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On a Generalization of the Padovan Numbers

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

This paper studies an extension of the classical Padovan sequence and that contains this as a particular case. Some very interesting formulas are found for the sum of these new sequences, for the sum of their squares as well as their self-convolution.

Keywords: Padovan numbers; generating function; self-convolution.

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1 Introduction

In this section we remember the Padovan numbers and study some of the results obtained for them that we will later adapt to our new numbers.

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The *Padovan sequence* [1,2] is the integer sequence *P(n)* defined by the recurrence relation

$P(n) = P(n-2) + P(n-3)$ with initial values P(0) = P(1) = P(2) = 1.

The first values of this sequence are P = {P(n)} = {1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28…}. This sequence is indexed in the OEIS [3] as A000931.

The Padovan sequence is named after Richard Padovan who attributed its discovery to Dutch architect Hans van der Laan in his 1994 essay *Dom. Hans van der Laan: Modern Primitive*. The sequence was described by Ian Stewart in [4].

1.1 Recurrence Relation

Among the many recurring relations verified by the Padovan relation, we choose the following two for its demonstration:

1) P(n) = P(n – 1) + P(n – 5)
2) P(n) = P(n – 2) + P(n – 4) + P(n – 8)

Proof of Formula 1). From the definition

$$\begin{aligned} P(n) &= P(n-2) + P(n-3) = P(n-2) + (P(n-1) - P(n-4)) \\ &= P(n-2) + P(n-1) - (P(n-2) - P(n-5)) = P(n-1) + P(n-5) \end{aligned}$$

Proof of Formula 2).

$$\begin{aligned} P(n) &= P(n-2) + P(n-3) = P(n-2) + (P(n-5) + P(n-6)) \\ &= P(n-2) + P(n-5) + (P(n-4) - P(n-7)) \\ &= P(n-2) + P(n-5) + P(n-4) - (P(n-5) - P(n-8)) = P(n-2) + P(n-4) + P(n-8) \end{aligned}$$

because P(n – 4) = P(n – 6) + P(n – 7) from where P(n – 6) = P(n – 4) – P(n – 7)

In the following theorem we give a formula to calculate the sum of the first *n* Padovan numbers.

Theorem 1. Sum of the Padovan numbers

The sum of the first n terms in the Padovan sequence is $S(n) = \sum_{j=0}^{n} P(j) = P(n+5) - 2$

Proof.

When the result of a mathematical operation is known, a simple way to demonstrate it is by the method of induction. For this reason, we will use this method several times in this article. And even more so when, as in this case, the formula to calculate any Padovan number is too complicated. Or we simply don't use it.

We will do the proof of this formula by the induction method.

For $n = 3$: $S(3) = \sum_{j=0}^{3} P(j) = P(0) + P(1) + P(2) + P(3) = 1 + 1 + 1 + 2 = 5$ and

P(n + 5) – 2 = P(8) – 2 = 7 – 2 = 5

Suppose this formula is true up to *n*. So it must be the same for $n + 1$:

$$S(n) = \sum_{j=0}^{n} P(j) = P(n+5) - 2$$

$$S(n+1) = \sum_{j=0}^{n+1} P(j) = \sum_{j=0}^{n} P(j) + P(n+1) = P(n+5) - 2 + P(n+1)$$

= P(n + 3) + P(n + 2) – 2 + P(n + 1) = P(n + 3) – 2 + P(n + 4) = P(n + 6) – 2

= P((n + 1) + 5) – 2 = S(n + 1)

The formulas for the sum of the even or the odd Padovan numbers can also be proven by induction: $S(2n) = \sum_{j=0}^{2n} P(j) = P(2n+3) - 1$ and $S(2n+1) = \sum_{j=0}^{2n+1} P(j) = P(2n+4) - 1$

1.2 Two formulas for the sum of the squares of the padovan numbers

In this subsection, we will give two formulas to calculate the sum of the squares of the Padovan numbers.

Theorem 2. First formula for the sum of the squares of the Padovan numbers

The sum of the squares of the Padovan numbers is $S_2(n) = \sum_{j=0}^{n} P(j)^2 = 2P(n)P(n+1) - P(n-2)^2$.

Again we will use the induction method. For *n* = *3*, $S_2(3) = \sum_{j=0}^{3} P(j)^2 = 1+1+1+4 = 7$ and the second hand right is 2P(3)P(4) – P(1)^2 = 2·2·2 – 1^2 = 7.

Let us suppose the formula is true up to *n*. Then

$$S_2(n+1) = \sum_{j=0}^{n+1} P(j)^2 = \sum_{j=0}^{n} P(j)^2 + P(n+1)^2$$

$$S_2(n+1) = \sum_{j=0}^{n+1} P(j)^2 = \sum_{j=0}^{n} P(j)^2 + P(n+1)^2$$

$$= 2P(n)\,P(n+1) - P(n-2)^2 + P(n+1)^2$$

$$= 2P(n)\,P(n+1) - \big(P(n+1) - P(n-1)\big)^2 + P(n+1)^2$$

$$= 2P(n)\,P(n+1) - P(n+1)^2 + 2P(n-1)P(n+1) - P(n-1)^2 + P(n+1)^2$$

$$= 2P(n+1)\big(P(n) + P(n-1)\big) - P(n-1)^2$$

$$= 2P(n+1)\,P(n+2) - P(n-1)^2 = S_2(n+1)$$

And thus the formula for calculating the sum of the squares of the Padovan numbers is demonstrated in a simple way.

Theorem 3. Second formula for the sum of the squares of the Padovan numbers

The sum of the squares of the Padovan numbers is $S_2(n) = \sum_{j=0}^{n} P(j)^2 = P(n+2)^2 - P(n-1)^2 - P(n-3)^2$

Proof.

Changing *n* by *n + 1* in the preceding formula and applying the relation of the definition for P(n + 2) = P(n) + P(n – 1)

$$S_2(n+1)=\sum_{j=0}^{n+1}P(j)^2=2P(n+1)P(n+2)-P(n-1)^2$$

$$=2P(n+1)\big(P(n)+P(n-1)\big)-P(n-1)^2$$

$$=2P(n)P(n+1)+2P(n-1)P(n+1)-P(n-1)^2$$

$$=\big(P(n+1)+P(n)\big)^2-P(n+1)^2-P(n)^2-P(n-1)^2+2P(n-1)P(n+1)$$

$$=\big(P(n+1)+P(n)\big)^2-P(n)^2-\big(P(n+1)-P(n)\big)^2=P(n+3)^2-P(n)^2-P(n-2)^2$$

This formula has the advantage over the first that all the addends of the result are squares.

Next, and in order to extend the indices of the Padovan numbers to the set of integers *Z*, we define the negative Padovan numbers.

Definition 1

Following the same recurrence relationship as in the definition of positive Padovan numbers, the negative index Padovan numbers are defined below:

P(–n + 3) = P(–n + 1) + P(–n) or that is the most used formula P(–n) = P(–n + 3) – P(–n + 1).

As consequence, we have instead of it:

P(–1) = P(2) – P(0) = 1 – 1 = 0
P(–2) = P(1) – P(–1) = 1 – 1 = 0
P(–3) = P(0) – P(–2) = 1 – 1 = 0
P(–4) = P(-1) – P(–3) = 0, etc.

In this way, the following table is obtained:

Table 1. Sequence of the padovan numbers

n	–8	–7	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7	8
P(n)	0	1	–1	1	0	0	1	0	1	1	1	2	2	3	4	5	7

Below we talk about matrices that can generate Padovan numbers through successive powers of an initial matrix. First of all, we will study a theorem in which a proof of this generation is given. The initial matrix or generating matrix is defined in [5]

Theorem 4. [Generating matrix]

The generating matrix of the Padovan numbers is $Q=\begin{pmatrix}0&1&0\\0&0&1\\1&1&0\end{pmatrix}$ *because it is verified that*

$$Q^n=\begin{pmatrix}P(n-5)&P(n-3)&P(n-4)\\P(n-4)&P(n-2)&P(n-3)\\P(n-3)&P(n-1)&P(n-2)\end{pmatrix}$$

Once again, we Will prove this theorem by induction.

For *n = 2* it is $Q^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} P(-3) & P(-1) & P(-2) \\ P(-2) & P(0) & P(-1) \\ P(-1) & P(1) & P(0) \end{pmatrix}$

Assuming that the formula is true for the power *n*, let us show that it is also true for *n + 1*:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} P(n-5) & P(n-3) & P(n-4) \\ P(n-4) & P(n-2) & P(n-3) \\ P(n-3) & P(n-1) & P(n-2) \end{pmatrix} = \begin{pmatrix} P(n-4) & P(n-2) & P(n-3) \\ P(n-3) & P(n-1) & P(n-2) \\ P(n-5)+P(n-4) & P(n-3)+P(n-2) & P(n-4)+P(n-3) \end{pmatrix}$$
$$= \begin{pmatrix} P(n-4) & P(n-2) & P(n-3) \\ P(n-3) & P(n-1) & P(n-2) \\ P(n-2) & P(n) & P(n-1) \end{pmatrix} = Q^{n+1}$$

as we wanted to prove.

A change in numbering allows us to present the previous matrix in its most common form

$$\begin{pmatrix} P(n+4) & P(n+2) & P(n+3) \\ P(n+3) & P(n+1) & P(n+2) \\ P(n+2) & P(n) & P(n+1) \end{pmatrix}$$

2 Generalized K-Padovan Sequence

The goal of this article is to study a generalization of the Padovan sequence that contains the classical one as a particular case.

In a similar way to what we have done in the generalized numbers *k*-Fibonacci and *k*-Lucas [6,7,8], we continue to define the generalized numbers k-Padovan with a recurrence relation similar to that of the previous ones and but very different initial conditions.

Definition 2.

Let k be a non-zero natural number. We define the generalized Padovan sequence of parameter *k* or *k*-Padovan sequence to the sequence defined by the recurrence relation

$P_k(n) = P_k(n-2) + P_k(n-3)$ with the initial conditions $P_k(0) = P_k(1) = P_k(2) = 1$.

Then, the first elements of the *k*-Padovan sequence are $P_k = \{P_k(n)\} = \{1,1,1,k+1,k+1,k^2+k+1,k^2+2k+1,k^3+k^2+2k+1,\ldots\}$

For *k = 1* the sequences obtained is the classical Padovan sequence, {1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21…} indexed in the OEIS as A000931.

For k = 2, the sequence obtained is the Pell-Padovan sequence {1, 1, 1, 3, 3, 7, 9, 17, 25, 43, 67, 111…} indexed in the OEIS as A066983. In this sequence it is verified the relation $P_2(n) = P_2(n-1) + P_2(n-2) + (-1)^n$ with the initial conditions $P_2(1) = P_2(2) = 1$.

No more *k*-Padovan sequence is indexed in the OEIS.

The characteristic equation of the recurrence relation of the definition is $r^3 - k\,r - 1 = 0$.

For *k = 1*, the classical Padovan sequence already studied in the previous section is obtained. The characteristic equation is $r^3 - r - 1 = 0$ admits only one real solution $\Psi = \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \approx 1.324718$ while the other two are complex. This value of Ψ is called plastic number (or plastic ratio or plastic constant or silver number). It is easy to prove that the limit of the quotient of two consecutive numbers is the plastic number:

$$\lim_{n\to\infty} \frac{P(n+1)}{P(n)} = \lim_{n\to\infty} \frac{P(n-1)+P(n-2)}{P(n)} = \lim_{n\to\infty} \frac{\frac{P(n-1)}{P(n-2)} + 1}{\frac{P(n)}{P(n-1)}\,\frac{P(n-1)}{P(n-2)}}$$

If $\lim_{n\to\infty} \frac{P(n+1)}{P(n)} = \lim_{n\to\infty} \frac{P(n)}{P(n-1)} = \lim_{n\to\infty} \frac{P(n-1)}{P(n-2)} = L$ then $L = \frac{L+1}{L \cdot L} \to L^3 - L - 1 = 0$ and the real solution is the plastic number Ψ.

2.1 On the Characteristic Roots

We have already seen that the characteristic equation associated with the recurrence relation of the *k*-Padovan sequence is $r^3 - k\,r - 1 = 0$. Applying the results obtained in [2], the discriminant associated with this equation is $\Delta = 4k^3 - 27$ and the equation has three different real solutions if Δ < 0, while it has one real and two complexes if Δ > 0. Therefore, if Δ > 0 then $4k^3 - 27 > 0 \to k > \frac{3}{\sqrt[3]{4}} \approx 1.88988 \to k \geq 2$ since *k* is a non-null natural number.

Galois theory allows proving that when the three roots are real, and none is rational (casus irreducibilis), one cannot express the roots in terms of real radicals. Nevertheless, purely real expressions of the solutions may be obtained using trigonometric functions, specifically in terms of cosines [1].
In short:

- For every non-zero natural number k there is always a real root.
- For *k = 1* there are other two complex roots and are the only complex characteristic roots for any value of *k*.
- There is only an integer root *r = –1* for *k = 2*.
- If *k > 2*, the three roots are irrational and can be calculated by mean of the formula

$$r_m = 2\sqrt{\frac{k}{3}}\cos\left(\frac{1}{3}\arccos\left(\sqrt{\frac{27}{4k^3}}\right) - \frac{2\pi\,m}{3}\right) \text{ for m = 0, 1 or 2 [9]}$$

Example 1. ***Find the characteristic roots for k = 3***

For *k = 3*, the preceding formula is $r_m = 2\cos\left(\frac{1}{3}\arccos\left(\frac{1}{2}\right) - \frac{2\pi\,m}{3}\right) = 2\cos\left(\frac{\pi}{9} - \frac{2\pi\,m}{3}\right)$ so

(1) $m = 0 \to r_0 = 2\cos\left(\frac{\pi}{9}\right) \approx 1.87939$

(2) $m = 1 \to r_1 = 2\cos\left(\frac{\pi}{9} - \frac{2\pi}{3}\right) = 2\cos\left(-\frac{5\pi}{9}\right) \approx -0.347296$

$$(3) \quad m = 2 \to r_2 = 2\cos\left(\frac{\pi}{9} - \frac{4\pi}{3}\right) = 2\cos\left(-\frac{11\pi}{9}\right) \approx -1.53209$$

Example 2. *Find the characteristic roots for k = 4.*

Similarly, for k = 4, the roots verify the formula

$$r_m = 2\sqrt{\frac{4}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{3}{8}\sqrt{\frac{3}{4}}\right) - \frac{2\pi m}{3}\right)$$ and therefore the roots are

$$r_{0,1,2} = \{2.11491,\ 0.254102,\ 1.86081\}$$

2.2 Sum of the terms of the k-Padovan sequence

Given the *k*-Padovan sequence $P_k = \{P_k(n)\} = \{1,1,1,k+1,k+1,k^2+k+1,k^2+2k+1,k^3+k^2+2k+1,\ldots\}$ the sum sequence of the first *n* terms is $S_k = \{S_k(n)\} = \{1,\ 2,\ 3,\ k+4,\ 2k+5,\ k^2+3k+6,\ 2k^2+5k+7\ldots\}$

For n ≥ 4, the terms of this sum sequence verify the recurrence relation $S_k(n) = S_k(n-1) + k\,S_k(n-2) - (k-1)S_k(n-3) - S_k(n-4)$ with the initial conditions $S_k(0)=1,\ \ S_k(1)=2,\ \ S_k(2)=3,\ \ S_k(3)=k+4$ and we can prove it by induction.

Its characteristic equation is $r^4 - r^3 - k\,r^2 + (k-1)r + 1 = 0$ and its factorization is $(r-1)(r^3 - k\,r - 1) = 0$. Obviously, an integer root is *r = 1* and the factor $r^3 - k\,r - 1 = 0$ had been studied in the preceding subsection. Therefore, the general term of each of these sequences has the form $S_k(n) = C_1 + C_2 r_2^n + C_3 r_3^n + C_4 r_4^n$ with the preceding conditions. Each of the roots is calculated in the way indicated in the previous subsection. To find the constants C_i any mathematical program that allows the resolution of a 4 x 4 system must be used.
For *k = 3* the characteristic equation reduces to the third degree equation $r^3 - k\,r - 1 = 0$, studied in the first section.

Example 3. *Find the recurrence relation for the sums S_4*(n).

First characteristic root is *1* and the other three roots have been found in Exemple 2:

{2.11491, –0.254102, –1.86081}.

Then $S_4(n) = C_1 + C_2\,(2.11491)^n + C_3(-0.254102)^n + C_4(-1.86081)^n$.

For n = 0, 1, 2, 3 and the initial conditions $S_4(0) = 1,\ \ S_4(1) = 2\ \ S_4(2) = 3\ \ S_4(3) = 8$ we solve the linear system and find the recurrence relation $S_4(n) = 0.250005 + 0.722587(2.11491)^n + 0.169782(-0.254102)^n - 0.142374(1.86081)^n$

2.3 On the pell-padovan sequence

Taking into account that $r^3 - 2r - 1 = 0$ is the only equation that has an integer root (*r = –1*), the sequence P_2 constitutes a special case of the k-Padovan sequences. This sequence is for *k = 2: P_2 = {1, 1, 1, 3, 3, 7, 9, 17, 25, 43, 67, 111, 177, 289, 465, 755…}*: A066983 in the OEIS and is called the Pell-Padovan sequence [10].

The characteristic equation of the recurrence relation for $k = 2$ is $r^3 - 2r - 1 = 0$ and its solutions are $\left\{-1, \ \frac{1+\sqrt{5}}{2}, \ \frac{1-\sqrt{5}}{2}\right\}$.

From these characteristic roots it is possible to find Binet's formula to find the general term of the sequence. This must be of the form $P_2(n) = C_1(-1)^n + C_2\left(\frac{1+\sqrt{5}}{2}\right)^n + C_3\left(\frac{1-\sqrt{5}}{2}\right)^n$

For $n = 0, 1, 2$ the following system is obtained:

$$n = 0 \to P_2(0) = C_1 + C_2 + C_3 = 1$$

$$n = 1 \to P_2(1) = -C_1 + C_2\left(\frac{1+\sqrt{5}}{2}\right) + C_3\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$n = 2 \to P_2(2) = C_1 + C_2\left(\frac{1+\sqrt{5}}{2}\right)^2 + C_3\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1$$

The solution of this system is $C_1 = -1, \ C_2 = 1 - \frac{1}{\sqrt{5}}, \ C_3 = 1 + \frac{1}{\sqrt{5}}$ and so $P_2(n) = -(-1)^n + \left(1 - \frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(1 + \frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$

This last formula can be written as $P_2(n) = (-1)^{n+1} + 2\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}}\right)$ being $\alpha = \frac{1+\sqrt{5}}{2}$ the Golden Ratio and $\beta = \frac{1-\sqrt{5}}{2}$. So $P_2(n) = 2F(n) - \frac{1-(-1)^n}{2}$ where *F(n)* is the Fibonacci number of order *n*.

Moreover, $P_2(n)$ verify the recurrence relation $P_2(n+1) = P_2(n) + P_2(n-1) - \eta$ where $\eta = \frac{1-(-1)^n}{2}$.

Finally, the sum of the P_2-Padovan numbers is $S_2(n) = 2F(n) - \eta$.

2.4 k-Padovan numbers of negative indices

As with any sequence defined by a recurrence relation, *k*-Padovan numbers $P_k(n)$ for $n < 0$ can be defined by rewriting the recurrence relation as $P_k(n) = P_k(n+3) - k\,P_k(n+1)$. Then

$$P_k(-1) = P_k(2) - k\,P_k(0) = 1 - k$$

$$P_k(-2) = P_k(1) - k\,P_k(-1) = 1 - k(1-k) = k^2 - k + 1$$

$$P_k(-3) = P_k(0) - k\,P_k(-2) = 1 - k(k^2 - k + 1) = -k^3 + k^2 - k + 1$$

$$P_k(-4) = P_k(-1) - k\,P_k(-3) = k^4 - k^3 + k^2 - 2k + 1$$

etc.

As a similar way than in the classical Padovan numbers, it is

Table 2. Sequence of the generalized *k*-Padovan numbers

n	−2	−1	0	1	2	3	4	5
P_k(n)	$k^2 - k + 1$	$1 - k$	1	1	1	$k + 1$	$k + 1$	$k^2 - k + 1$

In the following theorem we study the generating function *p(k, x)* of the *k*-Padovan sequence so that $p(k,x)=\sum_{n=0}^{\infty} P_k(n)x^n$.

As we have seen in subsection 2.1, the characteristic equation associated with the definition of the *k*-Padovan numbers is $r^3 - k\,r - 1 = 0$. If we change r for $\frac{1}{x}$, the polynomial equation $1 - k\,x^2 - x^3 = 0$ results and this equation shows us the steps to follow to find the generating function of the *k*-Padovan sequence:

(1) The function $p(k,x)=\sum_{n=0}^{\infty} P_k(n)x^n$ is developed in a power series of x
(2) Multiply *p(k, x)* by $-k\,x^2$
(3) Multiply *p(k, x)* by $-x^2$
(4) Add these three equations
(5) On the left hand side, take out *p(k, x)* common factor
(6) On the right side, due to the recurrence relation of the definition of the *k*-Padovan numbers, all addends starting from the fourth are null.
(7) In this way, the rational function *p(k, x)* is obtained, which is the desired generating function.

With these indications, let us face the problem of finding the generating function of the *k*-Padovan numbers.

Theorem 5. Generating function

The generating function of the k-Padovan numbers is $p(k,x)=\dfrac{1+x+(1-k)x}{1-k\,x^2-x^3}$

Proof. We will follow the process indicated in the previous paragraph.

(1) $$p(k,x)=\sum_{n=0}^{\infty} P_k(n)x^n = P_k(0)+P_k(1)x+P_k(2)x^2+P_k(3)x^3+$$
$$+\cdots+P_k(n-1)x^{n-1}+P_k(n)x^n$$

(2) $$k\,x^2\,p(k,x)= \quad k\,P_k(0)x^2+k\,P_k(1)x^3+k\,P_k(2)x^4+$$
$$+\cdots+k\,P_k(n-3)x^{n-1}+k\,P_k(n-2)x^n+\cdots$$

(3) $$x^3\,p(k,x)= \quad P_k(0)x^3+P_k(1)x^4+P_k(2)x^5+P_k(3)x^6+$$
$$+\cdots+P_k(n-4)x^{n-1}+P_k(n-3)x^n+\cdots$$

(4,5, 6) $$(1-k\,x^2-x^3)\,p(k,x)=P_k(0)+P_k(1)x+\left(P_k(2)-k\,P_k(0)\right)x^2$$
$$=1+x+\left(1-k\right)x^2$$

(7) $$p(k,x)=\frac{1+x+(1-k)x^2}{1-k\,x^2-x^3}$$

The generating function is useful not only to find the terms of the corresponding numerical sequence, but also to solve other problems in a simple way. As an example, we indicate the following: if we do $x = \frac{1}{r}$,we can calculate the following sum: $\sum_{n=0}^{n} \frac{P_k(n)}{r^n} = \frac{1+\frac{k}{r}+\frac{k^2-k}{r^2}}{1-\frac{k}{r^2}-\frac{1}{r^3}} = \frac{r^3+k\,r^2+(k^2-k)r}{r^3-k\,r-1}$

As a particular case of the latter, if *k = 1* and *r = 2*, then $\sum_{n=0}^{\infty} \frac{P(n)}{2^n} = \frac{12}{5}$.

Furthermore, for any fixed value of *k*, this quotient tends to 1 as *n* increases.

A convolution of two numerical sequences (equal or different) is a mathematical operation of these sequences in such a way that a new sequence is produced. This means that the terms of each of the sequences are modified in accordance with the terms of the other. Graphically, it expresses how the "shape" of one function is modified by the other.

The convolution of the numerical sequences A = $\{a_n\}$and B = $\{b_n\}$ is defined as the new sequence $A \otimes B = \{a_n \otimes b_n\} = \sum_{j=0}^{n} a_j b_{n-j}$. If the convolution is a sequence with itself, it is usually called self-convolution.

Next, we study the self-convolution of the *k*-Padovan numbers.

2.5 Self-convolution of the *k*-Padovan sequence

According to the previous definition, the self-convolution of the *k*-Padovan sequence is $C_k(n) = \sum_{j=0}^{n} P_k(j) P_k(n-j)$.

The first terms of this sequence are $C_k(n) = \{1, 2, 3, 2k+4, 4k+5, 2k^2+6k+6, 5k^2+10k+7, 2k^3+8k^2+14k+8...\}$ This sequence verifies the recurrence relation $C_k(n) = 2k\,C_k(n-2) + 2C_k(n-3) - k^2 C_k(n-4) - 2k\,C_k(n-5) - C_k(n-6)$

For *k = 1*, the classical Padovan sequence is $P = \{1,1,1,2,2,3,4,5,7,9,12...\}$ and its self-convolution is the sequence $P \otimes P = \{1,2,3,6,9,14,22,32,48,70...\}$, A228364 in the OEIS [3].

For *k = 2*, the Pell-Padovan sequence is $P_2 = \{1,1,1,3,3,7,9,17,25,43...\}$ so its self-convolution is $P_2 \otimes P_2 = \{1,2,3,8,13,26,47,84,153,266...\}$ and it is not indexed in the OEIS.

The self-convolution of the *k*-Padovan sequences verify the recurrence relation $C(k,n) = 2k\,C(k,n-2) + 2C(k,n-3) - k^2 C(k,n-4) - 2k\,C(k,n-5) - C(k,n-6)$. Then, for the classical Padovan sequence it is $C(n) = 2C(n-2) + 2C(n-3) - C(n-4) - 2C(n-5) - C(n-6)$. And in similar form for the Pell-Padovan sequence.

3 Conclusion

We have recalled the Padovan numbers and proven some of their properties. Next, this concept has been generalized by means of a parameter *k* and some of the properties of the new numbers have been proven. The generating function of this new sequence has been found and has been particularized for the classical Padovan

sequence as well as for that the Pell-Padovan. We finish the article with a small foray into the convolution of the *k*-Padovan numbers that may be the subject of new research.
We keep doors open for future research on this topic.

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Competing Interests

Author has declared that no competing interests exist.

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