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Fixed Point Theorems for Kannan Interpolative, Riech Interpolative and Dass-Gupta Interpolative Rational type Contractions in A-Metric Spaces

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

(λ, α)-interpolative Kannan contraction, $(\lambda\ \alpha, \beta)$-interpolative Kannan contraction, $(\lambda, \alpha, \beta, \gamma)$-interpolative Riech contraction and $(\lambda\ \alpha, \beta)$-interpolative Dass-Gupta rational contraction are presented in this study. Furthermore, we prove a few fixed-point theorems for interpolative contractions in complete A-metric spaces. These theorems also extend and apply to an A-metric setting several interesting results from metric fixed-point theory.

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1 Introduction and Preliminaries

Fixed point theory is a fascinating field of research in analysis and topology. In 1922, Banach [1] proposed an important result that became known as the Banach contraction principle. Its relevance to metric fixed-point theory was investigated. Let $(\mathfrak{D}, d)$ be a full metric space and Υ a self-map on a nonempty set D. If there exists a constant $c \in [0, 1)$ such that.

$$d(\Upsilon\sigma, \Upsilon\varsigma) \leq c\, d(\sigma, \varsigma), \text{ for all } \sigma, \varsigma \in \mathfrak{D}, \quad (1)$$

then it possesses a unique fixed point in D. The Banach contraction principle was then widely generalized in the literature (see [2,3]). Both pure and applied mathematics make extensive use of it. Kannan [4] defined a new variation of this theory in 1968 and eliminated the continuity condition from it.

Theorem 1.1 (see [4]). *Let $(\mathfrak{D}, d)$ be a complete metric space and a self-map $\Upsilon: \mathfrak{D} \longrightarrow \mathfrak{D}$ be a Kannan contraction mapping, i.e.,*

$$d(\Upsilon\sigma, \Upsilon\varsigma) \leq k[d(\sigma, \Upsilon\sigma) + d(\varsigma, \Upsilon\varsigma)], \quad (2)$$

for all $\sigma, \varsigma \in \mathfrak{D}$, where $k \in [0, 1/2)$. Then, Υ admits a unique fixed point in $\mathfrak{D}$.

The idea of b-metric space, which is a generalization of the well-known Banach contraction mapping principle, was first presented by Bakhtin [5] in 1989. In 1993, Czerwik [6,7] expanded on the idea of b-metric space. "Kannan fixed-point theorem is the first significant variant of the outstanding result of Banach on the metric fixed-point theory" [1].

The concept of A-metric space was first developed by Abbas et al. [8] in 2015.

Definition 1.2 (see [1]) Let $\mathfrak{D}$ be a nonempty set. A mapping $A: \mathfrak{D}^n \to [0,+\infty)$ is called an A-metric on $\mathfrak{D}$ if and only if for all $\sigma_i, a \in \mathfrak{D}, i = 1,2,3,..n$: the following conditions hold:

(A1). $A(\sigma_1, \sigma_2, \sigma_3, ..., \sigma_{n-1}, \sigma_n) \geq 0$,
(A2). $A(\sigma_1, \sigma_2, \sigma_3, ..., \sigma_{n-1}, \sigma_n) = 0$ if and only if $\sigma_1 = \sigma_2 = \cdots = \sigma_{n-1} = \sigma_n$,
(A3). $A(\sigma_1, \sigma_2, \sigma_3, ..., \sigma_{n-1}, \sigma_n) \leq A(\sigma_1, \sigma_1, \sigma_1, ..., (\sigma_1)_{n-1}, a)$
$+A(\sigma_2, \sigma_2, \sigma_2, ..., (\sigma_2)_{n-1}, a)$
$+A(\sigma_3, \sigma_3, \sigma_3, ..., (\sigma_3)_{n-1}, a) + \cdots$
$+A(\sigma_{n-1}, \sigma_{n-1}, \sigma_{n-1}, ..., (\sigma_{n-1})_{n-1}, a)$
$+A(\sigma_n, \sigma_n, \sigma_n, ..., (\sigma_n)_{n-1}, a)]$

The pair $(\mathfrak{D}, A)$ is called an A-metric space.

The following is the intuitive geometric example for A-metric spaces.

Example 1.3 (see [8]) Let $\mathfrak{D} = [1,+\infty)$. Define $A: \mathfrak{D}^n \to [0,+\infty)$ by

$$A(\sigma_1, \sigma_2, \sigma_3, ..., \sigma_{n-1}, \sigma_n) = \sum_{i=1}^{n} \sum_{i<j} |\sigma_i - \sigma_j|$$

for all $\sigma_i \in \mathfrak{D}, i = 1,2, ... n$.

Example 1.4 (see [8]) Let $\mathfrak{D} = \mathbb{R}$. Define $A: \mathfrak{D}^n \to [0,+\infty)$ by

$$A(\sigma_1,\sigma_2,\sigma_3,\dots,\sigma_{n-1},\sigma_n) = \left|\sum_{i=n}^{2}\sigma_i-(n-1)\sigma_1\right| + \left|\sum_{i=n}^{3}\sigma_i-(n-2)\sigma_2\right| + \cdots + \left|\sum_{i=n}^{n-3}\sigma_i-3\sigma_{n-3}\right| + \left|\sum_{i=n}^{n-2}\sigma_i-2\sigma_{n-2}\right| + |\sigma_n-\sigma_{n-1}|$$

for all $\sigma_i \in \mathfrak{D}, i = 1,2,\dots n$.

Lemma 1.5 (see [8]) Let $(\mathfrak{D}, A)$ be an A-metric space. Then for all $\sigma, \varsigma \in \mathfrak{D}$,

$$A(\sigma,\sigma,\sigma,\sigma,\dots,(\sigma)_{n-1},\varsigma) = A(\varsigma,\varsigma,\varsigma,\varsigma\dots,(\varsigma)_{n-1},\sigma)$$

Lemma 1.6 (see [8]) Let $(\mathfrak{D}, A)$ be an A-metric space. Then for all $\sigma, \varsigma, z \in \mathfrak{D}$,

$$A(\sigma,\sigma,\sigma,\sigma,\dots,(\sigma)_{n-1},z) \le (n-1)A(\sigma,\sigma,\sigma,\sigma,\dots,(\sigma)_{n-1},\varsigma) + A(z,z,z,z,\dots,(z)_{n-1},\varsigma)$$

and

$$A(\sigma,\sigma,\sigma,\sigma,\dots,(\sigma)_{n-1},z) \le (n-1)A(\sigma,\sigma,\sigma,\sigma,\dots,(\sigma)_{n-1},\varsigma) + A(\varsigma,\varsigma,\varsigma,\varsigma,\dots,(\varsigma)_{n-1},z)$$

Lemma 1.7 (see [8]) Let $(\mathfrak{D}, A)$ be an A-metric space. Then $(\mathfrak{D}\times\mathfrak{D}, d_A)$ is an A-metric space on $\mathfrak{D}\times\mathfrak{D}$, where d_A is given by for all $\sigma_i, \varsigma_j \in \mathfrak{D}, i, j = 1,2,\dots,n$:

$$d_A\big((\sigma_1,\varsigma_1),(\sigma_2,\varsigma_2),(\sigma_3,\varsigma_3),\dots,(\sigma_n,\varsigma_n)\big) = A(\sigma_1,\sigma_2,\sigma_3,\dots,\sigma_n) + A(\varsigma_1,\varsigma_2,\varsigma_3,\dots,\varsigma_n).$$

Definition 1.7 (see [8]) Let $(\mathfrak{D}, A)$ be an A-metric space. Then

1. A sequence $\{\sigma_k\}$ is called convergent to σ in $(\mathfrak{D}, A)$ if $\lim_{k\to+\infty} A(\sigma_k,\sigma_k,\sigma_k,\sigma_k,..(\sigma_k)_{n-1},\sigma) = 0$. That is, for each $\epsilon \ge 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k \ge n_0$, $A(\sigma_k,\sigma_k,\sigma_k,\sigma_k,..(\sigma_k)_{n-1},\sigma) \le \epsilon$ and we write $\lim_{k\to+\infty} \sigma_k = \sigma$.

2. A sequence $\{\sigma_k\}$ is called Cauchy in $(\mathfrak{D}, A)$ if $\lim_{k,m\to+\infty} A(\sigma_k,\sigma_k,\sigma_k,\sigma_k,..(\sigma_k)_{n-1},\sigma_m) = 0$. That is, for each $\epsilon \ge 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k, m \ge n_0$, $A(\sigma_k,\sigma_k,\sigma_k,\sigma_k,..(\sigma_k)_{n-1},\sigma_m) \le \epsilon$.

3. $(\mathfrak{D}, A)$ is said to be complete if every Cauchy sequence in $(\mathfrak{D}, A)$ is a convergent.

Lemma 1.8 (see [8]) Let $(\mathfrak{D}, A)$ be an A-metric space. If the sequence $\{\sigma_k\}$ in $\mathfrak{D}$ converges to σ, then σ is unique.

Lemma 1.9 (see [8]) Every convergent sequence in A-metric space $(\mathfrak{D}, A)$ is a Cauchy sequence.

This study defines and discusses Kannan, Riech, and Dass-Gupta rational types interpolative contraction within the context of A-metric space. Furthermore, the concept of interpolation is used to establish a few popular fixed-point results. These theorems also extend and apply to an A-metric setting a number of interesting results from metric fixed-point theory (see [4, 9, 10, 11, 12, 13, 14]).

2 Main Results

We begin by defining the terms below.

Definition 2.1 Let $(\mathfrak{D}, A)$ be an A-metric space. A mapping $\Upsilon: \mathfrak{D} \longrightarrow \mathfrak{D}$ is called a (λ, α)-interpolative Kannan contraction, if there exist $\lambda \in [0,1), \alpha \in (0,1)$ such that

$$A\Big(\underbrace{\Upsilon\sigma,\Upsilon\sigma,\dots,\Upsilon\sigma}_{(n-1)\ times},\Upsilon\varsigma\Big) \le \lambda\left(A\Big(\underbrace{\sigma,\sigma,\dots,\sigma}_{(n-1)\ times},\Upsilon\sigma\Big)\right)^{\alpha}\left(A\Big(\underbrace{\varsigma,\varsigma,\dots,\varsigma}_{(n-1)\ times},\Upsilon\varsigma\Big)\right)^{1-\alpha} \quad (3)$$

for all $\sigma, \varsigma \in \mathfrak{D}$, with $\sigma \neq \varsigma$.

Definition 2.2 Let $(\mathfrak{D}, A)$ be an A-metric space. A mapping $\Upsilon: \mathfrak{D} \longrightarrow \mathfrak{D}$ is called a (λ, α, β)-interpolative Kannan contraction, if there exist $\lambda \in [0,1), \alpha, \beta \in (0,1)$, $\alpha + \beta < 1$ such that

$$A\left(\underbrace{\Upsilon\sigma, \Upsilon\sigma, \dots, \Upsilon\sigma}_{(n-1)\ times}, \Upsilon\varsigma\right) \leq \lambda\left(A\left(\underbrace{\sigma, \sigma, \dots, \sigma}_{(n-1)\ times}, \Upsilon\sigma\right)\right)^{\alpha}\left(A\left(\underbrace{\varsigma, \varsigma, \dots, \varsigma}_{(n-1)\ times}, \Upsilon\varsigma\right)\right)^{\beta} \tag{4}$$

for all $\sigma, \varsigma \in \mathfrak{D}$, with $\sigma \neq \varsigma$.

Definition 2.3 Let $(\mathfrak{D}, A)$ be an A-metric space. A mapping $\Upsilon: \mathfrak{D} \longrightarrow \mathfrak{D}$ is called a $(\lambda, \alpha, \beta, \gamma)$-interpolative Reich contraction, if there exist $\lambda \in [0,1), \alpha, \beta, \gamma \in (0,1)$, $\alpha + \beta + \gamma < 1$ such that

$$A\left(\underbrace{\Upsilon\sigma, \Upsilon\sigma, \dots, \Upsilon\sigma}_{(n-1)\ times}, \Upsilon\varsigma\right) \leq \lambda\left(A\left(\underbrace{\sigma, \sigma, \dots, \sigma}_{(n-1)\ times}, \varsigma\right)\right)^{\alpha}\left(A\left(\underbrace{\sigma, \sigma, \dots, \sigma}_{(n-1)\ times}, \Upsilon\sigma\right)\right)^{\beta}\left(A\left(\underbrace{\varsigma, \varsigma, \dots, \varsigma}_{(n-1)\ times}, \Upsilon\varsigma\right)\right)^{\gamma} \tag{5}$$

for all $\sigma, \varsigma \in \mathfrak{D}$, with $\sigma \neq \varsigma$.

Definition 2.4 Let $(\mathfrak{D}, A)$ be an A-metric space. A mapping $\Upsilon: \mathfrak{D} \longrightarrow \mathfrak{D}$ is (λ, α, β)-interpolative Dass-Gupta rational contraction, if there exist $\lambda \in [0,1), \alpha, \beta \in (0,1)$, $\alpha + \beta < 1$ such that

$$A\left(\underbrace{\Upsilon\sigma, \Upsilon\sigma, \dots, \Upsilon\sigma}_{(n-1)\ times}, \Upsilon\varsigma\right) \leq \lambda\left(A\left(\underbrace{\sigma, \sigma, \dots, \sigma}_{(n-1)\ times}, \varsigma\right)\right)^{\alpha}\left(\frac{\left[1 + A\left(\underbrace{\sigma, \sigma, \dots, \sigma}_{(n-1)\ times}, \Upsilon\sigma\right)\right] A\left(\underbrace{\varsigma, \varsigma, \dots, \varsigma}_{(n-1)\ times}, \Upsilon\varsigma\right)}{1 + A\left(\underbrace{\sigma, \sigma, \dots, \sigma}_{(n-1)\ times}, \varsigma\right)}\right)^{\beta} \tag{6}$$

for all $\sigma, \varsigma \in \mathfrak{D}$, with $\sigma \neq \varsigma$.

Our first main result as follows.

Theorem 2.5 *Let $(\mathfrak{D}, A)$ be a complete A-metric space. Let $\Upsilon: \mathfrak{D} \longrightarrow \mathfrak{D}$ be a (λ, α)-interpolative Kannan contraction. Then, Υ admits a unique fixed point in $\mathfrak{D}$.*

Proof. Let $\sigma_0 \in \mathfrak{D}$ be initial point. Define $\sigma_{n+1} = \Upsilon\sigma_n, \forall\, n \in \mathbb{N}$. Obviously, if $\exists\, n_0 \in \mathbb{N}$ for which $\sigma_{n_0+1} = \sigma_{n_0}$, then $\Upsilon\sigma_{n_0} = \sigma_{n_0}$, and the proof is finished. Thus, we suppose that $\sigma_{n+1} \neq \sigma_n$ for each $n \in \mathbb{N}$. Thus, by (3), we have

$$\begin{aligned} A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma_{n+1}\right) &= A\left(\underbrace{\Upsilon\sigma_{n-1}, \Upsilon\sigma_{n-1}, \dots, \Upsilon\sigma_{n-1}}_{(n-1)\ times}, \Upsilon\sigma_n\right) \\ &\leq \lambda\left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \Upsilon\sigma_{n-1}\right)\right)^{\alpha}\left(A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \Upsilon\sigma_n\right)\right)^{1-\alpha} \\ &= \lambda\left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\right)\right)^{\alpha}\left(A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma_{n+1}\right)\right)^{1-\alpha} \end{aligned}$$

The last inequality gives

$$\left(A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\ times},\sigma_{n+1}\right)\right)^{\alpha}\le\lambda\left(A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\ times},\sigma_n\right)\right)^{\alpha} \tag{7}$$

Since $\alpha<1$, we have

$$\begin{aligned}A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\ times},\sigma_{n+1}\right)&\le\lambda^{\frac{1}{\alpha}}A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\ times},\sigma_n\right)\\&\le\lambda A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\ times},\sigma_n\right)\end{aligned}$$

and then

$$\begin{aligned}A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\ times},\sigma_{n+1}\right)&\le\lambda A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\ times},\sigma_n\right)\\&\le\lambda^2A\left(\underbrace{\sigma_{n-2},\sigma_{n-2},\dots,\sigma_{n-2}}_{(n-1)\ times},\sigma_{n-1}\right)\\&\le\cdots\le\lambda^nA\left(\underbrace{\sigma_0,\sigma_0,\dots,\sigma_0}_{(n-1)\ times},\sigma_1\right)\end{aligned} \tag{8}$$

For all $n,m\in\mathbb{N}$ and $n<m$, we have

$$\begin{aligned}A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\ times},\sigma_m\right)&\le(n-1)A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\ times},\sigma_{n+1}\right)+A\left(\underbrace{\sigma_m,\sigma_m,\dots,\sigma_m}_{(n-1)\ times},\sigma_{n+1}\right)\\&=(n-1)A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\ times},\sigma_{n+1}\right)+A\left(\underbrace{\sigma_{n+1},\sigma_{n+1},\dots,\sigma_{n+1}}_{(n-1)\ times},\sigma_m\right)\\&\le(n-1)A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\ times},\sigma_{n+1}\right)+(n-1)A\left(\underbrace{\sigma_{n+1},\sigma_{n+1},\dots,\sigma_{n+1}}_{(n-1)\ times},\sigma_{n+2}\right)\\&+A\left(\underbrace{\sigma_m,\sigma_m,\dots,\sigma_m}_{(n-1)\ times},\sigma_{n+2}\right)\\&\le(n-1)A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\ times},\sigma_{n+1}\right)+(n-1)A\left(\underbrace{\sigma_{n+1},\sigma_{n+1},\dots,\sigma_{n+1}}_{(n-1)\ times},\sigma_{n+2}\right)\\&+A\left(\underbrace{\sigma_{n+2},\sigma_{n+2},\dots,\sigma_{n+2}}_{(n-1)\ times},\sigma_m\right)\\&\le(n-1)A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\ times},\sigma_{n+1}\right)+(n-1)A\left(\underbrace{\sigma_{n+1},\sigma_{n+1},\dots,\sigma_{n+1}}_{(n-1)\ times},\sigma_{n+2}\right)+\cdots\\&+(n-1)A\left(\underbrace{\sigma_{m-2},\sigma_{m-2},\dots,\sigma_{m-2}}_{(n-1)\ times},\sigma_{m-1}\right)+A\left(\underbrace{\sigma_{m-1},\sigma_{m-1},\dots,\sigma_{m-1}}_{(n-1)\ times},\sigma_m\right)\\&\le(n-1)[\lambda^n+\lambda^{n+1}+\cdots+\lambda^{m-2}]A\left(\underbrace{\sigma_0,\sigma_0,\dots,\sigma_0}_{(n-1)\ times},\sigma_1\right)+\lambda^{m-2}A\left(\underbrace{\sigma_0,\sigma_0,\dots,\sigma_0}_{(n-1)\ times},\sigma_1\right)\\&\le(n-1)[\lambda^n+\lambda^{n+1}+\cdots+\lambda^{m-2}+\cdots]A\left(\underbrace{\sigma_0,\sigma_0,\dots,\sigma_0}_{(n-1)\ times},\sigma_1\right)\end{aligned}$$

$$\leq (n-1)[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-2} + \cdots] A\left(\underbrace{\sigma_0, \sigma_0, \dots, \sigma_0}_{(n-1)\ times}, \sigma_1\right)$$
$$\leq (n-1)\frac{\lambda^n}{1-\lambda} A\left(\underbrace{\sigma_0, \sigma_0, \dots, \sigma_0}_{(n-1)\ times}, \sigma_1\right)$$

Letting $n, m \longrightarrow \infty$, we obtain

$$\lim_{n,m\to\infty} A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma_m\right) = 0 \tag{9}$$

Thus, the sequence $\{\sigma_n\}$ is Cauchy in the complete A-metric space $(\mathfrak{D}, A)$. So, there is some $\sigma^* \in \mathfrak{D}$. So that

$$\lim_{n\to\infty} A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma^*\right) = 0; \tag{10}$$

that is, $\sigma_n \longrightarrow \sigma^*$ as $n \longrightarrow \infty$. Now, we will prove that $\sigma^* \in \mathfrak{D}$ is a fixed point of Υ. By (3) and condition (A3), we get

$$A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1)\ times}, \Upsilon\sigma^*\right) \leq (n-1)A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1)\ times}, \sigma_{n+1}\right) + A\left(\underbrace{\Upsilon\sigma^*, \Upsilon\sigma^*, \dots, \Upsilon\sigma^*}_{(n-1)\ times}, \sigma_{n+1}\right)$$
$$= (n-1)A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1)\ times}, \sigma_{n+1}\right) + A\left(\underbrace{\Upsilon\sigma^*, \Upsilon\sigma^*, \dots, \Upsilon\sigma^*}_{(n-1)\ times}, \Upsilon\sigma_n\right)$$
$$\leq (n-1)A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1)\ times}, \sigma_{n+1}\right) + \lambda\left(\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1)\ times}, \Upsilon\sigma^*\right)\right)^{\alpha}\left(\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \Upsilon\sigma_n\right)\right)^{1-\alpha}$$
$$\leq (n-1)A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1)\ times}, \sigma_{n+1}\right) + \lambda\left(\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1)\ times}, \Upsilon\sigma^*\right)\right)^{\alpha}\left(\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma_{n+1}\right)\right)^{1-\alpha} \tag{11}$$

Taking the limit as $n \longrightarrow \infty$, we obtain that

$$A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1)\ times}, \Upsilon\sigma^*\right) = 0 \tag{12}$$

This yields that $\sigma^* = \Upsilon\sigma^*$. Now, we prove the uniqueness of σ^*. Let ς^* be another fixed point of Υ in $\mathfrak{D}$, then $\Upsilon\varsigma^* = \varsigma^*$. Now, by (3), we have

$$A\left(\underbrace{\sigma^*, \sigma^*, , \dots, \sigma^*}_{(n-1)\ times}, \varsigma^*\right) = A\left(\underbrace{\Upsilon\sigma^*, \Upsilon\sigma^*, \dots, \Upsilon\sigma^*}_{(n-1)\ times}, \Upsilon\varsigma^*\right)$$
$$\leq \lambda\left(A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1)\ times}, \Upsilon\sigma^*\right)\right)^{\alpha}\left(A\left(\underbrace{\varsigma^*, \varsigma^*, \dots, \varsigma^*}_{(n-1)\ times}, \Upsilon\varsigma^*\right)\right)^{1-\alpha} = 0 \tag{13}$$

This yields that $\sigma^* = \varsigma^*$. It completes the proof.

Theorem 2.6 *Let $(\mathfrak{D}, A)$ be a complete A-metric space. Let $\Upsilon: \mathfrak{D} \longrightarrow \mathfrak{D}$ be a (λ, α, β)-interpolative Kannan contraction. Then, Υ admits a unique fixed point in $\mathfrak{D}$.*

Proof Following the steps of proof of Theorem 2.5, we construct the sequence $\{\sigma_n\}$ by iterating

$$\sigma_{n+1} = \Upsilon\sigma_n, \forall\, n \in \mathbb{N},$$

where $\sigma_0 \in \mathfrak{D}$ is arbitrary starting point. Then, by (4), we have

$$\begin{aligned} A\Big(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma_{n+1}\Big) &= A\Big(\underbrace{\Upsilon\sigma_{n-1}, \Upsilon\sigma_{n-1}, \dots, \Upsilon\sigma_{n-1}}_{(n-1)\ times}, \Upsilon\sigma_n\Big) \\ &\leq \lambda\left(A\Big(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \Upsilon\sigma_{n-1}\Big)\right)^{\alpha}\left(A\Big(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \Upsilon\sigma_n\Big)\right)^{\beta} \\ &= \lambda\left(A\Big(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\Big)\right)^{\alpha}\left(A\Big(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma_{n+1}\Big)\right)^{\beta} \end{aligned}$$

Since $\alpha < 1 - \beta$, the last inequality gives

$$\begin{aligned} \left(A\Big(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma_{n+1}\Big)\right)^{1-\beta} &\leq \lambda\left(A\Big(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\Big)\right)^{\alpha} \\ &\leq \lambda\left(A\Big(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\Big)\right)^{1-\beta} \end{aligned} \tag{14}$$

$$\begin{aligned} A\Big(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma_{n+1}\Big) &\leq \lambda^{\frac{1}{1-\beta}} A\Big(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\Big) \\ &\leq \lambda A\Big(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\Big) \end{aligned}$$

and then

$$\begin{aligned} A\Big(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma_{n+1}\Big) &\leq \lambda A\Big(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\Big) \\ &\leq \lambda^2 A\Big(\underbrace{\sigma_{n-2}, \sigma_{n-2}, \dots, \sigma_{n-2}}_{(n-1)\ times}, \sigma_{n-1}\Big) \\ &\leq \cdots \leq \lambda^n A\Big(\underbrace{\sigma_0, \sigma_0, \dots, \sigma_0}_{(n-1)\ times}, \sigma_1\Big) \end{aligned} \tag{15}$$

A fixed-point $\sigma^* \in \mathfrak{D}$ is produced by the classical process, as was previously shown in the proof of Theorem 2.5. Now, we prove the uniqueness of σ^*. Let ς^* be another fixed point of Υ in $\mathfrak{D}$, then $\Upsilon\varsigma^* = \varsigma^*$. Now, by (4), we have

$$\begin{aligned} A\Big(\underbrace{\sigma^*, \sigma^*,, \dots, \sigma^*}_{(n-1)\ times}, \varsigma^*\Big) &= A\Big(\underbrace{\Upsilon\sigma^*, \Upsilon\sigma^*, \dots, \Upsilon\sigma^*}_{(n-1)\ times}, \Upsilon\varsigma^*\Big) \\ &\leq \lambda\left(A\Big(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1)\ times}, \Upsilon\sigma^*\Big)\right)^{\alpha}\left(A\Big(\underbrace{\varsigma^*, \varsigma^*, \dots, \varsigma^*}_{(n-1)\ times}, \Upsilon\varsigma^*\Big)\right)^{\beta} = 0 \end{aligned} \tag{16}$$

This yields that $\sigma^* = \varsigma^*$. This completes the proof.

Theorem 2.7 *Let $(\mathfrak{D}, A)$ be a complete A-metric space. Let $\Upsilon: \mathfrak{D} \longrightarrow \mathfrak{D}$ be a $(\lambda, \alpha, \beta, \gamma)$-interpolative Reich contraction. Then, Υ admits a unique fixed point in $\mathfrak{D}$.*

Proof Following the steps of proof of Theorem 2.5, we construct the sequence $\{\sigma_n\}$ by iterating

$$\sigma_{n+1} = \Upsilon\sigma_n, \forall\, n \in \mathbb{N},$$

where $\sigma_0 \in D$ is arbitrary starting point. Then, by (5), we have

$$\begin{aligned} A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma_{n+1}\right) &= A\left(\underbrace{\Upsilon\sigma_{n-1}, \Upsilon\sigma_{n-1}, \dots, \Upsilon\sigma_{n-1}}_{(n-1)\ times}, \Upsilon\sigma_n\right) \\ &\le \lambda\left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\right)\right)^{\alpha}\left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \Upsilon\sigma_{n-1}\right)\right)^{\beta}\left(A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \Upsilon\sigma_n\right)\right)^{\gamma} \\ &\le \lambda\left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\right)\right)^{\alpha}\left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\right)\right)^{\beta}\left(A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma_{n+1}\right)\right)^{\gamma} \end{aligned}$$

Since $\alpha + \beta < 1 - \gamma$, the last inequality gives

$$\begin{aligned} \left(A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma_{n+1}\right)\right)^{1-\gamma} &\le \lambda\left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\right)\right)^{\alpha+\beta} \\ &\le \lambda\left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\right)\right)^{1-\gamma} \end{aligned} \tag{18}$$

$$\begin{aligned} A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma_{n+1}\right) &\le \lambda^{\frac{1}{1-\gamma}} A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\right) \\ &\le \lambda A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\right) \end{aligned}$$

and then

$$\begin{aligned} A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1)\ times}, \sigma_{n+1}\right) &\le \lambda A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1)\ times}, \sigma_n\right) \\ &\le \lambda^2 A\left(\underbrace{\sigma_{n-2}, \sigma_{n-2}, \dots, \sigma_{n-2}}_{(n-1)\ times}, \sigma_{n-1}\right) \\ &\le \cdots \le \lambda^n A\left(\underbrace{\sigma_0, \sigma_0, \dots, \sigma_0}_{(n-1)\ times}, \sigma_1\right) \end{aligned} \tag{19}$$

A fixed-point $\sigma^* \in \mathfrak{D}$ is produced by the classical process, as was previously explained in the proof of Theorem 2.5. We now demonstrate σ^*'s uniqueness. If ς^* be another fixed point of Υ in $\mathfrak{D}$, then $\Upsilon\varsigma^* = \varsigma^*$. As of (5), we now have

$$A\left(\underbrace{\sigma^*, \sigma^*, , \dots, \sigma^*}_{(n-1)\ times}, \varsigma^*\right) = A\left(\underbrace{\Upsilon\sigma^*, \Upsilon\sigma^*, \dots, \Upsilon\sigma^*}_{(n-1)\ times}, \Upsilon\varsigma^*\right)$$

$$\leq \lambda\left(A\left(\underbrace{\sigma^*,\sigma^*,,\dots,\sigma^*}_{(n-1)\,times},\varsigma^*\right)\right)^{\alpha}\left(A\left(\underbrace{\sigma^*,\sigma^*,\dots,\sigma^*}_{(n-1)\,times},\Upsilon\sigma^*\right)\right)^{\beta}\left(A\left(\underbrace{\varsigma^*,\varsigma^*,\dots,\varsigma^*}_{(n-1)\,times},\Upsilon\varsigma^*\right)\right)^{\gamma}=0 \quad (20)$$

This yields that $\sigma^* = \varsigma^*$. This completes the proof.

Theorem 2.8 *Let $(\mathfrak{D}, A)$ be a complete A-metric space. Let $\Upsilon: \mathfrak{D} \longrightarrow \mathfrak{D}$ be a (λ, α, β)-interpolative Dass-Gupta rational contraction. Then, Υ admits a unique fixed point in $\mathfrak{D}$.*

Proof Following the steps of proof of Theorem 2.5, we construct the sequence $\{\sigma_n\}$ by iterating

$$\sigma_{n+1} = \Upsilon\sigma_n, \forall\, n \in \mathbb{N},$$

where $\sigma_0 \in \mathfrak{D}$ is arbitrary starting point. Then, by (6), we have

$$\begin{aligned}
A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\,times},\sigma_{n+1}\right) &= A\left(\underbrace{\Upsilon\sigma_{n-1},\Upsilon\sigma_{n-1},\dots,\Upsilon\sigma_{n-1}}_{(n-1)\,times},\Upsilon\sigma_n\right)\\
&\leq \lambda\left(A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\,times},\sigma_n\right)\right)^{\alpha}\left(\frac{\left[1+A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\,times},\Upsilon\sigma_{n-1}\right)\right]A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\,times},\Upsilon\sigma_n\right)}{1+A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\,times},\sigma_n\right)}\right)^{\beta}\\
&\leq \lambda\left(A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\,times},\sigma_n\right)\right)^{\alpha}\left(\frac{\left[1+A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\,times},\sigma_n\right)\right]A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\,times},\sigma_{n+1}\right)}{1+A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\,times},\sigma_n\right)}\right)^{\beta}\\
&= \lambda\left(A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\,times},\sigma_n\right)\right)^{\alpha}\left(A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\,times},\sigma_{n+1}\right)\right)^{\beta}
\end{aligned}$$

Since $\alpha + \beta < 1$, the last inequality gives

$$\begin{aligned}
\left(A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\,times},\sigma_{n+1}\right)\right)^{1-\beta} &\leq \lambda\left(A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\,times},\sigma_n\right)\right)^{\alpha}\\
&\leq \lambda\left(A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\,times},\sigma_n\right)\right)^{1-\beta} \quad (21)
\end{aligned}$$

$$\begin{aligned}
A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\,times},\sigma_{n+1}\right) &\leq \lambda^{\frac{1}{1-\beta}}A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\,times},\sigma_n\right)\\
&\leq \lambda A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\,times},\sigma_n\right)
\end{aligned}$$

and then

$$\begin{aligned}
A\left(\underbrace{\sigma_n,\sigma_n,\dots,\sigma_n}_{(n-1)\,times},\sigma_{n+1}\right) &\leq \lambda A\left(\underbrace{\sigma_{n-1},\sigma_{n-1},\dots,\sigma_{n-1}}_{(n-1)\,times},\sigma_n\right)\\
&\leq \lambda^2 A\left(\underbrace{\sigma_{n-2},\sigma_{n-2},\dots,\sigma_{n-2}}_{(n-1)\,times},\sigma_{n-1}\right)
\end{aligned}$$

$$\leq \cdots \leq \lambda^n A\Big(\underbrace{\sigma_0, \sigma_0, \dots, \sigma_0}_{(n-1)\ times}, \sigma_1\Big) \tag{22}$$

A fixed-point $\sigma^* \in D$ is produced by the classical process, as was previously shown in the proof of Theorem 2.5. Now, we prove the uniqueness of σ^*. Let ς^* be another fixed point of Υ in D, then $\Upsilon\varsigma^* = \varsigma^*$. Now, by (6), we have

$$A\Big(\underbrace{\sigma^*, \sigma^*,, \dots, \sigma^*}_{(n-1)\ times}, \varsigma^*\Big) = A\Big(\underbrace{\Upsilon\sigma^*, \Upsilon\sigma^*, \dots, \Upsilon\sigma^*}_{(n-1)\ times}, \Upsilon\varsigma^*\Big)$$

$$\leq \lambda\left(A\Big(\underbrace{\sigma^*, \sigma^*,, \dots, \sigma^*}_{(n-1)\ times}, \varsigma^*\Big)\right)^{\alpha}\left(\frac{\Big[1+A\Big(\underbrace{\sigma^*,\sigma^*,\dots,\sigma^*}_{(n-1)\ times},\Upsilon\sigma^*\Big)\Big]A\Big(\underbrace{\varsigma^*,\varsigma^*,\dots,\varsigma^*}_{(n-1)\ times},\Upsilon\varsigma^*\Big)}{1+A\Big(\underbrace{\sigma^*,\sigma^*,,\dots,\sigma^*}_{(n-1)\ times},\varsigma^*\Big)}\right)^{\beta}$$

$$= 0 \tag{23}$$

This yields that $\sigma^* = \varsigma^*$. This completes the proof.

4 Conclusion

In this work, we presented the notion of (λ, α)-interpolative Kannan contraction, $(\lambda\ \alpha, \beta)$-interpolative Kannan contraction and $(\lambda, \alpha, \beta, \gamma)$ -interpolative Riech contraction and $(\lambda\ \alpha, \beta)$ -interpolative Dass-Gupta rational contraction. We also demonstrated the existence of fixed points for self-mapping. All of these concepts were introduced using the new framework of A-metric spaces.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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