 0 \quad \text{in } \Omega, \\ &\frac{\partial y_2}{\partial \nu} = v_1 \quad \text{on } \Gamma_0, \quad y_2 = w_2 \quad \text{on } \Gamma_1, \end{aligned} \tag{18}$$

then y_1 and y_2 further satisfy the conditions (15).

Remark 4. The symmetric character of the roles played by y_1 and y_2 in the formulation of the controllability problem is obvious. Consequently, one could very well be satisfied with only one of these states in the definition of the problem, thus considering one or the other of problems (17) and (18) with the corresponding observation objective in (15). This is evidenced by the first part of the proof of Theorem 1.

As far as the present analysis is concerned, it is precisely this symmetrical nature of the roles of y_1 and y_2 that motivates their simultaneous use (which facilitates, perhaps for a short time, the continuation of the analysis), but also the wish to remain faithful to the framework of Cauchy's problem.

Remark 5. (Well-defined nature of the controllability problem). For $z \in L^2(\Omega)$ with $\Delta z = 0$, we know that

$$z|_\Gamma \in H^{-1/2}(\Gamma) \text{ and } \left.\frac{\partial z}{\partial \nu}\right|_\Gamma \in H^{-3/2}(\Gamma). \tag{19}$$

Thus, seeking, within the framework of problems of controllability, functions of $L^2(\Gamma_1)$ making it possible to reach, or if not, approaching, the targets fixed still in $L^2(\Gamma_0)$, it is necessary that the accessible states y_1 and y_2 be in $H^{3/2}(\Omega)$.

Hence, the necessity within the framework of the problem of optimal control of the elliptic Cauchy problem, to consider, beyond the assumption of nonvacuity $\mathscr{A} \neq \emptyset$, that it is the set

$$\left\{(v, z) \in \mathscr{A} : z \in H^{3/2}(\Omega)\right\}, \tag{20}$$

which is nonempty.

With these notations, conditions (15) become

$$\frac{\partial y_1}{\partial \nu}(v_0, w_1)|_{\Gamma_0} = v_1 \text{ and } y_2(v_1, w_2)|_{\Gamma_0} = v_0. \tag{21}$$

Finally, and to fix the vocabulary, we will say that the problems (17), (18), (21) constitute a problem of exact controllability and that the systems (17) and (18) are exactly controllable in (v_1, v_0) if it exists $w_1, w_2 \in L^2(\Gamma_1)$, satisfying (21).

Remark 6. By linearity of mappings

$$(v_0, w_1) \longmapsto y_1(v_0, w_1) = y_1(v_0, 0) + y_1(0, w_1), \tag{22}$$

and

$$(v_1, w_2) \longmapsto y_2(v_1, w_2) = y_2(v_1, 0) + y_2(0, w_2), \tag{23}$$

the exact controllability problems (17), (18), and (21) are equivalent to the following:

$$\begin{aligned} &\text{Find } w_1, w_2 \in L^2(\Gamma_1) \text{ such that the solutions} \\ &y_1(0, w_1) \quad \text{and} \quad y_2(0, w_2) \text{ verify} \\ &\tfrac{\partial y_1}{\partial \nu}(0, w_1)|_{\Gamma_0} = 0 \quad \text{and} \quad y_2(0, w_2)|_{\Gamma_0} = 0, \end{aligned} \tag{24}$$

translating the controllability of the system $(y_1(0, w_1), y_2(0, w_2))$ in $(0, 0)$.

A method to solve (24) is the method of approximate controllability, which consists of an approximation, by density, of the problem. This is reflecting in the following proposition:

Proposition 1. *(see. [2]). Let us denote by*

$$\begin{aligned} &E_1 = \left\{\tfrac{\partial y_1}{\partial \nu}(0, w_1)|_{\Gamma_0};\ w_1 \in L^2(\Gamma_1)\right\} \text{ and} \\ &E_2 = \left\{y_2(0, w_2)|_{\Gamma_0};\ w_2 \in L^2(\Gamma_1)\right\}, \end{aligned} \tag{25}$$

the sets of zero and one orders traces, on Γ_0, of the reachable states y_1 and y_2, respectively.

Then, we have that

$$\text{sets } E_1 \text{ and } E_2 \text{ are dense in } L^2(\Gamma_0), \tag{26}$$

and we then speak of the approximate controllability of the system $(y_1(0, w_1), y_2(0, w_2))$.

The following result is then immediate:

Corollary 1. *For all $\varepsilon > 0$, there are $w_{1\varepsilon}, w_{2\varepsilon} \in L^2(\Gamma_1)$, such that*

$$y_{1\varepsilon} = y_1(0, w_{1\varepsilon}),\ y_{2\varepsilon} = y_2(0, w_{2\varepsilon}) \in H^{3/2}(\Omega), \tag{27}$$

are unique solutions of

$$\begin{aligned} &\Delta y_{1\varepsilon} = 0 \quad \text{in } \Omega, \\ &y_{1\varepsilon} = 0 \quad \text{on } \Gamma_0, \quad \frac{\partial y_{1\varepsilon}}{\partial \nu} = w_{1\varepsilon} \quad \text{on } \Gamma_1, \end{aligned} \tag{28}$$

$$\begin{aligned} &\Delta y_{2\varepsilon} = 0 \quad \text{in } \Omega, \\ &\frac{\partial y_{2\varepsilon}}{\partial \nu} = 0 \quad \text{on } \Gamma_0, \quad y_{2\varepsilon} = w_{2\varepsilon} \quad \text{on } \Gamma_1, \end{aligned} \tag{29}$$

$$\left\|\frac{\partial y_{1\varepsilon}}{\partial \nu}\right\|_{L^2(\Gamma_0)} < \varepsilon \text{ and } \|y_{2\varepsilon}\|_{L^2(\Gamma_0)} < \varepsilon. \quad (30)$$

Starting from Remark 6, we deduce from the previous results the following:

Corollary 2. *For all* $v_0, v_1 \in L^2(\Gamma_0)$ *and* $\varepsilon > 0$, *there are* $w_{1\varepsilon}$, $w_{2\varepsilon} \in L^2(\Gamma_1)$ *such that*

$$y_1(v_0, w_{1\varepsilon}),\ y_2(v_1, w_{2\varepsilon}) \in H^{3/2}(\Omega), \quad (31)$$

are unique solutions of

$$\begin{aligned} &\Delta y_1(v_0, w_{1\varepsilon}) = 0 \quad \text{in } \Omega, \\ &y_1(v_0, w_{1\varepsilon}) = v_0 \quad \text{on } \Gamma_0, \quad \frac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) = w_{1\varepsilon} \quad \text{on } \Gamma_1, \end{aligned} \quad (32)$$

$$\begin{aligned} &\Delta y_2(v_1, w_{2\varepsilon}) = 0 \quad \text{in } \Omega, \\ &\frac{\partial y_2}{\partial \nu}(v_1, w_{2\varepsilon}) = v_1 \quad \text{on } \Gamma_0, \quad y_2(v_1, w_{2\varepsilon}) = w_{2\varepsilon} \quad \text{on } \Gamma_1, \end{aligned} \quad (33)$$

$$\left\|\frac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) - v_1\right\|_{L^2(\Gamma_0)} < \varepsilon \text{ and } \|y_2(v_1, w_{2\varepsilon}) - v_0\|_{L^2(\Gamma_0)} < \varepsilon. \quad (34)$$

Proof. Let $\varepsilon > 0$ and $v_0, v_1 \in L^2(\Gamma_0)$. From Corollary 1 we have that there are $w_{1\varepsilon}, w_{2\varepsilon} \in L^2(\Gamma_1)$ such that

$$y_{1\varepsilon} = y_1(0, w_{1\varepsilon}),\ y_{2\varepsilon} = y_2(0, w_{2\varepsilon}) \in H^{3/2}(\Omega), \quad (35)$$

are, respectively, unique solutions of (28) and (29), with (30).

So, by linearity, it immediately follows that

$$\begin{aligned} y_1(v_0, w_{1\varepsilon}) &= y_1(v_0, 0) + y_{1\varepsilon} \in H^{3/2}(\Omega) \text{ and } y_2(v_1, w_{2\varepsilon}) \\ &= y_2(v_1, 0) + y_{2\varepsilon} \in H^{3/2}(\Omega), \end{aligned} \quad (36)$$

are, respectively, unique solutions of (32) and (33), with

$$\frac{\partial y_1}{\partial \nu}(v_0, 0)|_{\Gamma_0},\ y_2(v_1, 0)|_{\Gamma_0} \in L^2(\Gamma_0). \quad (37)$$

Thus, by density of the sets E_1 and E_2 in $L^2(\Gamma_0)$, it follows that

$$\left(-\frac{\partial y_1}{\partial \nu}(v_0, 0)|_{\Gamma_0} + v_1\right) \in L^2(\Gamma_0) \quad (38)$$

implies the existence of $w_{1\varepsilon} = w_{1\varepsilon}(v_0, v_1) \in L^2(\Gamma_1)$ such that

$$\begin{aligned} &\left\|\frac{\partial y_1}{\partial \nu}(0, w_{1\varepsilon}) + \frac{\partial y_1}{\partial \nu}(v_0, 0) - v_1\right\|_{L^2(\Gamma_0)} < \varepsilon \\ &\text{i.e. } \left\|\frac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) - v_1\right\|_{L^2(\Gamma_0)} < \varepsilon. \end{aligned} \quad (39)$$

Likewise,

$$\left(-y_2(v_1, 0)|_{\Gamma_0} + v_0\right) \in L^2(\Gamma_0), \quad (40)$$

implies the existence of $w_{2\varepsilon} = w_{2\varepsilon}(v_0, v_1) \in L^2(\Gamma_1)$ such that

$$\begin{aligned} &\|y_2(0, w_{2\varepsilon}) + y_2(v_1, 0) - v_0\|_{L^2(\Gamma_0)} < \varepsilon \\ &\text{i.e. } \|y_2(v_1, w_{2\varepsilon}) - v_0\|_{L^2(\Gamma_0)} < \varepsilon. \end{aligned} \quad (41)$$

From where the result. □

Then we have the following theorem:

Theorem 1. *(see [2]). Given* $v = (v_0, v_1) \in (L^2(\Gamma_0))^2$, *the ill-posed Cauchy problem*

$$\begin{aligned} &\Delta z = 0 \quad \text{in } \Omega, \\ &z = v_0, \quad \frac{\partial z}{\partial \nu} = v_1 \text{ on } \Gamma_0, \end{aligned} \quad (42)$$

admits a regular solution $z \in H^{3/2}(\Omega)$ *if and only if either of the sequences* $(w_{1\varepsilon})_\varepsilon$ or $(w_{2\varepsilon})_\varepsilon$ *is bounded in* $L^2(\Gamma_1)$.

It follows from Theorem 1 the following corollary:

Corollary 3. *(see [2]). z being a regular solution of the Cauchy problem, then* $y_1 = z = y_2$.

3. The Flow Observation Problem

Let us start by recalling that we are interested in controlling the Cauchy problem for the Laplacian. That is to say, more precisely, we consider the problem

$$\begin{aligned} &\Delta z = 0 \quad \text{in } \Omega, \\ &z = v_0, \quad \frac{\partial z}{\partial \nu} = v_1 \text{ on } \Gamma_0, \end{aligned} \quad (43)$$

and, for all control-state pair (v, z), the cost function

$$J(v, z) = \frac{1}{2}\left\|\frac{\partial z}{\partial \nu} - z_d\right\|^2_{L^2(\Gamma_1)} + \frac{N_0}{2}\|v_0\|^2_{L^2(\Gamma_0)} + \frac{N_1}{2}\|v_1\|^2_{L^2(\Gamma_0)}, \quad (44)$$

being interested in the optimal control problem

$$\inf\{J(v, z); (v, z) \in \mathscr{A}\}. \quad (45)$$

We propose here to use the controllability method (cf. [9, p. 222]) to characterize the optimal solution (u, y) of the problem (43)–(45), without any other assumption than the "sufficient" one of nonvacuity of the set of admissible control-state pairs (cf. Remark 5). To the best of our knowledge, this method seems new.

3.1. The Controllability Method. Starting therefore from the assumption $\mathscr{A} \neq \emptyset$ and within the framework of Remark 5, we have, for all

$$v = (v_0, v_1) \in \mathscr{U}_{ad} \text{ and } \varepsilon > 0, \tag{46}$$

there exists

$$w_{1\varepsilon},\ w_{2\varepsilon} \in L^2(\Gamma_1) \text{ and } y_1(v_0, w_{1\varepsilon}),\ y_2(v_1, w_{2\varepsilon}) \in H^{3/2}(\Omega), \tag{47}$$

such that

$$\begin{aligned} &\Delta y_1(v_0, w_{1\varepsilon}) = 0 \quad \text{in } \Omega, \\ &y_1(v_0, w_{1\varepsilon}) = v_0 \quad \text{on } \Gamma_0,\ \frac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) = w_{1\varepsilon} \text{ on } \Gamma_1, \end{aligned} \tag{48}$$

$$\begin{aligned} &\Delta y_2(v_1, w_{2\varepsilon}) = 0 \quad \text{in } \Omega, \\ &\frac{\partial y_2}{\partial \nu}(v_1, w_{2\varepsilon}) = v_1 \quad \text{on } \Gamma_0,\ y_2(v_1, w_{2\varepsilon}) = w_{2\varepsilon} \text{ on } \Gamma_1, \end{aligned} \tag{49}$$

$$\left\| \frac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) - v_1 \right\|^2_{L^2(\Gamma_0)} < \varepsilon \text{ and } \| y_2(v_1, w_{2\varepsilon}) - v_0 \|^2_{L^2(\Gamma_0)} < \varepsilon. \tag{50}$$

Then we consider, for $\theta_1, \theta_2 \in \mathbb{R}_+ : \theta_1 + \theta_2 = 1$, the functional

$$\begin{aligned} J_\varepsilon(v_0, v_1) = {} & \frac{\theta_1}{2} \left\| \frac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) - z_d \right\|^2_{L^2(\Gamma_1)} \\ & + \frac{\theta_2}{2} \left\| \frac{\partial y_2}{\partial \nu}(v_1, w_{2\varepsilon}) - z_d \right\|^2_{L^2(\Gamma_1)} \\ & + \frac{N_0}{2} \| v_0 \|^2_{L^2(\Gamma_0)} + \frac{N_1}{2} \| v_1 \|^2_{L^2(\Gamma_0)}, \end{aligned} \tag{51}$$

being interested in the control problem

$$\inf \{ J_\varepsilon(v_0, v_1);\ v = (v_0, v_1) \in \mathscr{U}_{ad} \}. \tag{52}$$

The following result is then immediate

Proposition 2. *For all $\varepsilon > 0$, the control problem (52) admits a unique solution, the optimal control $\overline{u}_\varepsilon = (\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon})$.*

3.2. Convergence of the Method. Let $\varepsilon > 0$. Due to the existence of the optimal control $\overline{u}_\varepsilon = (\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon}) \in \mathscr{U}_{ad} \subset (L^2(\Gamma_0))^2$, and according to the results of the previous section, there exists

$$\overline{w}_{1\varepsilon},\ \overline{w}_{2\varepsilon} \in L^2(\Gamma_1) \text{ and } \overline{y}_{1\varepsilon},\ \overline{y}_{2\varepsilon} \in H^{3/2}(\Omega) \subset L^2(\Omega), \tag{53}$$

such that

$$\begin{aligned} &\Delta \overline{y}_{1\varepsilon} = 0 \quad \text{in } \Omega, \\ &\overline{y}_{1\varepsilon} = \overline{u}_{0\varepsilon} \quad \text{on } \Gamma_0,\ \frac{\partial \overline{y}_{1\varepsilon}}{\partial \nu} = \overline{w}_{1\varepsilon} \text{ on } \Gamma_1, \end{aligned} \tag{54}$$

$$\begin{aligned} &\Delta \overline{y}_{2\varepsilon} = 0 \quad \text{in } \Omega, \\ &\frac{\partial \overline{y}_{2\varepsilon}}{\partial \nu} = \overline{u}_{1\varepsilon} \quad \text{on } \Gamma_0,\ \overline{y}_{2\varepsilon} = \overline{w}_{2\varepsilon} \text{ on } \Gamma_1, \end{aligned} \tag{55}$$

$$\left\| \frac{\partial \overline{y}_{1\varepsilon}}{\partial \nu} - \overline{u}_{1\varepsilon} \right\|_{L^2(\Gamma_0)} < \varepsilon \text{ and } \| \overline{y}_{2\varepsilon} - \overline{u}_{0\varepsilon} \|_{L^2(\Gamma_0)} < \varepsilon, \tag{56}$$

with, for all $v \in \mathscr{U}_{ad}$,

$$J_\varepsilon(\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon}) \leq J_\varepsilon(v_0, v_1). \tag{57}$$

In particular

$$J_\varepsilon(\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon}) \leq J_\varepsilon(u_0, u_1), \tag{58}$$

where $u = (u_0, u_1)$ is the optimal solution of (43)–(45). We have in fact that $J_\varepsilon(u_0, u_1)$ is independent of ε. Indeed, let $(w^*_{1\varepsilon})_\varepsilon$ and $(w^*_{2\varepsilon})_\varepsilon$ be the constant sequences defined by

$$w^*_{1\varepsilon} = \left. \frac{\partial y}{\partial \nu} \right|_{\Gamma_1} \in L^2(\Gamma_1) \text{ and } w^*_{2\varepsilon} = y|_{\Gamma_1} \in L^2(\Gamma_1). \tag{59}$$

So we have
$y_1(u_0, w^*_{1\varepsilon}) = y^*_{1\varepsilon} = y$ verify:

$$\begin{aligned} &\Delta y^*_{1\varepsilon} = 0 \quad \text{in } \Omega, \\ &y^*_{1\varepsilon} = u_0 \quad \text{on } \Gamma_0,\ \partial \frac{y^*_{1\varepsilon}}{\partial \nu} = w^*_{1\varepsilon} \text{ on } \Gamma_1, \end{aligned} \tag{60}$$

$y_2(u_1, w^*_{2\varepsilon}) = y^*_{2\varepsilon} = y$ verify:

$$\begin{aligned} &\Delta y^*_{2\varepsilon} = 0 \quad \text{in } \Omega, \\ &\partial \frac{y^*_{2\varepsilon}}{\partial \nu} = u_1 \quad \text{on } \Gamma_0,\ y^*_{2\varepsilon} = w^*_{2\varepsilon} \text{ on } \Gamma_1, \end{aligned} \tag{61}$$

with

$$\partial \frac{y^*_{1\varepsilon}}{\partial \nu} = u_1 \text{ and } y^*_{2\varepsilon} = u_0 \text{ on } \Gamma_0. \tag{62}$$

Consequently,

$$J_\varepsilon(u_0,u_1) = \frac{\theta_1}{2}\left\|\frac{\partial y_{1\varepsilon}^*}{\partial\nu} - z_d\right\|^2_{L^2(\Gamma_1)} + \frac{\theta}{2}\left\|\frac{\partial y_{2\varepsilon}^*}{\partial\nu} - z_d\right\|^2_{L^2(\Gamma_1)} + \frac{N_0}{2}\|u_0\|^2_{L^2(\Gamma_0)} + \frac{N_1}{2}\|u_1\|^2_{L^2(\Gamma_0)}$$
$$= \frac{\theta_1}{2}\left\|\frac{\partial y}{\partial\nu} - z_d\right\|^2_{L^2(\Gamma_1)} + \frac{\theta_2}{2}\left\|\frac{\partial y}{\partial\nu} - z_d\right\|^2_{L^2(\Gamma_1)} + \frac{N_0}{2}\|u_0\|^2_{L^2(\Gamma_0)} + \frac{N_1}{2}\|u_1\|^2_{L^2(\Gamma_0)} \quad (63)$$

i.e.,

$$J_\varepsilon(u_0,u_1) = \frac{1}{2}\left\|\frac{\partial y}{\partial\nu} - z_d\right\|^2_{L^2(\Gamma_1)} + \frac{N_0}{2}\|u_0\|^2_{L^2(\Gamma_0)} + \frac{N_1}{2}\|u_1\|^2_{L^2(\Gamma_0)} = J(u,y) \quad (64)$$

Thus (58) becomes

$$J_\varepsilon(\overline{u}_{0\varepsilon},\overline{u}_{1\varepsilon}) \le J_\varepsilon(u_0,u_1) = J(u,y), \quad (65)$$

and it follows there exist constants $C_i \in \mathbb{R}_+^*$, independent of ε, such that

$$\left\|\frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu}\right\|_{L^2(\Gamma_1)} = \|\overline{w}_{1\varepsilon}\|_{L^2(\Gamma_1)} \le C_1, \quad \left\|\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu}\right\|_{L^2(\Gamma_1)} \le C_2,$$
$$\|\overline{u}_{0\varepsilon}\|_{L^2(\Gamma_0)} \le C_3, \quad \|\overline{u}_{1\varepsilon}\|_{L^2(\Gamma_0)} \le C_4, \quad (66)$$

since $\frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu} = \overline{w}_{1\varepsilon}$ on Γ_1.

So, we have, on the one hand, the sequence $(\overline{w}_{1\varepsilon})_\varepsilon$ being bounded in $L^2(\Gamma_1)$ and by using Theorem 1, there exist $\widehat{w}_1$, $\widehat{w}_2 \in L^2(\Gamma_1)$ and $\widehat{y}_1, \widehat{y}_2 \in H^{3/2}(\Omega)$ such that

$$\begin{aligned}\overline{w}_{1\varepsilon} &\longrightarrow \widehat{w}_1 \quad \text{weakly in } L^2(\Gamma_1),\\ \overline{w}_{2\varepsilon} &\longrightarrow \widehat{w}_2 \quad \text{weakly in } L^2(\Gamma_1),\\ \overline{y}_{1\varepsilon} &\longrightarrow \widehat{y}_1 \quad \text{weakly in } H^{3/2}(\Omega),\\ \overline{y}_{2\varepsilon} &\longrightarrow \widehat{y}_2 \quad \text{weakly in } H^{3/2}(\Omega).\end{aligned} \quad (67)$$

On the other hand, we immediately deduce, from (66), that there exist $\widehat{u}_0, \widehat{u}_1 \in L^2(\Gamma_0)$ such that

$$\begin{aligned}\overline{u}_{0\varepsilon} &\longrightarrow \widehat{u}_0 \quad \text{weakly in } L^2(\Gamma_0),\\ \overline{u}_{1\varepsilon} &\longrightarrow \widehat{u}_1 \quad \text{weakly in } L^2(\Gamma_0).\end{aligned} \quad (68)$$

Then it follows, on the one hand

$$\begin{aligned}&\Delta\widehat{y}_1 = 0 \text{ in } \Omega,\\ &\widehat{y}_1 = \widehat{u}_0 \text{ on } \Gamma_0, \quad \frac{\partial\widehat{y}_1}{\partial\nu} = \widehat{w}_1 \text{ on } \Gamma_1,\end{aligned} \quad (69)$$

and, on the other hand,

$$\begin{aligned}&\Delta\widehat{y}_1 = 0 \quad \text{in } \Omega,\\ &\widehat{y}_1 = \widehat{u}_0, \quad \frac{\partial\widehat{y}_1}{\partial\nu} = \widehat{u}_1 \quad \text{on } \Gamma_0,\end{aligned} \quad (70)$$

Analogously, we get

$$\begin{aligned}&\Delta\widehat{y}_2 = 0 \quad \text{in } \Omega,\\ &\widehat{y}_2 = \widehat{u}_0, \quad \frac{\partial\widehat{y}_2}{\partial\nu} = \widehat{u}_1 \quad \text{on } \Gamma_0.\end{aligned} \quad (71)$$

Then, by the uniqueness of the solution of the ill-posed Cauchy problem, we conclude that

$$\widehat{y}_1 = \widehat{y} = \widehat{y}_2. \quad (72)$$

At this stage, we have that there exist

$$\widehat{y} \in H^{3/2}(\Omega) \text{ and } \widehat{u} = (\widehat{u}_0, \widehat{u}_1) \in \mathcal{U}_{ad}, \quad (73)$$

such that

$$\begin{aligned}\overline{u}_{0\varepsilon} &\longrightarrow \widehat{u}_0 \quad \text{weakly in } L^2(\Gamma_0),\\ \overline{u}_{1\varepsilon} &\longrightarrow \widehat{u}_1 \quad \text{weakly in } L^2(\Gamma_0),\\ \overline{y}_{1\varepsilon} &\longrightarrow \widehat{y} \quad \text{weakly in } H^{3/2}(\Omega),\\ \overline{y}_{2\varepsilon} &\longrightarrow \widehat{y} \quad \text{weakly in } H^{3/2}(\Omega),\end{aligned} \quad (74)$$

the control-state pair $(\widehat{u}, \widehat{y})$ being admissible. So that it follows

$$J(u,y) \le J(\widehat{u},\widehat{y}). \quad (75)$$

Finally, passing to the limit in (65), we get

$$J(\widehat{u},\widehat{y}) \le J(u,y). \quad (76)$$

Hence it follows, by uniqueness of the optimal solution (u, y) to (43)–(45), that (75), and (76) leads to

$$J(\widehat{u},\widehat{y}) \le J(u,y) \le J(\widehat{u},\widehat{y}) \Longrightarrow J(\widehat{u},\widehat{y}) = J(u,y), \quad (77)$$

which implies

$$(\widehat{u},\widehat{y}) = (u,y). \quad (78)$$

Thereby, we have just proved the following result.

Proposition 3. *For all $\varepsilon>0$, the optimal control $(\overline{u}_{0\varepsilon},\overline{u}_{1\varepsilon})$, solution of (52), is such that $(\overline{u}_{\varepsilon},\overline{y}_{\varepsilon})$ verifies*

$$\begin{aligned} \overline{u}_{0\varepsilon} &\longrightarrow u_0 \quad \text{weakly in } L^2(\Gamma_0),\\ \overline{u}_{1\varepsilon} &\longrightarrow u_1 \quad \text{weakly in } L^2(\Gamma_0),\\ \overline{y}_{1\varepsilon} &\longrightarrow \mathrm{y} \quad \text{weakly in } H^{3/2}(\Omega),\\ \overline{y}_{2\varepsilon} &\longrightarrow \mathrm{y} \quad \text{weakly in } H^{3/2}(\Omega), \end{aligned} \tag{79}$$

where (u,y) is the optimal control-state pair of (43)–(45).

Moreover, we establish, as follows, the strong convergence of the optimal control-state pair $(\overline{u}_{\varepsilon},\overline{y}_{\varepsilon})$ toward the other one (u,y).

Theorem 2. *The optimal control $(\overline{u}_{0\varepsilon},\overline{u}_{1\varepsilon})$ solution of (52) and the associate optimal states $\overline{y}_{1\varepsilon}$ and $\overline{y}_{2\varepsilon}$ are such that, when $\varepsilon\longrightarrow 0$,*

$$\begin{aligned} \overline{u}_{0\varepsilon} &\longrightarrow u_0 \quad \text{strongly in } L^2(\Gamma_0),\\ \overline{u}_{1\varepsilon} &\longrightarrow u_1 \quad \text{strongly in } L^2(\Gamma_0), \end{aligned} \tag{80}$$

and

$$\begin{aligned} \overline{y}_{1\varepsilon} &\longrightarrow y \quad \text{strongly in } H^{3/2}(\Omega),\\ \overline{y}_{2\varepsilon} &\longrightarrow y \quad \text{strongly in } H^{3/2}(\Omega), \end{aligned} \tag{81}$$

where (u,y) is the optimal control-state pair of (43)–(45).

Proof. Let us start by noting that, by continuity of the trace operators γ_0 and γ_1,

$$\overline{y}_{1\varepsilon} \longrightarrow y \text{ weakly in } H^{3/2}(\Omega), \tag{82}$$

implies

$$\begin{aligned} \gamma_0\overline{y}_{1\varepsilon} &\longrightarrow \gamma_0 y \quad \text{weakly in } H^1(\Gamma),\\ \gamma_1\overline{y}_{2\varepsilon} &\longrightarrow \gamma_1 y \quad \text{weakly in } L^2(\Gamma), \end{aligned} \tag{83}$$

and, analogously, that

$$\overline{y}_{2\varepsilon} \longrightarrow y \text{ weakly in } H^{3/2}(\Omega), \tag{84}$$

implies that

$$\begin{aligned} \gamma_0\overline{y}_{2\varepsilon} &\longrightarrow \gamma_0 y \quad \text{weakly in } H^1(\Gamma),\\ \gamma_1\overline{y}_{2\varepsilon} &\longrightarrow \gamma_1 y \quad \text{weakly in } L^2(\Gamma). \end{aligned} \tag{85}$$

Thus, we have, with the previous results,

$$\begin{aligned} \overline{u}_{0\varepsilon} &\longrightarrow u_0 \quad \text{weakly in } L^2(\Gamma_0),\\ \overline{u}_{1\varepsilon} &\longrightarrow u_1 \quad \text{weakly in } L^2(\Gamma_0), \end{aligned} \tag{86}$$

$$\begin{aligned} \overline{w}_{1\varepsilon} = \left.\frac{\partial \overline{y}_{1\varepsilon}}{\partial \nu}\right|_{\Gamma_1} &\longrightarrow \widehat{w}_1 = \left.\frac{\partial y}{\partial \nu}\right|_{\Gamma_1} \quad \text{weakly in } L^2(\Gamma_1),\\ \overline{w}_{2\varepsilon} = \overline{y}_{2\varepsilon}|_{\Gamma_1} &\longrightarrow \widehat{w}_2 = y|_{\Gamma_1} \quad \text{weakly in } H^1(\Gamma_1), \end{aligned} \tag{87}$$

$$\begin{aligned} \overline{y}_{1\varepsilon} &\longrightarrow y \quad \text{weakly in } H^{3/2}(\Omega),\\ \overline{y}_{2\varepsilon} &\longrightarrow y \quad \text{weakly in } H^{3/2}(\Omega), \end{aligned} \tag{88}$$

but also that

$$\left.\frac{\partial \overline{y}_{2\varepsilon}}{\partial \nu}\right|_{\Gamma_1} \longrightarrow \left.\frac{\partial y}{\partial \nu}\right|_{\Gamma_1} \text{ weakly in } L^2(\Gamma_1); \tag{89}$$

with

$$J(u,y) \le \lim_{\varepsilon\to 0} J_\varepsilon(\overline{u}_{0\varepsilon},\overline{u}_{1\varepsilon}) \le J(u,y); \tag{90}$$

this last result can still be written

$$\begin{aligned} &\left\|\frac{\partial y}{\partial \nu} - z_d\right\|^2_{L^2(\Gamma_1)} + N_0\|u_0\|^2_{L^2(\Gamma_0)} + N_1\|u_1\|^2_{L^2(\Gamma_0)}\\ &\quad = \lim_{\varepsilon\to 0}\left(\theta_1\left\|\frac{\partial \overline{y}_{1\varepsilon}}{\partial \nu} - z_d\right\|^2_{L^2(\Gamma_1)} + \theta_2\left\|\frac{\partial \overline{y}_{2\varepsilon}}{\partial \nu} - z_d\right\|^2_{L^2(\Gamma_1)}\right.\\ &\qquad \left. + N_0\|\overline{u}_{0\varepsilon}\|^2_{L^2(\Gamma_0)} + N_1\|\overline{u}_{1\varepsilon}\|^2_{L^2(\Gamma_0)}\right). \end{aligned} \tag{91}$$

But then, the norms being continuous, a fortiori weakly lower semicontinuous, we have with (86), $(87)_1$ and (89), that

$$\begin{aligned} \left\|\frac{\partial y}{\partial \nu} - z_d\right\|^2_{L^2(\Gamma_1)} &\le \liminf_{\varepsilon\to 0}\left(\theta_1\left\|\frac{\partial \overline{y}_{1\varepsilon}}{\partial \nu} - z_d\right\|^2_{L^2(\Gamma_1)} + \theta_2\left\|\frac{\partial \overline{y}_{2\varepsilon}}{\partial \nu} - z_d\right\|^2_{L^2(\Gamma_1)}\right),\\ \|u_0\|^2_{L^2(\Gamma_0)} &\le \liminf_{\varepsilon\to 0}\ \|\overline{u}_{0\varepsilon}\|^2_{L^2(\Gamma_0)},\\ \|u_1\|^2_{L^2(\Gamma_0)} &\le \liminf_{\varepsilon\to 0}\ \|\overline{u}_{1\varepsilon}\|^2_{L^2(\Gamma_0)}. \end{aligned} \tag{92}$$

So that (91) and (92) lead to

$$\begin{aligned} \left\|\frac{\partial y}{\partial \nu} - z_d\right\|^2_{L^2(\Gamma_1)} &= \lim_{\varepsilon\to 0}\left(\theta_1\left\|\frac{\partial \overline{y}_{1\varepsilon}}{\partial \nu} - z_d\right\|^2_{L^2(\Gamma_1)} + \theta_2\left\|\frac{\partial \overline{y}_{2\varepsilon}}{\partial \nu} - z_d\right\|^2_{L^2(\Gamma_1)}\right),\\ \|u_0\|^2_{L^2(\Gamma_0)} &= \lim_{\varepsilon\to 0}\ \|\overline{u}_{0\varepsilon}\|^2_{L^2(\Gamma_0)},\\ \|u_1\|^2_{L^2(\Gamma_0)} &= \lim_{\varepsilon\to 0}\ \|\overline{u}_{1\varepsilon}\|^2_{L^2(\Gamma_0)}. \end{aligned} \tag{93}$$

But then, since

$$\begin{aligned}\|\overline{u}_{0\varepsilon}-u_0\|^2_{L^2(\Gamma_0)} &+\|\overline{u}_{1\varepsilon}-u_1\|^2_{L^2(\Gamma_0)}\\ &=\|\overline{u}_{0\varepsilon}\|^2_{L^2(\Gamma_0)}+\|u_0\|^2_{L^2(\Gamma_0)}+\|\overline{u}_{1\varepsilon}\|^2_{L^2(\Gamma_0)}+\|u_1\|^2_{L^2(\Gamma_0)}\\ &\quad-2(\overline{u}_{0\varepsilon},\ u_0)_{L^2(\Gamma_0)}-2(\overline{u}_{1\varepsilon},\ u_1)_{L^2(\Gamma_0)},\end{aligned} \tag{94}$$

we obtain, passing to the limit with (86), $(93)_2$ and $(93)_3$, that

$$\lim_{\varepsilon\to 0}\left(\|\overline{u}_{0\varepsilon}-u_0\|^2_{L^2(\Gamma_0)}+\|\overline{u}_{1\varepsilon}-u_1\|^2_{L^2(\Gamma_0)}\right)=0, \tag{95}$$

which leads well to

$$\begin{aligned}\overline{u}_{0\varepsilon}&\longrightarrow u_0 \quad \text{strongly in } L^2(\Gamma_0),\\ \overline{u}_{1\varepsilon}&\longrightarrow u_1 \quad \text{strongly in } L^2(\Gamma_0).\end{aligned} \tag{96}$$

Otherwise, let note that we can take, in $(93)_1$, successively

$$(\theta_1=1,\ \theta_2=0)\ \text{then}(\theta_1=0,\ \theta_2=1), \tag{97}$$

to get that

$$\begin{aligned}\lim_{\varepsilon\to 0}\left\|\frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu}-z_d\right\|^2_{L^2(\Gamma_1)} &=\left\|\frac{\partial y}{\partial\nu}-z_d\right\|^2_{L^2(\Gamma_1)}\\ &=\lim_{\varepsilon\to 0}\left\|\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu}-z_d\right\|^2_{L^2(\Gamma_1)}.\end{aligned} \tag{98}$$

Then we have, on the one hand, that

$$\begin{aligned}\left\|\frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu}-\frac{\partial y}{\partial\nu}\right\|^2_{L^2(\Gamma_1)} &=\left\|\frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu}-z_d\right\|^2_{L^2(\Gamma_1)}+\left\|\frac{\partial y}{\partial\nu}-z_d\right\|^2_{L^2(\Gamma_1)}\\ &\quad-2\left(\frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu}-z_d,\ \frac{\partial y}{\partial\nu}-z_d\right)_{L^2(\Gamma_1)},\end{aligned} \tag{99}$$

with $(87)_1$ and (98), imply that

$$\lim_{\varepsilon\to 0}\left\|\frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu}-\frac{\partial y}{\partial\nu}\right\|^2_{L^2(\Gamma_1)}=0, \tag{100}$$

i.e.,

$$\frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu}\longrightarrow\frac{\partial y}{\partial\nu}\ \text{strongly in } L^2(\Gamma_1), \tag{101}$$

and likewise

$$\begin{aligned}\left\|\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu}-\frac{\partial y}{\partial\nu}\right\|^2_{L^2(\Gamma_1)} &=\left\|\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu}-z_d\right\|^2_{L^2(\Gamma_1)}+\left\|\frac{\partial y}{\partial\nu}-z_d\right\|^2_{L^2(\Gamma_1)}\\ &\quad-2\left(\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu}-z_d,\ \frac{\partial y}{\partial\nu}-z_d\right)_{L^2(\Gamma_1)},\end{aligned} \tag{102}$$

with (89) and (98) imply that

$$\lim_{\varepsilon\to 0}\left\|\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu}-\frac{\partial y}{\partial\nu}\right\|^2_{L^2(\Gamma_1)}=0, \tag{103}$$

i.e.,

$$\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu}\longrightarrow\frac{\partial y}{\partial\nu}\ \text{strongly in } L^2(\Gamma_1). \tag{104}$$

Therefore, (54) being well-posed with

$$\begin{aligned}\overline{u}_{0\varepsilon}&\longrightarrow u_0 &&\text{strongly in } L^2(\Gamma_0),\\ \overline{w}_{1\varepsilon}=\left.\frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu}\right|_{\Gamma_1}&\longrightarrow \left.\frac{\partial y}{\partial\nu}\right|_{\Gamma_1} &&\text{strongly in } L^2(\Gamma_1),\\ \overline{y}_{1\varepsilon}&\longrightarrow y &&\text{weakly in } H^{3/2}(\Omega),\end{aligned} \tag{105}$$

we can deduce that

$$\overline{y}_{1\varepsilon}\longrightarrow y\ \text{strongly in } H^{3/2}(\Omega). \tag{106}$$

Otherwise, $(56)_2$ and $(96)_1$, with

$$\|\overline{y}_{2\varepsilon}-u_0\|^2_{L^2(\Gamma_0)}\le\|\overline{y}_{2\varepsilon}-\overline{u}_{0\varepsilon}\|^2_{L^2(\Gamma_0)}+\|\overline{u}_{0\varepsilon}-u_0\|^2_{L^2(\Gamma_0)}, \tag{107}$$

lead to

$$\overline{y}_{2\varepsilon}\longrightarrow u_0\ \text{strongly in } L^2(\Gamma_0). \tag{108}$$

Thus,

$$\begin{aligned}&\Delta\psi=0 \text{ in } \Omega,\\ &\psi=\zeta \text{ on } \Gamma_0,\quad \frac{\partial\psi}{\partial\nu}=\xi \text{ on } \Gamma_1,\end{aligned} \tag{109}$$

being well posed, with $\overline{y}_{2\varepsilon}$ as solution, for

$$\zeta=\overline{y}_{2\varepsilon}|_{\Gamma_0}\in L^2(\Gamma_0)\ \text{and}\ \xi=\left.\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu}\right|_{\Gamma_1}\in L^2(\Gamma_1), \tag{110}$$

it follows from

$$\begin{aligned} \bar{y}_{2\varepsilon}|_{\Gamma_0} &\longrightarrow u_0 && \text{strongly in } L^2(\Gamma_0), \\ \left.\frac{\partial \bar{y}_{2\varepsilon}}{\partial \nu}\right|_{\Gamma_1} &\longrightarrow \left.\frac{\partial y}{\partial \nu}\right|_{\Gamma_1} && \text{strongly in } L^2(\Gamma_1), \\ \bar{y}_{2\varepsilon} &\longrightarrow y && \text{weakly in } H^{3/2}(\Omega), \end{aligned} \tag{111}$$

that

$$\bar{y}_{2\varepsilon} \longrightarrow y \text{ strongly in } H^{3/2}(\Omega). \tag{112}$$

Which ends up proving the announced result. □

3.3. Approached Optimality System. We prove the following result:

Theorem 3. *Let $\varepsilon > 0$. The control $\bar{u}_\varepsilon = (\bar{u}_{0\varepsilon}, \bar{u}_{1\varepsilon})$ is unique solution of (52) if and only if there exists*

$$\bar{w}_{1\varepsilon}, \bar{w}_{2\varepsilon} \in L^2(\Gamma_1),\ \bar{y}_{1\varepsilon}, \bar{y}_{2\varepsilon} \in H^{3/2}(\Omega) \text{ and } p_{2\varepsilon} \in H^{3/2}(\Omega), \tag{113}$$

such that the quadruplet

$$\{(\bar{u}_{0\varepsilon}, \bar{u}_{1\varepsilon}), (\bar{w}_{1\varepsilon}, \bar{w}_{2\varepsilon}), (\bar{y}_{1\varepsilon}, \bar{y}_{2\varepsilon}), p_{2\varepsilon}\}, \tag{114}$$

is a solution of the optimality system defined by systems

$$\begin{aligned} &\Delta \bar{y}_{1\varepsilon} = 0 \quad \text{in } \Omega, \\ &\bar{y}_{1\varepsilon} = \bar{u}_{0\varepsilon} \quad \text{on } \Gamma_0, \quad \frac{\partial \bar{y}_{1\varepsilon}}{\partial \nu} = \bar{w}_{1\varepsilon} \quad \text{on } \Gamma_1, \end{aligned} \tag{115}$$

$$\begin{aligned} &\Delta \bar{y}_{2\varepsilon} = 0 \quad \text{in } \Omega, \\ &\frac{\partial \bar{y}_{2\varepsilon}}{\partial \nu} = \bar{u}_{1\varepsilon} \quad \text{on } \Gamma_0, \quad \bar{y}_{2\varepsilon} = \bar{w}_{2\varepsilon} \quad \text{on } \Gamma_1, \end{aligned} \tag{116}$$

$$\begin{aligned} &\Delta p_{2\varepsilon} = 0 \quad \text{in } \Omega, \\ &\frac{\partial p_{2\varepsilon}}{\partial \nu} = 0 \quad \text{on } \Gamma_0, \quad p_{1\varepsilon} = -\theta_2 \left(\frac{\partial \bar{y}_{2\varepsilon}}{\partial \nu} - z_d \right) \quad \text{on } \Gamma_1, \end{aligned} \tag{117}$$

the estimates

$$\left\| \frac{\partial \bar{y}_{1\varepsilon}}{\partial \nu} - \bar{u}_{1\varepsilon} \right\|_{L^2(\Gamma_0)} < \varepsilon \text{ and } \|\bar{y}_{2\varepsilon} - \bar{u}_{0\varepsilon}\|_{L^2(\Gamma_0)} < \varepsilon, \tag{118}$$

and the variational inequalities system

$$\begin{aligned} &\forall v = (v_0, v_1) \in \mathcal{U}_{ad}, \\ &(N_0 \bar{u}_{0\varepsilon},\ v_0 - \bar{u}_{0\varepsilon})_{L^2(\Gamma_0)} \geq 0, \\ &(N_1 \bar{u}_{1\varepsilon} + p_{2\varepsilon},\ v_1 - \bar{u}_{1\varepsilon})_{L^2(\Gamma_0)} \geq 0. \end{aligned} \tag{119}$$

Proof. For all $\varepsilon > 0$, we have existence of

$$\bar{w}_{1\varepsilon}, \bar{w}_{2\varepsilon} \in L^2(\Gamma_1) \text{ and } \bar{y}_{1\varepsilon}, \bar{y}_{2\varepsilon} \in L^2(\Omega), \tag{120}$$

such that

$$\begin{aligned} &\Delta \bar{y}_{1\varepsilon} = 0 \quad \text{in } \Omega, \\ &\bar{y}_{1\varepsilon} = \bar{u}_{0\varepsilon} \quad \text{on } \Gamma_0, \quad \frac{\partial \bar{y}_{1\varepsilon}}{\partial \nu} = \bar{w}_{1\varepsilon} \quad \text{on } \Gamma_1, \end{aligned} \tag{121}$$

$$\begin{aligned} &\Delta \bar{y}_{2\varepsilon} = 0 \quad \text{in } \Omega, \\ &\frac{\partial \bar{y}_{2\varepsilon}}{\partial \nu} = \bar{u}_{1\varepsilon} \quad \text{on } \Gamma_0, \quad \bar{y}_{2\varepsilon} = \bar{w}_{2\varepsilon} \quad \text{on } \Gamma_1, \end{aligned} \tag{122}$$

$$\left\| \frac{\partial \bar{y}_{1\varepsilon}}{\partial \nu} - \bar{u}_{1\varepsilon} \right\|_{L^2(\Gamma_0)} < \varepsilon \text{ and } \|\bar{y}_{2\varepsilon} - \bar{u}_{0\varepsilon}\|_{L^2(\Gamma_1)} < \varepsilon. \tag{123}$$

So let $v = (v_0, v_1) \in \mathcal{U}_{ad}$ and $\lambda \in \mathbb{R}^*$; posing

$$\begin{aligned} \phi_{1\varepsilon} &= y_1(v_0 - \bar{u}_{0\varepsilon}, \bar{w}_{1\varepsilon}) - y_1(0, \bar{w}_{1\varepsilon}) \text{ and } \phi_{2\varepsilon} \\ &= y_2(v_1 - \bar{u}_{1\varepsilon}, \bar{w}_{2\varepsilon}) - y_2(0, \bar{w}_{2\varepsilon}), \end{aligned} \tag{124}$$

let us begin by noting that

$$\begin{aligned} &\Delta \phi_{1\varepsilon} = 0 \text{ in } \Omega, \\ &\phi_{1\varepsilon}|_{\Gamma_0} = v_0 - \bar{u}_{0\varepsilon}, \ \left.\frac{\partial \phi_{1\varepsilon}}{\partial \nu}\right|_{\Gamma_1} = 0, \end{aligned} \tag{125}$$

and

$$\begin{aligned} &\Delta \phi_{2\varepsilon} = 0 \text{ in } \Omega, \\ &\left.\frac{\partial \phi_{2\varepsilon}}{\partial \nu}\right|_{\Gamma_0} = v_0 - \bar{u}_{0\varepsilon}, \ \phi_{2\varepsilon}|_{\Gamma_1} = 0. \end{aligned} \tag{126}$$

It therefore follows, on the one hand,

$$\begin{aligned} J_\varepsilon(\bar{u}_{0\varepsilon} + \lambda(v_0 - \bar{u}_{0\varepsilon}),\ \bar{u}_{1\varepsilon}) &= J_\varepsilon(\bar{u}_{0\varepsilon}, \bar{u}_{1\varepsilon}) \\ &+ \frac{\lambda^2 N_0}{2} \|v_0 - \bar{u}_{0\varepsilon}\|^2_{L^2(\Gamma_0)} + \lambda N_0 (\bar{u}_{0\varepsilon},\ v_0 - \bar{u}_{0\varepsilon})_{L^2(\Gamma_0)} \end{aligned} \tag{127}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} J_\varepsilon(\bar{u}_{0\varepsilon} + \lambda(v_0 - \bar{u}_{0\varepsilon}), \bar{u}_{1\varepsilon})_{|\lambda=0} = N_0 (\bar{u}_{0\varepsilon},\ v_0 - \bar{u}_{0\varepsilon})_{L^2(\Gamma_0)}, \tag{128}$$

and on the other hand,

$$J_\varepsilon(\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon} + \lambda(v_1 - \overline{u}_{1\varepsilon})) = J_\varepsilon(\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon}) + \frac{\lambda^2\theta_2}{2}\|\phi_{2\varepsilon}\|^2_{L^2(\Gamma_1)} + \frac{\lambda^2 N_1}{2}\|v_1 - \overline{u}_{1\varepsilon}\|^2_{L^2(\Gamma_0)} + \lambda\theta_2\left(\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu} - z_d, \frac{\partial\phi_{2\varepsilon}}{\partial\nu}\right)_{L^2(\Gamma_1)} + \lambda N_1(\overline{u}_{1\varepsilon}, v_1 - \overline{u}_{1\varepsilon})_{L^2(\Gamma_0)}, \tag{129}$$

and therefore

$$\frac{d}{d\lambda}J_\varepsilon(\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon} + \lambda(v_1 - \overline{u}_{1\varepsilon}))|_{\lambda=0} = \theta_2\left(\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu} - z_d, \frac{\partial\phi_{2\varepsilon}}{\partial\nu}\right)_{L^2(\Gamma_1)} + N_1(\overline{u}_{1\varepsilon}, v_1 - \overline{u}_{1\varepsilon})_{L^2(\Gamma_0)}. \tag{130}$$

So that with the first-order Euler-Lagrange conditions, we obtain that the optimal control $\overline{u}_\varepsilon = (\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon})$ is the unique element of $\mathcal{U}_{ad}$ satisfying.

$$\forall v = (v_0, v_1) \in \mathcal{U}_{ad}, N_0(\overline{u}_{0\varepsilon}, v_0 - \overline{u}_{0\varepsilon})_{L^2(\Gamma_0)} \geq 0, \theta_2\left(\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu} - z_d, \frac{\partial\phi_{2\varepsilon}}{\partial\nu}\right)_{L^2(\Gamma_1)} + N_1(\overline{u}_{1\varepsilon}, v_1 - \overline{u}_{1\varepsilon})_{L^2(\Gamma_0)} \geq 0. \tag{131}$$

Let us then introduce the adjunct state $p_{2\varepsilon}$ by

$$\Delta p_{2\varepsilon} = 0 \quad \text{in } \Omega, \qquad \frac{\partial p_{2\varepsilon}}{\partial\nu} = 0 \quad \text{on } \Gamma_0, \quad p_{1\varepsilon} = -\theta_2\left(\frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu} - z_d\right) \quad \text{on } \Gamma_1. \tag{132}$$

We immediately have, with (132),

$$(p_{2\varepsilon}, \Delta\phi_{2\varepsilon})_{L^2(\Omega)} = (p_{1\varepsilon}, v_1 - \overline{u}_{1\varepsilon})_{L^2(\Gamma_0)} - \theta_2\left(\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu} - z_d, \frac{\partial\phi_{2\varepsilon}}{\partial\nu}\right)_{L^2(\Gamma_1)}. \tag{133}$$

Then, the variational inequalities system (131) finally reduces to

$$\begin{aligned} &\forall v = (v_0, v_1) \in \mathcal{U}_{ad}, \\ &(N_0\overline{u}_{0\varepsilon}, v_0 - \overline{u}_{0\varepsilon})_{L^2(\Gamma_0)} \geq 0, \\ &(N_1\overline{u}_{1\varepsilon} + p_{2\varepsilon}, v_1 - \overline{u}_{1\varepsilon})_{L^2(\Gamma_0)} \geq 0, \end{aligned} \tag{134}$$

and we thus end up proving the announced result. □

3.4. Singular Optimality System. From the results of Section 3.2, we have

$$\begin{aligned} \overline{u}_{0\varepsilon} &\longrightarrow u_0 \quad \text{strongly in } L^2(\Gamma_0), \\ \overline{u}_{1\varepsilon} &\longrightarrow u_1 \quad \text{strongly in } L^2(\Gamma_0) \end{aligned} \tag{135}$$

and

$$\begin{aligned} \overline{y}_{1\varepsilon} &\longrightarrow y \quad \text{strongly in } L^2(\Omega), \\ \overline{y}_{2\varepsilon} &\longrightarrow y \quad \text{strongly in } L^2(\Omega), \end{aligned} \tag{136}$$

where (u, y) is the optimal control-state pair of (43)–(45).

Then it follows, from the fact that the mixed Dirichlet–Neumann problem (117) is well-posed, there exists $p_2 \in H^{3/2}(\Omega)$ such that

$$p_{2\varepsilon} \longrightarrow p_2 \text{ strongly in } L^2(\Omega). \tag{137}$$

Then, we easily pass to the limit, in the results of the previous theorem, to obtain the following characterization of the optimal pair (u, y).

Theorem 4. *The control-state pair (u, y) is unique solution of (43)–(45) if and only if the triple $\{u, y, p_2\}$ (with $p_2 \in H^{\frac{3}{2}}(\Omega)$ given by (137)), is solution of the singular optimality system defined by systems*

$$\Delta y = 0 \quad \text{in } \Omega, \qquad y = u_0, \quad \frac{\partial y}{\partial\nu} = u_1 \text{ on } \Gamma_0, \tag{138}$$

and

$$\Delta p_2 = 0 \quad \text{in } \Omega, \qquad \frac{\partial p_2}{\partial\nu} = 0 \quad \text{on } \Gamma_0, \quad p_2 = -\theta_2\left(\frac{\partial y}{\partial\nu} - z_d\right) \quad \text{on } \Gamma_1, \tag{139}$$

and the variational inequalities system

$$\begin{aligned} &\forall v = (v_0, v_1) \in \mathcal{U}_{ad}, \\ &(N_0 u_0, v_0 - u_0)_{L^2(\Gamma_0)} \geq 0, \\ &(N_1 u_1 + p_2, v_1 - u_1)_{L^2(\Gamma_0)} \geq 0. \end{aligned} \tag{140}$$

Remark 7. As we indicated earlier, the present analysis addresses the question of the control of the Cauchy problem without using any other assumption than the sufficient ones of nonvacuity, convexity, and closure of the sets of admissible controls. The density results obtained by the interpretation made of the initial problem being enough to achieve convergence of the process. Moreover, the sole intervention of the adjunct state p_2 in the optimality system characterizing the optimal control-state pair confirms the intuition that we could be satisfied only with the state y_2 in the interpretation that we make of the initial system as an inverse problem. That is to say, just consider the system (18) with the corresponding observation objective in (15).

4. The Distributed Observation Problem

Let us consider here the cost function

$$J(v,z)=\frac{1}{2}\|z-z_d\|^2_{L^2(\Omega)}+\frac{N_0}{2}\|v_0\|^2_{L^2(\Gamma_0)}+\frac{N_1}{2}\|v_1\|^2_{L^2(\Gamma_0)}, \tag{141}$$

being interested in the control problem

$$\inf\{J(v,z);\ (v,z)\in\mathscr{A}\}. \tag{142}$$

We begin again by noting, as is obvious, that

Theorem 5. *The optimal control problem (142) admits a unique solution, the optimal control-state pair (u,y).*

It follows, again by the Euler–Lagrange first-order optimality condition, that the optimal control-state pair (u,y) is then characterized by the variational inequality

$$\begin{aligned}&(y-z_d,\ z-y)_{L^2(\Omega)}+N_0(u_0,\ v_0-u_0)_{L^2(\Gamma_0)}\\&\quad+N_1(u_1,\ v_1-u_1)_{L^2(\Gamma_0)}\geq 0,\ \forall\,(v,z)\in\mathscr{A}.\end{aligned} \tag{143}$$

Let us now, by the controllability method, define a singular optimality system where state and control are independent, characterizing the optimal solution (u,y).

4.1. The Controllability Method. We still assume that $\mathscr{A}\neq\emptyset$. Then, for all

$$v=(v_0,v_1)\in\mathscr{U}_{ad}\text{ and }\varepsilon>0, \tag{144}$$

we have that, there exist

$$w_{1\varepsilon},w_{2\varepsilon}\in L^2(\Gamma_1)\text{ and }y_1(v_0,w_{1\varepsilon}),y_2(v_1,w_{2\varepsilon})\in H^{3/2}(\Omega), \tag{145}$$

such that

$$\begin{aligned}&\Delta y_1(v_0,w_{1\varepsilon})=0\quad\text{in }\Omega,\\&y_1(v_0,w_{1\varepsilon})=v_0\quad\text{on }\Gamma_0,\quad\frac{\partial y_1}{\partial\nu}(v_0,w_{1\varepsilon})=w_{1\varepsilon}\quad\text{on }\Gamma_1,\end{aligned} \tag{146}$$

$$\begin{aligned}&\Delta y_2(v_1,w_{2\varepsilon})=0\quad\text{in }\Omega,\\&\frac{\partial y_2}{\partial\nu}(v_1,w_{2\varepsilon})=v_1\quad\text{on }\Gamma_0,\quad y_2(v_1,w_{2\varepsilon})=w_{2\varepsilon}\quad\text{on }\Gamma_1,\end{aligned} \tag{147}$$

$$\left\|\frac{\partial y_1}{\partial\nu}(v_0,w_{1\varepsilon})-v_1\right\|_{L^2(\Gamma_0)}<\varepsilon\text{ and }\|y_2(v_1,w_{2\varepsilon})-v_0\|_{L^2(\Gamma_0)}<\varepsilon. \tag{148}$$

Then, we consider, for $\theta_1,\theta_2\in\mathbb{R}_+:\theta_1+\theta_2=1$, the functional

$$\begin{aligned}J_\varepsilon(v_0,v_1)=&\frac{\theta_1}{2}\|y_1(v_0,w_{1\varepsilon})-z_d\|^2_{L^2(\Omega)}\\&+\frac{\theta_2}{2}\|y_2(v_1,w_{2\varepsilon})-z_d\|^2_{L^2(\Omega)}\\&+\frac{N_0}{2}\|v_0\|^2_{L^2(\Gamma_0)}+\frac{N_1}{2}\|v_1\|_{L^2(\Gamma_0)},\end{aligned} \tag{149}$$

being interested in the control problem

$$\inf\{J_\varepsilon(v_0,v_1);\ v=(v_0,v_1)\in\mathscr{U}_{ad}\}. \tag{150}$$

We immediately have the following result.

Proposition 4. *For all $\varepsilon>0$, the control problem (150) admits a unique solution, the optimal control $\overline{u}_\varepsilon=(\overline{u}_{0\varepsilon},\overline{u}_{1\varepsilon})$.*

4.2. Convergence of the Method. Let $\varepsilon>0$; we have

$$\overline{w}_{1\varepsilon},\overline{w}_{2\varepsilon}\in L^2(\Gamma_1),\text{ and }\overline{y}_{1\varepsilon},\overline{y}_{2\varepsilon}\in H^{3/2}(\Omega), \tag{151}$$

such that

$$\begin{aligned}&\Delta\overline{y}_{1\varepsilon}=0\quad\text{in }\Omega,\\&\overline{y}_{1\varepsilon}=\overline{u}_{0\varepsilon}\quad\text{on }\Gamma_0,\quad\frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu}=\overline{w}_{1\varepsilon}\quad\text{on }\Gamma_1,\end{aligned} \tag{152}$$

$$\begin{aligned}&\Delta\overline{y}_{2\varepsilon}=0\quad\text{in }\Omega,\\&\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu}=\overline{u}_{1\varepsilon}\quad\text{on }\Gamma_0,\quad\overline{y}_{2\varepsilon}=\overline{w}_{2\varepsilon}\quad\text{on }\Gamma_1,\end{aligned} \tag{153}$$

$$\left\|\frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu}-\overline{u}_{1\varepsilon}\right\|_{L^2(\Gamma_0)}<\varepsilon\text{ and }\|\overline{y}_{2\varepsilon}-\overline{u}_{0\varepsilon}\|_{L^2(\Gamma_0)}<\varepsilon, \tag{154}$$

with

$$J_\varepsilon(\overline{u}_{0\varepsilon},\overline{u}_{1\varepsilon})\leq J_\varepsilon(v_0,v_1),\ \forall\,v=(v_0,v_1)\in\mathscr{U}_{ad}. \tag{155}$$

We verify here again that we have

$$J_\varepsilon(\overline{u}_{0\varepsilon},\overline{u}_{1\varepsilon})\leq J_\varepsilon(u_0,u_1)=J(u,y), \tag{156}$$

from where we again deduce the existence of constants $C_i\in\mathbb{R}^*_+$, independent of ε, such that

$$\begin{aligned}&\|\overline{y}_{1\varepsilon}\|_{L^2(\Omega)}\leq C_1,\quad\|\overline{y}_{2\varepsilon}\|_{L^2(\Omega)}\leq C_2,\\&\|\overline{u}_{0\varepsilon}\|_{L^2(\Gamma_0)}\leq C_3,\quad\|\overline{u}_{1\varepsilon}\|_{L^2(\Gamma_0)}\leq C_4.\end{aligned} \tag{157}$$

From which, it follows although there exists

$$\widehat{u}_0,\ \widehat{u}_1\in L^2(\Gamma_0)\text{ and }\widehat{y}_1=\widehat{y}=\widehat{y}_2\in L^2(\Omega), \tag{158}$$

such that

$$\begin{aligned} \overline{u}_{0\varepsilon} &\longrightarrow \widehat{u}_0 \quad \text{weakly in } L^2(\Gamma_0), \\ \overline{u}_{1\varepsilon} &\longrightarrow \widehat{u}_1 \quad \text{weakly in } L^2(\Gamma_0), \end{aligned} \tag{159}$$

and

$$\begin{aligned} \overline{y}_{1\varepsilon} &\longrightarrow \widehat{y}_1 \quad \text{weakly in } L^2(\Omega), \\ \overline{y}_{2\varepsilon} &\longrightarrow \widehat{y}_2 \quad \text{weakly in } L^2(\Omega), \end{aligned} \tag{160}$$

with $(\widehat{u}, \widehat{y})$ admissible.

Thus, it follows that

$$J(u, y) \leq J(\widehat{u}, \widehat{y}), \tag{161}$$

so, by passing to the limit in (156), that

$$J(\widehat{u}, \widehat{y}) \leq J(u, y) \leq J(\widehat{u}, \widehat{y}), \tag{162}$$

and consequently

$$(\widehat{u}, \widehat{y}) = (u, y), \tag{163}$$

ending there to prove the following proposition:

Proposition 5. *For all $\varepsilon > 0$, the optimal control $\overline{u}_\varepsilon = (\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon})$, solution of (150)is such that the control-state pair $(\overline{u}_\varepsilon, \overline{y}_\varepsilon)$ satisfied*

$$\begin{aligned} \overline{u}_{0\varepsilon} &\longrightarrow u_0 \quad \text{weakly in } L^2(\Gamma_0), \\ \overline{u}_{1\varepsilon} &\longrightarrow u_1 \quad \text{weakly in } L^2(\Gamma_0), \\ \overline{y}_{1\varepsilon} &\longrightarrow y \quad \text{weakly in } L^2(\Omega), \\ \overline{y}_{2\varepsilon} &\longrightarrow y \quad \text{weakly in } L^2(\Omega), \end{aligned} \tag{164}$$

where (u, y) is the optimal control-state pair of (43), (141), (142).

Moreover, we have the following theorem:

Theorem 6. *The optimal control $\overline{u}_\varepsilon = (\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon})$ and the corresponding state $\overline{y}_\varepsilon = (\overline{y}_{1\varepsilon}, \overline{y}_{2\varepsilon})$ are such that, when $\varepsilon \longrightarrow 0$,*

$$\begin{aligned} \overline{u}_{0\varepsilon} &\longrightarrow u_0 \quad \text{strongly in } L^2(\Gamma_0), \\ \overline{u}_{1\varepsilon} &\longrightarrow u_1 \quad \text{strongly in } L^2(\Gamma_0), \end{aligned} \tag{165}$$

and

$$\begin{aligned} \overline{y}_{1\varepsilon} &\longrightarrow y \quad \text{strongly in } L^2(\Omega), \\ \overline{y}_{2\varepsilon} &\longrightarrow y \quad \text{strongly in } L^2(\Omega), \end{aligned} \tag{166}$$

where (u, y) is the optimal control-state for the control problem (43), (141), (142).

Proof. From the previous results, we have that

$$\begin{aligned} \overline{u}_{0\varepsilon} &\longrightarrow u_0 \quad \text{weakly in } L^2(\Gamma_0), \\ \overline{u}_{1\varepsilon} &\longrightarrow u_1 \quad \text{weakly in } L^2(\Gamma_0), \end{aligned} \tag{167}$$

$$\begin{aligned} \overline{y}_{1\varepsilon} &\longrightarrow y \quad \text{weakly in } L^2(\Omega), \\ \overline{y}_{2\varepsilon} &\longrightarrow y \quad \text{weakly in } L^2(\Omega), \end{aligned} \tag{168}$$

and

$$J(u, y) \leq \lim_{\varepsilon \to 0} J_\varepsilon(\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon}) \leq J(u, y), \tag{169}$$

this last result can still be written

$$\begin{aligned} &\|y - z_d\|^2_{L^2(\Omega)} + N_0 \|u_0\|^2_{L^2(\Gamma_0)} + N_1 \|u_1\|^2_{L^2(\Gamma_0)} \\ &\quad = \lim_{\varepsilon \to 0} \Big(\theta_1 \|\overline{y}_{1\varepsilon} - z_d\|^2_{L^2(\Omega)} + \theta_2 \|\overline{y}_{2\varepsilon} - z_d\|^2_{L^2(\Omega)} \\ &\qquad + N_0 \|\overline{u}_{0\varepsilon}\|^2_{L^2(\Gamma_0)} + N_1 \|\overline{u}_{1\varepsilon}\|^2_{L^2(\Gamma_0)} \Big). \end{aligned} \tag{170}$$

But then, the norms being continuous, a fortiori weakly lower semicontinuous, we have with (167) and (168) that

$$\begin{aligned} \|u_0\|^2_{L^2(\Gamma_0)} &\leq \liminf_{\varepsilon \to 0} \|\overline{u}_{0\varepsilon}\|^2_{L^2(\Gamma_0)}, \\ \|u_1\|^2_{L^2(\Gamma_0)} &\leq \liminf_{\varepsilon \to 0} \|\overline{u}_{1\varepsilon}\|^2_{L^2(\Gamma_0)}, \\ \|y - z_d\|^2_{L^2(\Omega)} &\leq \liminf_{\varepsilon \to 0} \|\overline{y}_{1\varepsilon} - z_d\|^2_{L^2(\Omega)}, \\ \|y - z_d\|^2_{L^2(\Omega)} &\leq \liminf_{\varepsilon \to 0} \|\overline{y}_{2\varepsilon} - z_d\|^2_{L^2(\Omega)}, \end{aligned} \tag{171}$$

so also

$$\begin{aligned} \|u_0\|^2_{L^2(\Gamma_0)} &\leq \liminf_{\varepsilon \to 0} \|\overline{u}_{0\varepsilon}\|^2_{L^2(\Gamma_0)}, \\ \|u_1\|^2_{L^2(\Gamma_0)} &\leq \liminf_{\varepsilon \to 0} \|\overline{u}_{1\varepsilon}\|^2_{L^2(\Gamma_0)}, \\ \|y - z_d\|^2_{L^2(\Omega)} &\leq \liminf_{\varepsilon \to 0} \Big(\theta_1 \|\overline{y}_{1\varepsilon} - z_d\|^2_{L^2(\Omega)} \\ &\qquad + \theta_1 \|\overline{y}_{2\varepsilon} - z_d\|^2_{L^2(\Omega)} \Big). \end{aligned} \tag{172}$$

So that, with (170), it follows:

$$\begin{aligned} \|u_0\|^2_{L^2(\Gamma_0)} &= \lim_{\varepsilon \to 0} \|\overline{u}_{0\varepsilon}\|^2_{L^2(\Gamma_0)}, \\ \|u_1\|^2_{L^2(\Gamma_0)} &= \lim_{\varepsilon \to 0} \|\overline{u}_{1\varepsilon}\|^2_{L^2(\Gamma_0)}, \\ \|y - z_d\|^2_{L^2(\Omega)} &= \lim_{\varepsilon \to 0} \Big(\theta_1 \|\overline{y}_{1\varepsilon} - z_d\|^2_{L^2(\Omega)} + \theta_2 \|\overline{y}_{2\varepsilon} - z_d\|^2_{L^2(\Omega)} \Big). \end{aligned} \tag{173}$$

But then, since

$$\|\overline{u}_{0\varepsilon}-u_0\|^2_{L^2(\Gamma_0)}+\|\overline{u}_{1\varepsilon}-u_1\|^2_{L^2(\Gamma_0)}=\|\overline{u}_{0\varepsilon}\|^2_{L^2(\Gamma_0)}+\|u_0\|^2_{L^2(\Gamma_0)}+\|\overline{u}_{1\varepsilon}\|^2_{L^2(\Gamma_0)}+\|u_1\|^2_{L^2(\Gamma_0)} - 2(\overline{u}_{0\varepsilon},\ u_0)_{L^2(\Gamma_0)}-2(\overline{u}_{1\varepsilon},\ u_1)_{L^2(\Gamma_0)}, \tag{174}$$

we obtain, to the limit, with (167), $(173)_1$, and $(173)_2$, that

$$\lim_{\varepsilon\to0}\left(\|\overline{u}_{0\varepsilon}-u_0\|^2_{L^2(\Gamma_0)}+\|\overline{u}_{1\varepsilon}-u_1\|^2_{L^2(\Gamma_0)}\right)=0, \tag{175}$$

i.e.,

$$\begin{aligned}\overline{u}_{0\varepsilon}&\longrightarrow u_0 \quad \text{strongly in } L^2(\Gamma_0),\\ \overline{u}_{1\varepsilon}&\longrightarrow u_1 \quad \text{strongly in } L^2(\Gamma_0).\end{aligned} \tag{176}$$

Moreover, noting that we deduce from $(173)_3$ that

$$\lim_{\varepsilon\to0}\|\overline{y}_{1\varepsilon}-z_d\|^2_{L^2(\Omega)}=\|y-z_d\|^2_{L^2(\Omega)}=\lim_{\varepsilon\to0}\|\overline{y}_{2\varepsilon}-z_d\|^2_{L^2(\Omega)}, \tag{177}$$

it follows well with

$$\|\overline{y}_{1\varepsilon}-y\|^2_{L^2(\Omega)}=\|\overline{y}_{1\varepsilon}-z_d\|^2_{L^2(\Omega)}+\|y-z_d\|^2_{L^2(\Omega)} - 2(\overline{y}_{1\varepsilon}-z_d,\ y-z_d)_{L^2(\Omega)}, \tag{178}$$

and

$$\|\overline{y}_{2\varepsilon}-y\|^2_{L^2(\Omega)}=\|\overline{y}_{2\varepsilon}-z_d\|^2_{L^2(\Omega)}+\|y-z_d\|^2_{L^2(\Omega)} - 2(\overline{y}_{2\varepsilon}-z_d,\ y-z_d)_{L^2(\Omega)}, \tag{179}$$

that

$$\begin{aligned}\overline{y}_{1\varepsilon}&\longrightarrow y \quad \text{strongly in } L^2(\Omega),\\ \overline{y}_{2\varepsilon}&\longrightarrow y \quad \text{strongly in } L^2(\Omega).\end{aligned} \tag{180}$$

□

4.3. Approached Optimality System. We show here that, for all $\varepsilon>0$, the control $\overline{u}_\varepsilon=(\overline{u}_{0\varepsilon},\overline{u}_{1\varepsilon})$, optimal solution of the problem (150), is characterized by the optimality system defined by the following theorem:

Theorem 7. *Let $\varepsilon>0$. The control $\overline{u}_\varepsilon=(\overline{u}_{0\varepsilon},\overline{u}_{1\varepsilon})$ is unique solution of the problem (150) if and only if there exists*

$$\overline{w}_{1\varepsilon},\overline{w}_{2\varepsilon}\in L^2(\Gamma_1),\ \overline{y}_{1\varepsilon},\overline{y}_{2\varepsilon}\in H^{3/2}(\Omega),\ \text{and } p_{1\varepsilon},p_{2\varepsilon}\in H^{3/2}(\Omega), \tag{181}$$

such that the quadruplet

$$\{(\overline{u}_{0\varepsilon},\overline{u}_{1\varepsilon}),(\overline{w}_{1\varepsilon},\overline{w}_{2\varepsilon}),(\overline{y}_{1\varepsilon},\overline{y}_{2\varepsilon}),(p_{1\varepsilon},p_{2\varepsilon})\}, \tag{182}$$

is a solution of the optimality system defined by systems

$$\begin{aligned}&\Delta\overline{y}_{1\varepsilon}=0 \quad \text{in } \Omega,\\ &\overline{y}_{1\varepsilon}=\overline{u}_{0\varepsilon} \quad \text{on } \Gamma_0,\quad \frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu}=\overline{w}_{1\varepsilon} \quad \text{on } \Gamma_1,\end{aligned} \tag{183}$$

$$\begin{aligned}&\Delta\overline{y}_{2\varepsilon}=0 \quad \text{in } \Omega,\\ &\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu}=\overline{u}_{1\varepsilon} \quad \text{on } \Gamma_0,\quad \overline{y}_{2\varepsilon}=\overline{w}_{2\varepsilon} \quad \text{on } \Gamma_1,\end{aligned} \tag{184}$$

$$\begin{aligned}&\Delta p_{1\varepsilon}=\theta_1(\overline{y}_{1\varepsilon}-z_d) \quad \text{in } \Omega,\\ &p_{1\varepsilon}=0 \text{ on } \Gamma_0,\quad \frac{\partial p_{1\varepsilon}}{\partial\nu}=0 \text{ on } \Gamma_1,\end{aligned} \tag{185}$$

$$\begin{aligned}&\Delta p_{2\varepsilon}=\theta_2(\overline{y}_{2\varepsilon}-z_d) \quad \text{in } \Omega,\\ &\frac{\partial p_{2\varepsilon}}{\partial\nu}=0 \text{ on } \Gamma_0,\quad p_{2\varepsilon}=0 \text{ on } \Gamma_1,\end{aligned} \tag{186}$$

the estimates

$$\left\|\frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu}-\overline{u}_{1\varepsilon}\right\|_{L^2(\Gamma_0)}<\varepsilon \text{ and } \|\overline{y}_{2\varepsilon}-\overline{u}_{0\varepsilon}\|_{L^2(\Gamma_0)}<\varepsilon, \tag{187}$$

and the system of variational inequalities

$$\begin{aligned}&\forall\, v=(v_0,v_1)\in\mathcal{U}_{ad},\\ &\left(N_0\overline{u}_{0\varepsilon}+\frac{\partial p_{1\varepsilon}}{\partial\nu},\ v_0-\overline{u}_{0\varepsilon}\right)_{L^2(\Gamma_0)}\geq0,\\ &(N_1\overline{u}_{1\varepsilon}-p_{2\varepsilon},\ v_1-\overline{u}_{1\varepsilon})_{L^2(\Gamma_0)}\geq0.\end{aligned} \tag{188}$$

Proof. Let $\varepsilon>0$. We have that there exists a unique control $\overline{u}_\varepsilon=(\overline{u}_{0\varepsilon},\overline{u}_{1\varepsilon})\in\mathcal{U}_{ad}$ optimal solution of (150), with

$$\overline{w}_{1\varepsilon},\overline{w}_{2\varepsilon}\in L^2(\Gamma_1) \text{ and } \overline{y}_{1\varepsilon},\overline{y}_{2\varepsilon}\in H^{\frac{3}{2}}(\Omega), \tag{189}$$

such that

$$\begin{aligned}&\Delta\overline{y}_{1\varepsilon}=0 \quad \text{in } \Omega,\\ &\overline{y}_{1\varepsilon}=\overline{u}_{0\varepsilon} \quad \text{on } \Gamma_0,\quad \frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu}=\overline{w}_{1\varepsilon} \quad \text{on } \Gamma_1,\end{aligned} \tag{190}$$

$$\begin{aligned}&\Delta\overline{y}_{2\varepsilon}=0 \quad \text{in } \Omega,\\ &\frac{\partial\overline{y}_{2\varepsilon}}{\partial\nu}=\overline{u}_{1\varepsilon} \quad \text{on } \Gamma_0,\quad \overline{y}_{2\varepsilon}=\overline{w}_{2\varepsilon} \quad \text{on } \Gamma_1,\end{aligned} \tag{191}$$

$$\left\|\frac{\partial\overline{y}_{1\varepsilon}}{\partial\nu}-\overline{u}_{1\varepsilon}\right\|_{L^2(\Gamma_0)}<\varepsilon \text{ and } \|\overline{y}_{2\varepsilon}-\overline{u}_{0\varepsilon}\|_{L^2(\Gamma_0)}<\varepsilon. \tag{192}$$

So let $v=(v_0,v_1)\in\mathcal{U}_{ad}$ and $\lambda\in\mathbb{R}^*$; one easily checks that the functional J_ε is differentiable with respect to $\overline{u}_{0\varepsilon}$ and $\overline{u}_{1\varepsilon}$, with

$$\frac{d}{d\lambda}J_\varepsilon\left(\overline{u}_{0\varepsilon}+\lambda(v_0-\overline{u}_{0\varepsilon}),\overline{u}_{1\varepsilon}\right)|_{\lambda=0}=\theta_1(\overline{y}_{1\varepsilon}-z_d,\phi_{1\varepsilon})_{L^2(\Omega)}+N_0(\overline{u}_{0\varepsilon},v_0-\overline{u}_{0\varepsilon})_{L^2(\Gamma_0)}, \tag{193}$$

where we denote by $\phi_{1\varepsilon}=y_1\left(v_0-\overline{u}_{0\varepsilon},\overline{w}_{1\varepsilon}\right)-y_1\left(0,\overline{w}_{1\varepsilon}\right)$ the solution to

$$\begin{aligned}&\Delta\phi_{1\varepsilon}=0 \quad \text{in } \Omega,\\ &\phi_{1\varepsilon}=v_0-\overline{u}_{0\varepsilon} \quad \text{on } \Gamma_0, \quad \frac{\partial\phi_{1\varepsilon}}{\partial\nu}=0 \quad \text{on } \Gamma_1,\end{aligned} \tag{194}$$

and

$$\frac{d}{d\lambda}J_\varepsilon\left(u_{0\varepsilon},\overline{u}_{1\varepsilon}+\lambda(v_1-\overline{u}_{1\varepsilon})\right)|_{\lambda=0}=\theta_2(\overline{y}_{2\varepsilon}-z_d,\ \phi_{2\varepsilon})_{L^2(\Omega)}+N_1(\overline{u}_{1\varepsilon},\ v_1-\overline{u}_{1\varepsilon})_{L^2(\Gamma_0)}, \tag{195}$$

where $\phi_{2\varepsilon}=y_2\left(v_1-\overline{u}_{1\varepsilon},\overline{w}_{2\varepsilon}\right)-y_2\left(0,\overline{w}_{2\varepsilon}\right)$ is given by

$$\begin{aligned}&\Delta\phi_{2\varepsilon}=0 \quad \text{in } \Omega,\\ &\frac{\partial\phi_{2\varepsilon}}{\partial\nu}=v_1-\overline{u}_{1\varepsilon} \quad \text{on } \Gamma_0, \quad \phi_{2\varepsilon}=0 \quad \text{on } \Gamma_1.\end{aligned} \tag{196}$$

Thus, the Euler–Lagrange first-order optimality conditions make it possible to obtain that the optimal control $\overline{u}_\varepsilon=(\overline{u}_{0\varepsilon},\overline{u}_{1\varepsilon})$ is the unique element of $\mathcal{U}_{ad}$ satisfying the optimality condition

$$\begin{aligned}&\forall\, v=(v_0,v_1)\in\mathcal{U}_{ad},\\ &\theta_1(\overline{y}_{1\varepsilon}-z_d,\ \phi_{1\varepsilon})_{L^2(\Omega)}+N_0(\overline{u}_{0\varepsilon},\ v_0-\overline{u}_{0\varepsilon})_{L^2(\Gamma_0)}\geq 0,\\ &\theta_2(\overline{y}_{2\varepsilon}-z_d,\ \phi_{2\varepsilon})_{L^2(\Omega)}+N_1(\overline{u}_{1\varepsilon},\ v_1-\overline{u}_{1\varepsilon})_{L^2(\Gamma_0)}\geq 0.\end{aligned} \tag{197}$$

Let us then introduce the adjunct states $p_{1\varepsilon}$ and $p_{2\varepsilon}$, respectively, defined by

$$\begin{aligned}&\Delta p_{1\varepsilon}=\theta_1(\overline{y}_{1\varepsilon}-z_d) \quad \text{in } \Omega,\\ &p_{1\varepsilon}=0 \text{ on } \Gamma_0, \quad \frac{\partial p_{1\varepsilon}}{\partial\nu}=0 \text{ on } \Gamma_1,\end{aligned} \tag{198}$$

and

$$\begin{aligned}&\Delta p_{2\varepsilon}=\theta_2(\overline{y}_{2\varepsilon}-z_d) \quad \text{in } \Omega,\\ &\frac{\partial p_{2\varepsilon}}{\partial\nu}=0 \text{ on } \Gamma_0, \quad p_{2\varepsilon}=0 \text{ on } \Gamma_1.\end{aligned} \tag{199}$$

Hence, it follows, according to (194) and (198), that

$$\theta_1(\overline{y}_{1\varepsilon}-z_d\ ,\ \phi_{1\varepsilon})_{L^2(\Omega)}=\left(\frac{\partial p_{1\varepsilon}}{\partial\nu}\ ,\ v_0-\overline{u}_{0\varepsilon}\right)_{L^2(\Gamma_0)}, \tag{200}$$

and, from (196) to (199), that

$$\theta_2(\overline{y}_{2\varepsilon}-z_d,\ \phi_{2\varepsilon})_{L^2(\Omega)}=(-p_{2\varepsilon},\ v_1-\overline{u}_{1\varepsilon})_{L^2(\Gamma_0)}. \tag{201}$$

Which gives that the optimality condition (197) is rewritten

$$\begin{aligned}&\forall\, v=(v_0,v_1)\in\mathcal{U}_{ad},\\ &\left(N_0\overline{u}_{0\varepsilon}+\frac{\partial p_{1\varepsilon}}{\partial\nu},\ v_0-\overline{u}_{0\varepsilon}\right)_{L^2(\Gamma_0)}\geq 0,\\ &(N_1\overline{u}_{1\varepsilon}-p_{2\varepsilon},\ v_1-\overline{u}_{1\varepsilon})_{L^2(\Gamma_0)}\geq 0.\end{aligned} \tag{202}$$

We thus obtained the announced result. □

Passing to the limit in the last results above and calling on those of Section 4.2, we succeed, in the last section below, in defining the singular optimality system characterizing the control-state pair (u,y), optimal solution of (142).

4.4. Singular Optimality System. From the results of Section 4.2, we have that

$$\begin{aligned}&\overline{u}_{0\varepsilon}\longrightarrow u_0 \quad \text{strongly in } L^2(\Gamma_0),\\ &\overline{u}_{1\varepsilon}\longrightarrow u_1 \quad \text{strongly in } L^2(\Gamma_0)\end{aligned} \tag{203}$$

and

$$\begin{aligned}&\overline{y}_{1\varepsilon}\longrightarrow y \quad \text{strongly in } L^2(\Omega),\\ &\overline{y}_{2\varepsilon}\longrightarrow y \quad \text{strongly in } L^2(\Omega).\end{aligned} \tag{204}$$

Then, the problems (185) and (186) being well-posed, it follows that there exists

$$p_1,p_2\in H^{3/2}(\Omega), \tag{205}$$

such that

$$\begin{aligned}&p_{1\varepsilon}\longrightarrow p_1 \quad \text{strongly in } H^{3/2}(\Omega),\\ &p_{2\varepsilon}\longrightarrow p_2 \quad \text{strongly in } H^{3/2}(\Omega).\end{aligned} \tag{206}$$

Thus, the singular optimality system for the optimal solution (u,y) of (142), is obtained as follows, by passing to the limit in the results of Theorem 7.

Theorem 8. *The control-state pair (u,y) is a unique solution of (142) if and only if the triple $\{u,y,p\}$, with*

$$p = (p_1, p_2) \in \left(H^{\frac{3}{2}}(\Omega)\right)^2, \tag{207}$$

given by (205) and (206), is a solution of the optimality system defined by the systems

$$\begin{aligned} &\Delta y = 0 \quad \text{in}\,\Omega, \\ &y = u_0, \quad \frac{\partial y}{\partial \nu} = u_1 \ \text{on}\,\Gamma_0, \end{aligned} \tag{208}$$

$$\begin{aligned} &\Delta p_1 = \theta_1(y - z_d) \quad \text{in}\,\Omega, \\ &p_1 = 0 \ \text{on}\,\Gamma_0, \qquad \frac{\partial p_1}{\partial \nu} = 0 \ \text{on}\,\Gamma_1, \end{aligned} \tag{209}$$

$$\begin{aligned} &\Delta p_2 = \theta_2(y - z_d) \quad \text{in}\,\Omega, \\ &\frac{\partial p_2}{\partial \nu} = 0 \ \text{on}\,\Gamma_0, \qquad p_2 = 0 \ \text{on}\,\Gamma_1, \end{aligned} \tag{210}$$

and the variational inequalities system

$$\begin{aligned} &\forall v = (v_0, v_1) \in \mathscr{U}_{ad}, \\ &\left(N_0 u_0 + \frac{\partial p_1}{\partial \nu},\ v_0 - u_0\right)_{L^2(\Gamma_0)} \quad \geq 0, \\ &(N_1 u_1 - p_2,\ v_1 - u_1)_{L^2(\Gamma_0)} \quad \geq 0. \end{aligned} \tag{211}$$

We end well here, with this last case of distributed observation, the analysis of the control of the elliptic Cauchy problem by the controllability method. The results obtained above end up consolidating the intuition mentioned in the introduction and clarifying the point of view proposed here. This point of view, consisting of interpreting the Cauchy problem as an inverse problem, makes it possible to dispense with the Slater-type assumption (10).

Finally, we note, as underlined in the introduction, that in the case of distributed observation, the interpretation of the problem could be satisfied, depending on whether it is easier to observe/control one or the other of the Cauchy data from only one of the systems (146) and (147), with the corresponding observation objective in (148).

5. Conclusion

In this work, we succeed in characterizing the optimal control-state pair of the control problem for the elliptic ill-posed Cauchy problem, using the controllability concept. The method consists of interpreting the initial problem as a system of inverse problems and, therefore, a system of controllability problems. An approach that allows us to obtain, in the general case with constraints on the control, a strong and decoupled singular optimality system. And that, without using any additional assumption, such as that of nonvacuity of the interior of the sets of admissible controls, a Slater-type assumption that many analyses have had to use. Beyond that, the results obtained here confirm the intuition, announced in [2], that the analysis by controllability can be satisfied with a single inverse problem. In sum, therefore,

(i) for the control problem with boundary observation of the state, the analysis could be content with the system (13) with the corresponding observation objective in (15);
(ii) for the control problem with an observation of the flow, the analysis could be content with the system (14) and the corresponding observation objective in (15);
(iii) finally, for the problem with distributed observation, either of the systems (13) or (14), with the corresponding observation objective in (15), should suffice.

We think that the difficulty to circumvent will consist in knowing how to obtain strong convergence of the process.

Data Availability

No underlying data were collected and or produced in this study.

Disclosure

A preprint has previously been published (cf. [10]).

Conflicts of Interest

The author declares that there is no conflicts of interest regarding the publication of this paper.

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