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A Detailed Study on Matrix Sequence of Generalized Narayana Numbers

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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ABSTRACT

In this paper, we introduce and investigate the generalized Narayana matrix sequence and we deal with, in detail, three special cases of this sequence which we call them Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences. We present Binet's formulas, generating functions, and the summation formulas for these sequences. We present the proofs to indicate how these sum formulas, in general, were discovered. Of course, all the listed sum formulas may be proved by induction, but that method of proof gives no clue about their discovery. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there always exist interrelation between generalized Narayana, Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences.

Keywords: Narayana numbers; Narayana sequence; Narayana matrix sequence; Narayana-Lucas matrix sequence.

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1 INTRODUCTION AND PRELIMINARIES

In this paper we define generalized Narayana matrix sequence and investigate their properties. First, we present some information on generalized Narayana sequence and its special cases.

The Narayana numbers was introduced by the Indian mathematician Narayana in the 14th century, while studying the problem of a herd of cows and calves, see [1,2] for details. Narayana's cows problem is a problem similar to the Fibonacci's rabbit problem which can be given as follows: A cow produces one calf every year and beginning in its fourth year, each calf produces one calf at the beginning of each year. How many calves are there altogether after 20 years? This problem can be solved in the same way that Fibonacci solved its problem about rabbits (see [3]). If n is the year, then the Narayana problem can be modelled by the recurrence $N_{n+3} = N_{n+2} + N_n$, with $n \geq 0$, $N_0 = 0, N_1 = 1, N_2 = 1$, see [1]. The first few terms are $0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28...$, (the sequence A000930 in [4]). This sequence is called Narayana sequence. Recently, there has been considerable interest in the Narayana sequence and its generalizations (for more details, see [1,5,6,7,8,9,10,11,12 and the references given therein]).

A generalized Narayana sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-1} + V_{n-3} \tag{1.1}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero. The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-2)} + V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.For more details on generalized Narayana numbers, see [11]. Binet formula of generalized Narayana numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.2}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{1.3}$$

Here, α, β and γ are the roots of the cubic equation $x^3 - x^2 - 1 = 0$. Moreover

$$\begin{aligned}
\alpha &= \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\beta &= \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\gamma &= \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3},
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha + \beta + \gamma &= 1, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= 0, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

The first few generalized Narayana numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Narayana numbers

n	V_n	V_{-n}
0	V_0	...
1	V_1	$V_2 - V_1$
2	V_2	$V_1 - V_0$
3	$V_2 + V_0$	$-V_2 + V_1 + V_0$
4	$V_2 + V_1 + V_0$	$V_2 - 2V_1 + V_0$
5	$2V_2 + V_1 + V_0$	$V_2 - 2V_0$
6	$3V_2 + V_1 + 2V_0$	$-2V_2 + 3V_1$
7	$4V_2 + 2V_1 + 3V_0$	$-2V_1 + 3V_0$
8	$6V_2 + 3V_1 + 4V_0$	$3V_2 - 3V_1 - 2V_0$
9	$9V_2 + 4V_1 + 6V_0$	$-2V_2 + 5V_1 - 3V_0$
10	$13V_2 + 6V_1 + 9V_0$	$-3V_2 + V_1 + 5V_0$
11	$19V_2 + 9V_1 + 13V_0$	$5V_2 - 8V_1 + V_0$
12	$28V_2 + 13V_1 + 19V_0$	$V_2 + 4V_1 - 8V_0$
13	$41V_2 + 19V_1 + 28V_0$	$-8V_2 + 9V_1 + 4V_0$

Now we define three special case of the sequence $\{V_n\}$. Narayana sequence $\{N_n\}_{n\geq 0}$, Narayana-Lucas sequence $\{U_n\}_{n\geq 0}$ and Narayana-Perrin sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} N_{n+3} &= N_{n+2} + N_n, \quad N_0 = 0, N_1 = 1, N_2 = 1, \\ U_{n+3} &= U_{n+2} + U_n, \quad U_0 = 3, U_1 = 1, U_2 = 1, \\ H_{n+3} &= H_{n+2} + H_n, \quad H_0 = 3, H_1 = 0, H_2 = 2, \end{aligned}$$

The sequences $\{N_n\}_{n\geq 0}, \{U_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} N_{-n} &= -N_{-(n-2)} + N_{-(n-3)} \\ U_{-n} &= -U_{-(n-2)} + U_{-(n-3)} \\ H_{-n} &= -H_{-(n-2)} + H_{-(n-3)} \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively.

Note that N_n is the sequence A000930 in [4] associated with the Narayana's cows sequence and the sequence A078012 in [4] associated with the expansion of $(1-x)/(1-x-x^3)$ and U_n is the sequence A001609 in [4].

Next, we present the first few values of the Narayana, Narayana-Lucas and Narayana-Perrin numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
N_n	0	1	1	1	2	3	4	6	9	13	19	28	41	60
N_{-n}		0	1	0	-1	1	1	-2	0	3	-2	-3	5	1
U_n	3	1	1	4	5	6	10	15	21	31	46	67	98	144
U_{-n}		0	-2	3	2	-5	1	7	-6	-6	13	0	-19	13
H_n	3	0	2	5	5	7	12	17	24	36	53	77	113	166
H_{-n}		2	-3	1	5	-4	-4	9	0	-13	9	13	-22	-4

For all integers $n,$ Narayana, Narayana-Lucas and Narayana-Perrin numbers (using initial conditions in (1.3)) can be expressed using Binet's formulas as

$$\begin{array}{rcl} N_n &=& \dfrac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\dfrac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\dfrac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\ U_n &=& \alpha^n+\beta^n+\gamma^n, \\ H_n &=& \dfrac{(3+2\alpha)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)}+\dfrac{(3+2\beta)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)}+\dfrac{(3+2\gamma)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)}, \end{array}$$

respectively.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 1.1. *Suppose that $f_{V_n}(x)=\sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized Narayana sequence $\{V_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n=\frac{V_0+(V_1-V_0)x+(V_2-V_1)x^2}{1-x-x^3}. \tag{1.4}$$

The previous lemma gives the following results as particular examples.

Corollary 1.2. *Generated functions of Narayana, Narayana-Lucas and Narayana-Perrin numbers are*

$$\begin{array}{rcl} \sum_{n=0}^{\infty} N_n x^n &=& \dfrac{x}{1-x-x^3}, \\ \sum_{n=0}^{\infty} U_n x^n &=& \dfrac{3-2x}{1-x-x^3}, \\ \sum_{n=0}^{\infty} H_n x^n &=& \dfrac{3-3x+2x^2}{1-x-x^3}, \end{array}$$

respectively.

2 THE MATRIX SEQUENCES OF NARAYANA AND NARAYANA-LUCAS NUMBERS

Recently, there have been so many studies of the sequences of numbers in the literature that concern about subsequences of the Horadam (generalized Fibonacci) numbers and generalized Tribonacci numbers such as Fibonacci, Lucas, Pell and Jacobsthal numbers; third-order Pell, third-order Pell-Lucas, Padovan, Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas numbers. The sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. On the other hand, the matrix sequences have taken so much interest for different type of numbers. We present some works on matrix sequences of the numbers in the following Table 3.

Table 3. A few special study on the matrix sequences of the numbers

Name of sequence	work on the matrix sequences of the numbers
Generalized Fibonacci	[13,14,15,16,17,18,19,20,21]
Generalized Tribonacci	[22,23,24,25,26,27,28]
Generalized Tetranacci	[29]

In this section we define generalized Narayana matrix sequence and investigate its properties.

Definition 2.1. For any integer $n \geq 0$, the generalized Narayana matrix $(\mathcal{V}_n)$ is defined by

$$\mathcal{V}_n = \mathcal{V}_{n-1} + \mathcal{V}_{n-3} \tag{2.1}$$

with initial conditions

$$\begin{aligned}
\mathcal{V}_0 &= \begin{pmatrix} V_1 & V_2 - V_1 & V_0 \\ V_0 & V_1 - V_0 & V_2 - V_1 \\ V_2 - V_1 & V_0 + V_1 - V_2 & V_1 - V_0 \end{pmatrix}, \\
\mathcal{V}_1 &= \begin{pmatrix} V_2 & V_0 & V_1 \\ V_1 & V_2 - V_1 & V_0 \\ V_0 & V_1 - V_0 & V_2 - V_1 \end{pmatrix}, \\
\mathcal{V}_2 &= \begin{pmatrix} V_0 + V_2 & V_1 & V_2 \\ V_2 & V_0 & V_1 \\ V_1 & V_2 - V_1 & V_0 \end{pmatrix}.
\end{aligned}$$

The sequence $\{\mathcal{V}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{V}_{-n} = -\mathcal{V}_{-(n-2)} + \mathcal{V}_{-(n-3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.1) holds for all integers n.

Three special cases of generalized Narayana matrix sequence (take $V_n = N_n, V_n = U_n, V_n = H_n$, respectively) can be defined as follows.

Definition 2.2. For any integer $n \geq 0$, the Narayana matrix $(\mathcal{N}_n)$ and Narayana-Lucas matrix $(\mathcal{U}_n)$ and Narayana-Perrin matrix $(\mathcal{H}_n)$ are defined by

$$\begin{aligned}
\mathcal{N}_n &= \mathcal{N}_{n-1} + \mathcal{N}_{n-3}, \\
\mathcal{U}_n &= \mathcal{U}_{n-1} + \mathcal{U}_{n-3}, \\
\mathcal{H}_n &= \mathcal{H}_{n-1} + \mathcal{H}_{n-3},
\end{aligned}$$

respectively, with initial conditions

$$\begin{aligned}
\mathcal{N}_0 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{N}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{N}_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\
\mathcal{U}_0 &= \begin{pmatrix} 1 & 0 & 3 \\ 3 & -2 & 0 \\ 0 & 3 & -2 \end{pmatrix}, \mathcal{U}_1 = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 3 \\ 3 & -2 & 0 \end{pmatrix}, \mathcal{U}_2 = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 0 & 3 \end{pmatrix}, \\
\mathcal{H}_0 &= \begin{pmatrix} 0 & 2 & 3 \\ 3 & -3 & 2 \\ 2 & 1 & -3 \end{pmatrix}, \mathcal{H}_1 = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \\ 3 & -3 & 2 \end{pmatrix}, \mathcal{H}_2 = \begin{pmatrix} 5 & 0 & 2 \\ 2 & 3 & 0 \\ 0 & 2 & 3 \end{pmatrix}
\end{aligned}$$

The sequences $\{\mathcal{N}_n\}_{n\geq 0}$, $\{\mathcal{U}_n\}_{n\geq 0}$ and $\{\mathcal{H}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
\mathcal{N}_{-n} &= -\mathcal{N}_{-(n-2)} + \mathcal{N}_{-(n-3)}, \\
\mathcal{U}_{-n} &= -\mathcal{U}_{-(n-2)} + \mathcal{U}_{-(n-3)}, \\
\mathcal{H}_{-n} &= -\mathcal{H}_{-(n-2)} + \mathcal{H}_{-(n-3)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively.

The Narayana matrix $(\mathcal{N}_n)$ and Narayana-Lucas matrix $(\mathcal{U}_n)$ were defined and studied in [25].

The following theorem gives the nth general terms of the generalized Narayana matrix sequence.

Theorem 2.1. *For any integer $n,$ we have the following formulas of the generalized Narayana matrix sequence:*

$$\mathcal{V}_n = \begin{pmatrix} V_{n+1} & V_{n-1} & V_n \\ V_n & V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-3} & V_{n-2} \end{pmatrix} \tag{2.2}$$

Proof. Suppose that $n \geq 0$. We prove (2.2) by strong mathematical induction on $n.$ If $n = 0$ then, since $V_{-1} = V_2 - V_1$, $V_{-2} = V_1 - V_0$, $V_{-3} = V_0 + V_1 - V_2$, we have

$$\mathcal{V}_0 = \begin{pmatrix} V_1 & V_{-1} & V_0 \\ V_0 & V_{-2} & V_{-1} \\ V_{-1} & V_{-3} & V_{-2} \end{pmatrix} = \begin{pmatrix} V_1 & V_2 - V_1 & V_0 \\ V_0 & V_1 - V_0 & V_2 - V_1 \\ V_2 - V_1 & V_0 + V_1 - V_2 & V_1 - V_0 \end{pmatrix}$$

which is true. Assume that the equality holds for $n \leq k$. For $n = k+1,$ we have

$$\begin{aligned}
\mathcal{V}_{k+1} &= \mathcal{V}_k + \mathcal{V}_{k-2} \\
&= \begin{pmatrix} V_{k+1} & V_{k-1} & V_k \\ V_k & V_{k-2} & V_{k-1} \\ V_{k-1} & V_{k-3} & V_{k-2} \end{pmatrix} + \begin{pmatrix} V_{k-1} & V_{k-3} & V_{k-2} \\ V_{k-2} & V_{k-4} & V_{k-3} \\ V_{k-3} & V_{k-5} & V_{k-4} \end{pmatrix} \\
&= \begin{pmatrix} V_{k-1} + V_{k+1} & V_{k-1} + V_{k-3} & V_k + V_{k-2} \\ V_k + V_{k-2} & V_{k-2} + V_{k-4} & V_{k-1} + V_{k-3} \\ V_{k-1} + V_{k-3} & V_{k-3} + V_{k-5} & V_{k-2} + V_{k-4} \end{pmatrix} \\
&= \begin{pmatrix} V_{k+2} & V_k & V_{k+1} \\ V_{k+1} & V_{k-1} & V_k \\ V_k & V_{k-2} & V_{k-1} \end{pmatrix} \\
&= \begin{pmatrix} V_{k+1+1} & V_{k+1-1} & V_{k+1} \\ V_{k+1} & V_{k+1-2} & V_{k+1-1} \\ V_{k+1-1} & V_{k+1-3} & V_{k+1-2} \end{pmatrix}.
\end{aligned}$$

Thus, by strong induction on $k+1$, this proves (2.2).

For the case $n \leq 0$, similarly,(2.2) can be proved by strong mathematical induction on $n.$ □

The following theorem gives the nth general terms of the Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences.

Corollary 2.2. *For any integer $n,$ we have the following formulas of the matrix sequences:*

$$\begin{aligned}
\mathcal{N}_n &= \begin{pmatrix} N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \end{pmatrix}, \\
\mathcal{U}_n &= \begin{pmatrix} U_{n+1} & U_{n-1} & U_n \\ U_n & U_{n-2} & U_{n-1} \\ U_{n-1} & U_{n-3} & U_{n-2} \end{pmatrix}, \\
\mathcal{H}_n &= \begin{pmatrix} H_{n+1} & H_{n-1} & H_n \\ H_n & H_{n-2} & H_{n-1} \\ H_{n-1} & H_{n-3} & H_{n-2} \end{pmatrix}.
\end{aligned}$$

We now give the Binet's formula for the generalized Narayana matrix sequence.

Theorem 2.3. *For every integer $n,$ the Binet's formula of the generalized Narayana matrix sequence are given by*

$$\mathcal{V}_n = A\alpha^n + B\beta^n + C\gamma^n$$

where

$$A=\frac{\alpha\mathcal{V}_2+\alpha(\alpha-1)\mathcal{V}_1+\mathcal{V}_0}{\alpha\left(\alpha-\gamma\right)\left(\alpha-\beta\right)},B=\frac{\beta\mathcal{V}_2+\beta(\beta-1)\mathcal{V}_1+\mathcal{V}_0}{\beta\left(\beta-\gamma\right)\left(\beta-\alpha\right)},C=\frac{\gamma\mathcal{V}_2+\gamma(\gamma-1)\mathcal{V}_1+\mathcal{V}_0}{\gamma\left(\gamma-\beta\right)\left(\gamma-\alpha\right)}$$

Proof. We need to prove the theorem only for $n\geq 0$. By the assumption, the characteristic equation of (2.1) is $x^3-x^2-x-1=0$ and the roots of it are α,β and γ. So it's general solution is given by

$$\mathcal{V}_n=A\alpha^n+B\beta^n+C\gamma^n.$$

Using initial condition which is given in Definition 2.1, and also applying lineer algebra operations, we obtain the matrices A,B,C as desired. This gives the formula for $\mathcal{V}_n$. $\square$

The following theorem gives the Binet's formulas of the Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences.

Corollary 2.4. *For every integer $n,$ the Binet formulas of the Narayana and Narayana-Lucas matrix sequences are given by*

$$\begin{array}{rcl}\mathcal{N}_n&=&A_1\alpha^n+B_1\beta^n+C_1\gamma^n,\\ \mathcal{U}_n&=&A_2\alpha^n+B_2\beta^n+C_2\gamma^n,\\ \mathcal{H}_n&=&A_3\alpha^n+B_3\beta^n+C_3\gamma^n,\end{array}$$

where

$$\begin{array}{rcl}A_1&=&\dfrac{\alpha\mathcal{N}_2+\alpha(\alpha-1)\mathcal{N}_1+\mathcal{N}_0}{\alpha\left(\alpha-\gamma\right)\left(\alpha-\beta\right)},B_1=\dfrac{\beta\mathcal{N}_2+\beta(\beta-1)\mathcal{N}_1+\mathcal{N}_0}{\beta\left(\beta-\gamma\right)\left(\beta-\alpha\right)},C_1=\dfrac{\gamma\mathcal{N}_2+\gamma(\gamma-1)\mathcal{N}_1+\mathcal{N}_0}{\gamma\left(\gamma-\beta\right)\left(\gamma-\alpha\right)},\\ A_2&=&\dfrac{\alpha\mathcal{U}_2+\alpha(\alpha-1)\mathcal{U}_1+\mathcal{U}_0}{\alpha\left(\alpha-\gamma\right)\left(\alpha-\beta\right)},B_2=\dfrac{\beta\mathcal{U}_2+\beta(\beta-1)\mathcal{U}_1+\mathcal{U}_0}{\beta\left(\beta-\gamma\right)\left(\beta-\alpha\right)},C_2=\dfrac{\gamma\mathcal{U}_2+\gamma(\gamma-1)\mathcal{U}_1+\mathcal{U}_0}{\gamma\left(\gamma-\beta\right)\left(\gamma-\alpha\right)},\\ A_3&=&\dfrac{\alpha\mathcal{H}_2+\alpha(\alpha-1)\mathcal{H}_1+\mathcal{H}_0}{\alpha\left(\alpha-\gamma\right)\left(\alpha-\beta\right)},B_3=\dfrac{\beta\mathcal{H}_2+\beta(\beta-1)\mathcal{H}_1+\mathcal{H}_0}{\beta\left(\beta-\gamma\right)\left(\beta-\alpha\right)},C_3=\dfrac{\gamma\mathcal{H}_2+\gamma(\gamma-1)\mathcal{H}_1+\mathcal{H}_0}{\gamma\left(\gamma-\beta\right)\left(\gamma-\alpha\right)}.\end{array}$$

The well known Binet formulas for generalized Narayana numbers is given in (1.2). But, we will obtain these functions in terms of generalized Narayana matrix sequence as a consequence of Theorems 2.1 and 2.3. To do this, we will give the formulas for these numbers by means of the related matrix sequences. In fact, in the proof of next corollary, we will just compare the linear combination of the 2nd row and 1st column entries of the matrices.

Corollary 2.5. *For every integers $n,$ the Binet's formulas for the generalized Narayana numbers is given as*

$$V_n=\frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1=V_2-(\beta+\gamma)V_1+\beta\gamma V_0,\ b_2=V_2-(\alpha+\gamma)V_1+\alpha\gamma V_0,\ b_3=V_2-(\alpha+\beta)V_1+\alpha\beta V_0.$$

Proof. From Theorem 2.3, we have

$$\begin{aligned}
\mathcal{V}_n &= A\alpha^n + B\beta^n + C\gamma^n \\
&= \frac{\alpha\mathcal{V}_2 + \alpha(\alpha-1)\mathcal{V}_1 + \mathcal{V}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}\alpha^n + \frac{\beta\mathcal{V}_2 + \beta(\beta-1)\mathcal{V}_1 + \mathcal{V}_0}{\beta(\beta-\gamma)(\beta-\alpha)}\beta^n \\
&\quad + \frac{\gamma\mathcal{V}_2 + \gamma(\gamma-1)\mathcal{V}_1 + \mathcal{V}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\
&= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)}\begin{pmatrix} \cdot & \cdot & \cdot \\ \alpha V_2 + \alpha(\alpha-1)V_1 + V_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \\
&\quad + \frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)}\begin{pmatrix} \cdot & \cdot & \cdot \\ \beta V_2 + \beta(\beta-1)V_1 + V_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \\
&\quad + \frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)}\begin{pmatrix} \cdot & \cdot & \cdot \\ \gamma V_2 + \gamma(\gamma-1)V_1 + V_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}.
\end{aligned}$$

(we only write the 2nd row and 1st column entries of the matrices). By Theorem 2.1, we know that

$$\mathcal{V}_n = \begin{pmatrix} V_{n+1} & V_n + V_{n-1} & V_n \\ V_n & V_{n-1} + V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} & V_{n-2} \end{pmatrix}.$$

Now, if we compare the 2nd row and 1st column entries with the matrices in the above two equations, then we obtain

$$\begin{aligned}
V_n &= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)}(\alpha V_2 + \alpha(\alpha-1)V_1 + V_0) \\
&\quad + \frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)}(\beta V_2 + \beta(\beta-1)V_1 + V_0) \\
&\quad + \frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)}(\gamma V_2 + \gamma(\gamma-1)V_1 + V_0) \\
&= \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}
\end{aligned}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0.$$

Note that

$$\begin{aligned}
\alpha V_2 + \alpha(\alpha-1)V_1 + V_0 &= \alpha(V_2 + (\alpha-1)V_1 + \frac{1}{\alpha}V_0) \\
&= \alpha(V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0) = \alpha b_1, \\
\beta V_2 + \beta(\beta-1)V_1 + V_0 &= \beta(V_2 + (\beta-1)V_1 + \frac{1}{\beta}V_0) \\
&= \beta(V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0) = \beta b_2, \\
\gamma V_2 + \gamma(\gamma-1)V_1 + V_0 &= \gamma(V_2 + (\gamma-1)V_1 + \frac{1}{\gamma}V_0) \\
&= \gamma(V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0) = \gamma b_3.
\end{aligned}$$

□

Now, we present summation formulas for the generalized Narayana matrix sequence.

Theorem 2.6. *For all integers m, j we have*

$$\sum_{k=0}^{n-1} \mathcal{V}_{mk+j} = \frac{\mathcal{V}_{mn+m+j} + \mathcal{V}_{mn-m+j} + (1-U_m)\mathcal{V}_{mn+j} - \mathcal{V}_{m+j} - \mathcal{V}_{j-m} + (U_m - 1)\mathcal{V}_j}{U_m + (1 - U_{-m}) - 1} \tag{2.3}$$

Proof. *Note that*

$$\begin{aligned} \sum_{i=0}^{n-1} \mathcal{V}_{mi+j} &= \sum_{i=0}^{n-1} (A\alpha^{mi+j} + B\beta^{mi+j} + C\gamma^{mi+j}) \\ &= A\alpha^j \left(\frac{\alpha^{mn}-1}{\alpha^m-1}\right) + B\beta^j \left(\frac{\beta^{mn}-1}{\beta^m-1}\right) + C\gamma^j \left(\frac{\gamma^{mn}-1}{\gamma^m-1}\right) \end{aligned}$$

Simplifying and rearranging the last equalities in the last two expression imply (2.3) as required. □

As in Corollary 2.5, in the proof of next Corollary, we just compare the linear combination of the 2nd row and 1st column entries of the relevant matrices to obtain summation formula for the generalized Narayana sequence..

Corollary 2.7. *For all integers m, j we have*

$$\sum_{k=0}^{n-1} V_{mk+j} = \frac{V_{mn+m+j} + V_{mn-m+j} + (1-U_m)V_{mn+j} - V_{m+j} - V_{j-m} + (U_m - 1)V_j}{U_m + (1 - U_{-m}) - 1}.$$

We now give generating functions of $\mathcal{V}_n$.

Theorem 2.8. *The generating function for the generalized Narayana matrix sequences is given as*

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{V}_n x^n &= \frac{\mathcal{V}_0 + (\mathcal{V}_1 - \mathcal{V}_0)x + (\mathcal{V}_2 - \mathcal{V}_1)x^2}{1 - x - x^3} \\ &= \frac{1}{1-x-x^3} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} a_{11} &= V_0 x^2 + (-V_1 + V_2)\, x + V_1 \\ a_{21} &= (-V_1 + V_2)\, x^2 + (-V_0 + V_1)\, x + V_0 \\ a_{31} &= (-V_0 + V_1)\, x^2 + (V_0 + V_1 - V_2)\, x - V_1 + V_2 \end{aligned}$$

$$\begin{aligned} a_{12} &= (-V_0 + V_1)\, x^2 + (V_0 + V_1 - V_2)\, x - V_1 + V_2 \\ a_{22} &= (V_0 + V_1 - V_2)\, x^2 + (V_0 - 2V_1 + V_2)\, x - V_0 + V_1 \\ a_{32} &= V_0 + V_1 - V_2 + x^2\, (V_0 - 2V_1 + V_2) + x\, (V_2 - 2V_0) \end{aligned}$$

$$\begin{aligned} a_{13} &= (-V_1 + V_2)\, x^2 + (-V_0 + V_1)\, x + V_0 \\ a_{23} &= (-V_0 + V_1)\, x^2 + (V_0 + V_1 - V_2)\, x - V_1 + V_2 \\ a_{33} &= (V_0 + V_1 - V_2)\, x^2 + (V_0 - 2V_1 + V_2)\, x - V_0 + V_1 \end{aligned}$$

Proof. Suppose that $g(x) = \sum_{n=0}^{\infty} \mathcal{V}_n x^n$ is the generating function for the sequence $\{\mathcal{V}_n\}_{n\geq 0}$. Using the definition of the matrix sequence of generalized Narayana numbers (2.1), and substracting $x\sum_{n=0}^{\infty} \mathcal{V}_n x^n$ and $x^3\sum_{n=0}^{\infty} \mathcal{V}_n x^n$ from $\sum_{n=0}^{\infty} \mathcal{V}_n x^n$ we obtain

$$\begin{aligned}
(1-x-x^3)\sum_{n=0}^{\infty}\mathcal{V}_n x^n &= \sum_{n=0}^{\infty}\mathcal{V}_n x^n - x\sum_{n=0}^{\infty}\mathcal{V}_n x^n - x^3\sum_{n=0}^{\infty}\mathcal{V}_n x^n \\
&= \sum_{n=0}^{\infty}\mathcal{V}_n x^n - \sum_{n=0}^{\infty}\mathcal{V}_n x^{n+1} - \sum_{n=0}^{\infty}\mathcal{V}_n x^{n+3} \\
&= \sum_{n=0}^{\infty}\mathcal{V}_n x^n - \sum_{n=1}^{\infty}\mathcal{V}_{n-1} x^{n} - \sum_{n=3}^{\infty}\mathcal{V}_{n-3} x^{n} \\
&= (\mathcal{V}_0+\mathcal{V}_1 x+\mathcal{V}_2 x^2) - (\mathcal{V}_0 x+\mathcal{V}_1 x^2) + \sum_{n=3}^{\infty}(\mathcal{V}_n-\mathcal{V}_{n-1}-\mathcal{V}_{n-3})x^n \\
&= \mathcal{V}_0+\mathcal{V}_1 x+\mathcal{V}_2 x^2 - \mathcal{V}_0 x-\mathcal{V}_1 x^2 \\
&= \mathcal{V}_0+(\mathcal{V}_1-\mathcal{V}_0) x+(\mathcal{V}_2-\mathcal{V}_1) x^2.
\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty}\mathcal{V}_n x^n = \frac{\mathcal{V}_0+(\mathcal{V}_1-\mathcal{V}_0) x+(\mathcal{V}_2-\mathcal{V}_1) x^2}{1-x-x^3}$$

which equals the $\sum_{n=0}^{\infty}\mathcal{V}_n x^n$ in the Theorem. This completes the proof. □

The following corollary gives the generating functions of the Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences.

Corollary 2.9. *The generating functions for the Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences are given as*

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{N}_n x^n &= \frac{1}{1-x-x^3}\begin{pmatrix} 1 & x^2 & x \\ x & 1-x & x^2 \\ x^2 & x-x^2 & 1-x \end{pmatrix}, \\
\sum_{n=0}^{\infty}\mathcal{U}_n x^n &= \frac{1}{1-x-x^3}\begin{pmatrix} 3x^2+1 & 3x-2x^2 & 3-2x \\ 3-2x & 3x^2+2x-2 & 3x-2x^2 \\ 3x-2x^2 & 2x^2-5x+3 & 3x^2+2x-2 \end{pmatrix}, \\
\sum_{n=0}^{\infty}\mathcal{H}_n x^n &= \frac{1}{1-x-x^3}\begin{pmatrix} 3x^2+2x & -3x^2+x+2 & 2x^2-3x+3 \\ 2x^2-3x+3 & x^2+5x-3 & -3x^2+x+2 \\ -3x^2+x+2 & 5x^2-4x+1 & x^2+5x-3 \end{pmatrix}.
\end{aligned}$$

The well known generating function for generalized Narayana numbers is as in (1.4). However, we will obtain these functions in terms of generalized Narayana matrix sequences as a consequence of Theorem 2.8. To do this, we will again compare the the 2nd row and 1st column entries with the matrices in Theorem 2.8. Thus we have the following corollary.

Corollary 2.10. *The generating function for the generalized Narayana sequence* $\{V_n\}$ *is given as*

$$\sum_{n=0}^{\infty}V_n x^n = \frac{V_0+(V_1-V_0)x+(V_2-V_1)x^2}{1-x-x^3}.$$

Using Theorem 2.1 and Corollary 2.2, we see that

$$\begin{aligned}
\mathcal{V}_{-1} &= \begin{pmatrix} V_0 & V_1 - V_0 & V_2 - V_1 \\ V_2 - V_1 & V_0 + V_1 - V_2 & V_1 - V_0 \\ V_1 - V_0 & V_0 - 2V_1 + V_2 & V_0 + V_1 - V_2 \end{pmatrix}, \\
\mathcal{V}_{-2} &= \begin{pmatrix} V_2 - V_1 & V_0 + V_1 - V_2 & V_1 - V_0 \\ V_1 - V_0 & V_0 - 2V_1 + V_2 & V_0 + V_1 - V_2 \\ V_0 + V_1 - V_2 & V_2 - 2V_0 & V_0 - 2V_1 + V_2 \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{N}_{-1} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \mathcal{N}_{-2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \\
\mathcal{U}_{-1} &= \begin{pmatrix} 3 & -2 & 0 \\ 0 & 3 & -2 \\ -2 & 2 & 3 \end{pmatrix}, \mathcal{U}_{-2} = \begin{pmatrix} 0 & 3 & -2 \\ -2 & 2 & 3 \\ 3 & -5 & 2 \end{pmatrix}, \\
\mathcal{H}_{-1} &= \begin{pmatrix} 3 & -3 & 2 \\ 2 & 1 & -3 \\ -3 & 5 & 1 \end{pmatrix}, \mathcal{H}_{-2} = \begin{pmatrix} 2 & 1 & -3 \\ -3 & 5 & 1 \\ 1 & -4 & 5 \end{pmatrix}.
\end{aligned}$$

We now give generating functions of the generalized Narayana matrix sequence $\mathcal{V}_n$ for negative indices.

Theorem 2.11. *For negative indices, the generating function for the generalized Narayana matrix sequence is given as*

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{V}_{-n} x^n &= \frac{\mathcal{V}_0 + (\mathcal{V}_0 + \mathcal{V}_{-1})x + (\mathcal{V}_{-1} + \mathcal{V}_{-2})x^2}{1 + x - x^3} \\
&= \frac{1}{1 + x - x^3} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
b_{11} &= (V_0 - V_1 + V_2)\, x^2 + (V_0 + V_1)\, x + V_1 \\
b_{21} &= (-V_0 + V_2)\, x^2 + (V_0 - V_1 + V_2)\, x + V_0 \\
b_{31} &= V_2 - V_1 - x\,(V_0 - V_2) - x^2\,(V_2 - 2V_1)
\end{aligned}$$

and

$$\begin{aligned}
b_{12} &= V_2 - V_1 - x\,(V_0 - V_2) - x^2\,(V_2 - 2V_1) \\
b_{22} &= V_1 - V_0 - x\,(V_2 - 2V_1) - x^2\,(V_1 - 2V_0) \\
b_{32} &= V_0 + V_1 - V_2 - x\,(V_1 - 2V_0) - x^2\,(V_0 + 2V_1 - 2V_2)
\end{aligned}$$

and

$$\begin{aligned}
b_{13} &= (-V_0 + V_2)\, x^2 + (V_0 - V_1 + V_2)\, x + V_0 \\
b_{23} &= V_2 - V_1 - x\,(V_0 - V_2) - x^2\,(V_2 - 2V_1) \\
b_{33} &= V_1 - V_0 - x\,(V_2 - 2V_1) - x^2\,(V_1 - 2V_0)
\end{aligned}$$

Proof. Then, using Definition 2.1, and adding $xg(x)$ to $g(x)$ and also substracting $x^3g(x)$ we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
(1+x-x^3)g(x) &= \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n + x\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n - x^3\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n \\
&= \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n + \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^{n+1} - \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^{n+3} \\
&= \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n + \sum_{n=1}^{\infty}\mathcal{V}_{-n+1}x^{n} - \sum_{n=3}^{\infty}\mathcal{V}_{-n+3}x^{n} \\
&= (\mathcal{V}_0+\mathcal{V}_{-1}x+\mathcal{V}_{-2}x^2)+(\mathcal{V}_0x+\mathcal{V}_{-1}x^2) \\
&\quad +\sum_{n=3}^{\infty}(\mathcal{V}_{-n}+\mathcal{V}_{-n+1}-\mathcal{V}_{-n+3})x^n \\
&= (\mathcal{V}_0+\mathcal{V}_{-1}x+\mathcal{V}_{-2}x^2)+(\mathcal{V}_0x+\mathcal{V}_{-1}x^2) \\
&= \mathcal{V}_0+(\mathcal{V}_0+\mathcal{V}_{-1})x+(\mathcal{V}_{-1}+\mathcal{V}_{-2})x^2
\end{aligned}$$

Rearranging above equation, we get

$$g(x)=\frac{\mathcal{V}_0+(\mathcal{V}_0+\mathcal{V}_{-1})x+(\mathcal{V}_{-1}+\mathcal{V}_{-2})x^2}{1+x-x^3}$$

which equals the $\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n$ in the Theorem. □

The following corollary gives the generating functions of the Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences with negative indices .

Corollary 2.12. *The generating functions for the Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences with negative indices are given as*

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{N}_{-n}x^n &= \frac{1}{1+x-x^3}\begin{pmatrix} x+1 & x^2+x & x^2 \\ x^2 & -x^2+x+1 & x^2+x \\ x^2+x & -x & -x^2+x+1 \end{pmatrix}, \\
\sum_{n=0}^{\infty}\mathcal{U}_{-n}x^n &= \frac{1}{1+x-x^3}\begin{pmatrix} 3x^2+4x+1 & x^2-2x & -2x^2+3x+3 \\ -2x^2+3x+3 & 5x^2+x-2 & x^2-2x \\ x^2-2x & -3x^2+5x+3 & 5x^2+x-2 \end{pmatrix}, \\
\sum_{n=0}^{\infty}\mathcal{H}_{-n}x^n &= \frac{1}{1+x-x^3}\begin{pmatrix} 5x^2+3x & -2x^2-x+2 & -x^2+5x+3 \\ -x^2+5x+3 & 6x^2-2x-3 & -2x^2-x+2 \\ -2x^2-x+2 & x^2+6x+1 & 6x^2-2x-3 \end{pmatrix},
\end{aligned}$$

respectively.

Now, we will obtain generating functions for generalized Narayana numbers in terms of generalized Narayana matrix sequences with negative indices as a consequence of Theorem 2.11. To do this, we will again compare the the 2nd row and 1st column entries with the matrices in Theorem 2.11. Thus we have the following corollary.

Corollary 2.13. *The generating functions for the generalized Narayana sequence* $\{V_{-n}\}_{n\geq 0}$ *is given as*

$$\sum_{n=0}^{\infty}V_{-n}x^n=\frac{V_0+(V_0-V_1+V_2)x+(-V_0+V_2)x^2}{1+x-x^3}.$$

The previous corollary gives the following results as particular examples.

Corollary 2.14. *Generated functions of Narayana, Narayana-Lucas and Narayana-Perrin numbers with negative indices are*

$$\begin{aligned}\sum_{n=0}^{\infty} N_{-n}x^n &= \frac{x^2}{1+x-x^3},\\ \sum_{n=0}^{\infty} U_{-n}x^n &= \frac{3+3x-2x^2}{1+x-x^3},\\ \sum_{n=0}^{\infty} H_{-n}x^n &= \frac{3+5x-x^2}{1+x-x^3},\end{aligned}$$

respectively.

3 SOME IDENTITIES

In this section, we assume that m and n are arbitrary integers, unless otherwise mentioned. In this section, we obtain some identities of generalized Narayana and Narayana, Narayana-Lucas and Narayana-Perrin numbers. We need these identities in the next section. First, we can give a few basic relations between $\{V_n\}$ and $\{N_n\}$.

Lemma 3.1. *The following equalities are true:*

(a) $V_n = (V_0 + V_1 - V_2)N_{n+4} + (V_2 - 2V_0)N_{n+3} + (V_0 - 2V_1 + V_2)N_{n+2}.$

(b) $V_n = (V_1 - V_0)N_{n+3} + (V_0 - 2V_1 + V_2)N_{n+2} + (V_0 + V_1 - V_2)N_{n+1}.$

(c) $V_n = (V_2 - V_1)N_{n+2} + (V_0 + V_1 - V_2)N_{n+1} + (V_1 - V_0)N_n.$

(d) $V_n = V_0N_{n+1} + (V_1 - V_0)N_n + (V_2 - V_1)N_{n-1}.$

(e) $V_n = V_1N_n + (V_2 - V_1)N_{n-1} + V_0N_{n-2}.$

Proof. Note that all the identities hold for all integers n. We prove (a). Writing

$$V_n = a \times N_{n+4} + b \times N_{n+3} + c \times N_{n+2}$$

and solving the system of equations

$$\begin{aligned}V_0 &= a \times N_4 + b \times N_3 + c \times N_2\\ V_1 &= a \times N_5 + b \times N_4 + c \times N_3\\ V_2 &= a \times N_6 + b \times N_5 + c \times N_4\end{aligned}$$

we find that $a = V_0 + V_1 - V_2, b = V_2 - 2V_0, c = V_0 - 2V_1 + V_2$. The other equalities can be proved similarly. □

Note that all the identities in the above lemma can be proved by induction as well.

Next, we present a few basic relations between $\{N_n\}$ and $\{V_n\}$.

Lemma 3.2. *The following equalities are true:*

(a) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)N_n = (V_2^2 - V_1V_2 - V_0V_1)V_{n+4} + (V_1^2 + V_1V_2 + V_0V_1 - V_2^2 - V_0V_2)V_{n+3} + (V_0^2 + V_2V_0 - V_1V_2)V_{n+2}.$

(b) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)N_n=(V_1^2-V_0V_2)V_{n+3}+(V_0^2+V_2V_0-V_1V_2)V_{n+2}+(V_2^2-V_1V_2-V_0V_1)V_{n+1}$.

(c) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)N_n=(V_0^2+V_1^2-V_2V_1)V_{n+2}+(V_2^2-V_1V_2-V_0V_1)V_{n+1}+(V_1^2-V_0V_2)V_n$.

(d) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)N_n=(V_0^2-V_0V_1+V_1^2-2V_1V_2+V_2^2)V_{n+1}+(V_1^2-V_0V_2)V_n+(V_0^2+V_1^2-V_2V_1)V_{n-1}$.

(e) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)N_n=(V_0^2-V_0V_1-V_0V_2+2V_1^2-2V_1V_2+V_2^2)V_n+(V_0^2+V_1^2-V_2V_1)V_{n-1}+(V_0^2-V_0V_1+V_1^2-2V_1V_2+V_2^2)V_{n-2}$.

Now, we give a few basic relations between $\{V_n\}$ and $\{U_n\}$.

Lemma 3.3. *The following equalities are true:*

(a) $31V_n=(V_0-14V_1+11V_2)U_{n+4}+(10V_0+15V_1-14V_2)U_{n+3}+(10V_1-14V_0+V_2)U_{n+2}$.

(b) $31V_n=(11V_0+V_1-3V_2)U_{n+3}+(10V_1-14V_0+V_2)U_{n+2}+(V_0-14V_1+11V_2)U_{n+1}$.

(c) $31V_n=(11V_1-3V_0-2V_2)U_{n+2}+(V_0-14V_1+11V_2)U_{n+1}+(11V_0+V_1-3V_2)U_n$.

(d) $31V_n=(9V_2-3V_1-2V_0)U_{n+1}+(11V_0+V_1-3V_2)U_n+(11V_1-3V_0-2V_2)U_{n-1}$.

(e) $31V_n=(9V_0-2V_1+6V_2)U_n+(11V_1-3V_0-2V_2)U_{n-1}+(9V_2-3V_1-2V_0)U_{n-2}$.

Next, we present a few basic relations between $\{U_n\}$ and $\{V_n\}$.

Lemma 3.4. *The following equalities are true:*

(a) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)U_n=(3V_1^2+2V_1V_2+2V_0V_1-2V_2^2-3V_0V_2)V_{n+4}+(3V_0^2-2V_0V_1+5V_0V_2-2V_1^2-5V_1V_2+2V_2^2)V_{n+3}+(-2V_0^2-2V_0V_2-3V_1V_0+3V_2^2-V_1V_2)V_{n+2}$.

(b) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)U_n=(3V_0^2+V_1^2+2V_0V_2-3V_1V_2)V_{n+3}+(-2V_0^2-2V_0V_2-3V_1V_0+3V_2^2-V_1V_2)V_{n+2}+(3V_1^2-2V_2^2+2V_0V_1-3V_0V_2+2V_1V_2)V_{n+1}$.

(c) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)U_n=(V_0^2-3V_0V_1+V_1^2-4V_1V_2+3V_2^2)V_{n+2}+(3V_1^2+2V_1V_2+2V_0V_1-2V_2^2-3V_0V_2)V_{n+1}+(3V_0^2+2V_2V_0+V_1^2-3V_2V_1)V_n$.

(d) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)U_n=(V_0^2-V_0V_1-3V_0V_2+4V_1^2-2V_1V_2+V_2^2)V_{n+1}+(3V_0^2+2V_2V_0+V_1^2-3V_2V_1)V_n+(V_0^2-3V_0V_1+V_1^2-4V_1V_2+3V_2^2)V_{n-1}$.

(e) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)U_n=(4V_0^2-V_0V_1-V_0V_2+5V_1^2-5V_1V_2+V_2^2)V_n+(V_0^2-3V_0V_1+V_1^2-4V_1V_2+3V_2^2)V_{n-1}+(V_0^2-V_0V_1-3V_0V_2+4V_1^2-2V_1V_2+V_2^2)V_{n-2}$.

Now, we give a few basic relations between $\{V_n\}$ and $\{H_n\}$.

Lemma 3.5. *The following equalities are true:*

(a) $53V_n=(15V_2-11V_1-10V_0)H_{n+4}+(25V_0+V_1-11V_2)H_{n+3}+(25V_1-11V_0-10V_2)H_{n+2}$.

(b) $53V_n=(15V_0-10V_1+4V_2)H_{n+3}+(25V_1-11V_0-10V_2)H_{n+2}+(15V_2-11V_1-10V_0)H_{n+1}$.

(c) $53V_n=(4V_0+15V_1-6V_2)H_{n+2}+(15V_2-11V_1-10V_0)H_{n+1}+(15V_0-10V_1+4V_2)H_n$.

(d) $53V_n=(4V_1-6V_0+9V_2)H_{n+1}+(15V_0-10V_1+4V_2)H_n+(4V_0+15V_1-6V_2)H_{n-1}$.

(e) $53V_n=(9V_0-6V_1+13V_2)H_n+(4V_0+15V_1-6V_2)H_{n-1}+(4V_1-6V_0+9V_2)H_{n-2}$.

Next, we present a few basic relations between $\{H_n\}$ and $\{V_n\}$.

Lemma 3.6. *The following equalities are true:*

(a) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)H_n=(2V_0^2+3V_0V_1-V_0V_2+3V_1^2+V_1V_2-3V_2^2)V_{n+4}+(V_0^2-5V_0V_1+4V_0V_2-3V_1^2-6V_1V_2+5V_2^2)V_{n+3}+(-3V_0^2-V_0V_1-5V_0V_2+2V_1^2+2V_1V_2+V_2^2)V_{n+2}$.

(b) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)H_n=(3V_0^2+3V_0V_2-2V_1V_0+2V_2^2-5V_1V_2)V_{n+3}+(-3V_0^2-V_0V_1-5V_0V_2+2V_1^2+2V_1V_2+V_2^2)V_{n+2}+(2V_0^2+3V_0V_1-V_0V_2+3V_1^2+V_1V_2-3V_2^2)V_{n+1}$.

(c) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)H_n=(2V_1^2-3V_1V_2-3V_0V_1+3V_2^2-2V_0V_2)V_{n+2}+(2V_0^2+3V_0V_1-V_0V_2+3V_1^2+V_1V_2-3V_2^2)V_{n+1}+(3V_0^2+3V_0V_2-2V_1V_0+2V_2^2-5V_1V_2)V_n$.

(d) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)H_n=(2V_0^2-3V_2V_0+5V_1^2-2V_2V_1)V_{n+1}+(3V_0^2+3V_0V_2-2V_1V_0+2V_2^2-5V_1V_2)V_n+(2V_1^2-3V_1V_2-3V_0V_1+3V_2^2-2V_0V_2)V_{n-1}$.

(e) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)H_n=(5V_0^2-2V_0V_1+5V_1^2-7V_1V_2+2V_2^2)V_n+(2V_1^2-3V_1V_2-3V_0V_1+3V_2^2-2V_0V_2)V_{n-1}+(2V_0^2-3V_2V_0+5V_1^2-2V_2V_1)V_{n-2}$.

4 RELATION BETWEEN GENERALIZED NARAYANA MATRIX SEQUENCES AND IT'S SPECIAL CASES

In this section, we assume that m and n are arbitrary integers, unless otherwise mentioned.

The following theorem shows that there always exist interrelation between generalized Narayana and Narayana matrix sequences.

Theorem 4.1. *For the matrix sequences $\{\mathcal{V}_n\}$ and $\{\mathcal{N}_n\}$, we have the following identities.*

(a) $\mathcal{V}_n=(V_0+V_1-V_2)\mathcal{N}_{n+4}+(V_2-2V_0)\mathcal{N}_{n+3}+(V_0-2V_1+V_2)\mathcal{N}_{n+2}$.

(b) $\mathcal{V}_n=(V_1-V_0)\mathcal{N}_{n+3}+(V_0-2V_1+V_2)\mathcal{N}_{n+2}+(V_0+V_1-V_2)\mathcal{N}_{n+1}$.

(c) $\mathcal{V}_n=(V_2-V_1)\mathcal{N}_{n+2}+(V_0+V_1-V_2)\mathcal{N}_{n+1}+(V_1-V_0)\mathcal{N}_n$.

(d) $\mathcal{V}_n=V_0\mathcal{N}_{n+1}+(V_1-V_0)\mathcal{N}_n+(V_2-V_1)\mathcal{N}_{n-1}$.

(e) $\mathcal{V}_n=V_1N_n+(V_2-V_1)N_{n-1}+V_0N_{n-2}$.

(f) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)\mathcal{N}_n=(V_2^2-V_1V_2-V_0V_1)\mathcal{V}_{n+4}+(V_1^2+V_1V_2+V_0V_1-V_2^2-V_0V_2)\mathcal{V}_{n+3}+(V_0^2+V_2V_0-V_1V_2)\mathcal{V}_{n+2}$.

(g) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)\mathcal{N}_n=(V_1^2-V_0V_2)\mathcal{V}_{n+3}+(V_0^2+V_2V_0-V_1V_2)\mathcal{V}_{n+2}+(V_2^2-V_1V_2-V_0V_1)\mathcal{V}_{n+1}$.

(h) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)\mathcal{N}_n=(V_0^2+V_1^2-V_2V_1)\mathcal{V}_{n+2}+(V_2^2-V_1V_2-V_0V_1)\mathcal{V}_{n+1}+(V_1^2-V_0V_2)\mathcal{V}_n$.

(i) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)\mathcal{N}_n=(V_0^2-V_0V_1+V_1^2-2V_1V_2+V_2^2)\mathcal{V}_{n+1}+(V_1^2-V_0V_2)\mathcal{V}_n+(V_0^2+V_1^2-V_2V_1)\mathcal{V}_{n-1}$.

(j) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)\mathcal{N}_n=(V_0^2-V_0V_1-V_0V_2+2V_1^2-2V_1V_2+V_2^2)\mathcal{V}_n+(V_0^2+V_1^2-V_2V_1)\mathcal{V}_{n-1}+(V_0^2-V_0V_1+V_1^2-2V_1V_2+V_2^2)\mathcal{V}_{n-2}$.

Proof. From Lemmas 3.1 and 3.2, (a)-(j) follow. □

The following theorem shows that there always exist interrelation between generalized Narayana and Narayana-Lucas matrix sequences.

Theorem 4.2. *For the matrix sequences $\{\mathcal{V}_n\}$ and $\{\mathcal{U}_n\}$, we have the following identities.*

(a) $31\mathcal{V}_n = (V_0 - 14V_1 + 11V_2)\mathcal{U}_{n+4} + (10V_0 + 15V_1 - 14V_2)\mathcal{U}_{n+3} + (10V_1 - 14V_0 + V_2)\mathcal{U}_{n+2}.$

(b) $31\mathcal{V}_n = (11V_0 + V_1 - 3V_2)\mathcal{U}_{n+3} + (10V_1 - 14V_0 + V_2)\mathcal{U}_{n+2} + (V_0 - 14V_1 + 11V_2)\mathcal{U}_{n+1}.$

(c) $31\mathcal{V}_n = (11V_1 - 3V_0 - 2V_2)\mathcal{U}_{n+2} + (V_0 - 14V_1 + 11V_2)\mathcal{U}_{n+1} + (11V_0 + V_1 - 3V_2)\mathcal{U}_n.$

(d) $31\mathcal{V}_n = (9V_2 - 3V_1 - 2V_0)\mathcal{U}_{n+1} + (11V_0 + V_1 - 3V_2)\mathcal{U}_n + (11V_1 - 3V_0 - 2V_2)\mathcal{U}_{n-1}.$

(e) $31\mathcal{V}_n = (9V_0 - 2V_1 + 6V_2)\mathcal{U}_n + (11V_1 - 3V_0 - 2V_2)\mathcal{U}_{n-1} + (9V_2 - 3V_1 - 2V_0)\mathcal{U}_{n-2}.$

(f) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{U}_n = (3V_1^2 + 2V_1V_2 + 2V_0V_1 - 2V_2^2 - 3V_0V_2)\mathcal{V}_{n+4} + (3V_0^2 - 2V_0V_1 + 5V_0V_2 - 2V_1^2 - 5V_1V_2 + 2V_2^2)\mathcal{V}_{n+3} + (-2V_0^2 - 2V_0V_2 - 3V_1V_0 + 3V_2^2 - V_1V_2)\mathcal{V}_{n+2}.$

(g) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{U}_n = (3V_0^2 + V_1^2 + 2V_0V_2 - 3V_1V_2)\mathcal{V}_{n+3} + (-2V_0^2 - 2V_0V_2 - 3V_1V_0 + 3V_2^2 - V_1V_2)\mathcal{V}_{n+2} + (3V_1^2 - 2V_2^2 + 2V_0V_1 - 3V_0V_2 + 2V_1V_2)\mathcal{V}_{n+1}.$

(h) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{U}_n = (V_0^2 - 3V_0V_1 + V_1^2 - 4V_1V_2 + 3V_2^2)\mathcal{V}_{n+2} + (3V_1^2 + 2V_1V_2 + 2V_0V_1 - 2V_2^2 - 3V_0V_2)\mathcal{V}_{n+1} + (3V_0^2 + 2V_2V_0 + V_1^2 - 3V_2V_1)\mathcal{V}_n.$

(i) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{U}_n = (V_0^2 - V_0V_1 - 3V_0V_2 + 4V_1^2 - 2V_1V_2 + V_2^2)\mathcal{V}_{n+1} + (3V_0^2 + 2V_2V_0 + V_1^2 - 3V_2V_1)\mathcal{V}_n + (V_0^2 - 3V_0V_1 + V_1^2 - 4V_1V_2 + 3V_2^2)\mathcal{V}_{n-1}.$

(j) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{U}_n = (4V_0^2 - V_0V_1 - V_0V_2 + 5V_1^2 - 5V_1V_2 + V_2^2)\mathcal{V}_n + (V_0^2 - 3V_0V_1 + V_1^2 - 4V_1V_2 + 3V_2^2)\mathcal{V}_{n-1} + (V_0^2 - V_0V_1 - 3V_0V_2 + 4V_1^2 - 2V_1V_2 + V_2^2)\mathcal{V}_{n-2}.$

Proof. From Lemmas 3.3 and 3.4, (a)-(j) follow. □

The following theorem shows that there always exist interrelation between generalized Narayana and Narayana-Perrin matrix sequences.

Theorem 4.3. *For the matrix sequences* $\{\mathcal{V}_n\}$ *and* $\{\mathcal{H}_n\}$*, we have the following identities.*

(a) $53\mathcal{V}_n = (15V_2 - 11V_1 - 10V_0)\mathcal{H}_{n+4} + (25V_0 + V_1 - 11V_2)\mathcal{H}_{n+3} + (25V_1 - 11V_0 - 10V_2)\mathcal{H}_{n+2}.$

(b) $53\mathcal{V}_n = (15V_0 - 10V_1 + 4V_2)\mathcal{H}_{n+3} + (25V_1 - 11V_0 - 10V_2)\mathcal{H}_{n+2} + (15V_2 - 11V_1 - 10V_0)\mathcal{H}_{n+1}.$

(c) $53\mathcal{V}_n = (4V_0 + 15V_1 - 6V_2)\mathcal{H}_{n+2} + (15V_2 - 11V_1 - 10V_0)\mathcal{H}_{n+1} + (15V_0 - 10V_1 + 4V_2)\mathcal{H}_n.$

(d) $53\mathcal{V}_n = (4V_1 - 6V_0 + 9V_2)\mathcal{H}_{n+1} + (15V_0 - 10V_1 + 4V_2)\mathcal{H}_n + (4V_0 + 15V_1 - 6V_2)\mathcal{H}_{n-1}.$

(e) $53\mathcal{V}_n = (9V_0 - 6V_1 + 13V_2)\mathcal{H}_n + (4V_0 + 15V_1 - 6V_2)\mathcal{H}_{n-1} + (4V_1 - 6V_0 + 9V_2)\mathcal{H}_{n-2}.$

(f) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{H}_n = (2V_0^2 + 3V_0V_1 - V_0V_2 + 3V_1^2 + V_1V_2 - 3V_2^2)\mathcal{V}_{n+4} + (V_0^2 - 5V_0V_1 + 4V_0V_2 - 3V_1^2 - 6V_1V_2 + 5V_2^2)\mathcal{V}_{n+3} + (-3V_0^2 - V_0V_1 - 5V_0V_2 + 2V_1^2 + 2V_1V_2 + V_2^2)\mathcal{V}_{n+2}.$

(g) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{H}_n = (3V_0^2 + 3V_0V_2 - 2V_1V_0 + 2V_2^2 - 5V_1V_2)\mathcal{V}_{n+3} + (-3V_0^2 - V_0V_1 - 5V_0V_2 + 2V_1^2 + 2V_1V_2 + V_2^2)\mathcal{V}_{n+2} + (2V_0^2 + 3V_0V_1 - V_0V_2 + 3V_1^2 + V_1V_2 - 3V_2^2)\mathcal{V}_{n+1}.$

(h) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{H}_n = (2V_1^2 - 3V_1V_2 - 3V_0V_1 + 3V_2^2 - 2V_0V_2)\mathcal{V}_{n+2} + (2V_0^2 + 3V_0V_1 - V_0V_2 + 3V_1^2 + V_1V_2 - 3V_2^2)\mathcal{V}_{n+1} + (3V_0^2 + 3V_0V_2 - 2V_1V_0 + 2V_2^2 - 5V_1V_2)\mathcal{V}_n.$

(i) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{H}_n = (2V_0^2 - 3V_2V_0 + 5V_1^2 - 2V_2V_1)\mathcal{V}_{n+1} + (3V_0^2 + 3V_0V_2 - 2V_1V_0 + 2V_2^2 - 5V_1V_2)\mathcal{V}_n + (2V_1^2 - 3V_1V_2 - 3V_0V_1 + 3V_2^2 - 2V_0V_2)\mathcal{V}_{n-1}.$

(j) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{H}_n = (5V_0^2 - 2V_0V_1 + 5V_1^2 - 7V_1V_2 + 2V_2^2)\mathcal{V}_n + (2V_1^2 - 3V_1V_2 - 3V_0V_1 + 3V_2^2 - 2V_0V_2)\mathcal{V}_{n-1} + (2V_0^2 - 3V_2V_0 + 5V_1^2 - 2V_2V_1)\mathcal{V}_{n-2}.$

Proof. From Lemmas 3.5 and 3.6, (a)-(j) follow. □

To prove the following Lemma 4.5 (c) we need the next lemma.

Lemma 4.4. *Let A, B, C as in Theorem 2.3 and $A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3$ as in Corollary 2.4. Then the following relations hold:*

$$\begin{aligned} A_1^2 &= A_1,\ B_1^2 = B_1,\ C_1^2 = C_1, \\ AB &= BA = AC = CA = CB = BC = (0)\,, \\ A_1B_1 &= B_1A_1 = A_1C_1 = C_1A_1 = C_1B_1 = B_1C_1 = (0)\,, \\ A_2B_2 &= B_2A_2 = A_2C_2 = C_2A_2 = C_2B_2 = B_2C_2 = (0)\,, \\ A_3B_3 &= B_3A_3 = A_3C_3 = C_3A_3 = C_3B_3 = B_3C_3 = (0)\,. \end{aligned}$$

Proof. Using $\alpha+\beta+\gamma = 1$, $\alpha\beta+\alpha\gamma+\beta\gamma = 0$ and $\alpha\beta\gamma = 1$, required equalities can be established by matrix calculations. See also [25]. □

Lemma 4.5. *For all integers m and n, we have the following identities.*

(a) $\mathcal{N}_0\mathcal{V}_n = \mathcal{V}_n\mathcal{N}_0 = \mathcal{V}_n$.

(b) $\mathcal{V}_0\mathcal{N}_n = \mathcal{N}_n\mathcal{V}_0 = \mathcal{V}_n$.

(c) $\mathcal{N}_m\mathcal{N}_n = \mathcal{N}_n\mathcal{N}_m = \mathcal{N}_{m+n}$.

(d) $\mathcal{N}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{N}_m = \mathcal{V}_{m+n}$.

(e) $\mathcal{N}_m\mathcal{U}_n = \mathcal{U}_n\mathcal{N}_m = \mathcal{U}_{m+n}$.

(f) $\mathcal{N}_m\mathcal{H}_n = \mathcal{H}_n\mathcal{N}_m = \mathcal{H}_{m+n}$.

(g) $\mathcal{V}_0\mathcal{V}_n = \mathcal{V}_n\mathcal{V}_0$.

(h) $\mathcal{V}_n\mathcal{V}_m = \mathcal{V}_m\mathcal{V}_n = \mathcal{V}_0\mathcal{V}_{m+n}$.

(i) $\mathcal{N}_{-n} = (\mathcal{N}_n)^{-1}$.

(j) $\mathcal{V}_{-n} = (\mathcal{V}_0)^{1-n}(\mathcal{V}_{-1})^n$

Proof. Identities can be established easily.

(a) Since $\mathcal{N}_0$ is the identity matrix, (a) follows.

(b) It can be seen by using Lemma 3.1.

(c) (c) is given in [25]. We supply the proof for completeness. Using Lemma 4.4 we obtain

$$\begin{aligned} \mathcal{N}_m\mathcal{N}_n &= (A_1\alpha^m + B_1\beta^m + C_1\gamma^m)(A_1\alpha^n + B_1\beta^n + C_1\gamma^n) \\ &= A_1^2\alpha^{m+n} + B_1^2\beta^{m+n} + C_1^2\gamma^{m+n} + A_1B_1\alpha^m\beta^n + B_1A_1\alpha^n\beta^m \\ &\quad + A_1C_1\alpha^m\gamma^n + C_1A_1\alpha^n\gamma^m + B_1C_1\beta^m\gamma^n + C_1B_1\beta^n\gamma^m \\ &= A_1\alpha^{m+n} + B_1\beta^{m+n} + C_1\gamma^{m+n} \\ &= \mathcal{N}_{m+n}. \end{aligned}$$

(d) From (b), we have

$$\mathcal{N}_m\mathcal{V}_n = \mathcal{N}_m\mathcal{N}_n\mathcal{U}_0.$$

Now from (c) and again from (b), we obtain $\mathcal{N}_m\mathcal{V}_n = \mathcal{N}_{m+n}\mathcal{V}_0 = \mathcal{V}_{m+n}$.

It can be shown similarly that $\mathcal{V}_n\mathcal{N}_m = \mathcal{V}_{m+n}$.

(e) Take $\mathcal{V}_n = \mathcal{U}_n$ in (d).

(f) Take $\mathcal{V}_n = \mathcal{H}_n$ in (d).

(g) After matrix multiplication, just compare the row and column entries of the matrices.

(h) Using (d) and (g) and (b) we get

$$\mathcal{V}_0\mathcal{V}_{m+n} = \mathcal{V}_0\mathcal{V}_n\mathcal{N}_m = \mathcal{V}_n\mathcal{V}_0\mathcal{N}_m = \mathcal{V}_n\mathcal{V}_m.$$

Again, using (d) and (g) and (b), we obtain

$$\mathcal{V}_0\mathcal{V}_{m+n} = \mathcal{V}_0\mathcal{V}_m\mathcal{N}_n = \mathcal{V}_m\mathcal{V}_0\mathcal{N}_n = \mathcal{V}_m\mathcal{V}_n.$$

This completes the proof of (h).

(i) Suppose first that $n \geq 0$. We prove by mathematical induction. If $n = 0$ then we have

$$\mathcal{N}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = (\mathcal{N}_0)^{-1}$$

which is true and

$$\mathcal{N}_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = (\mathcal{N}_1)^{-1}$$

which is true. Assume that the equality holds for $n \leq k$. For $n = k+1$, by using (c), we obtain

$$\begin{aligned}
(\mathcal{N}_{k+1})^{-1} &= (\mathcal{N}_k\mathcal{N}_1)^{-1} = (\mathcal{N}_1)^{-1}(\mathcal{N}_k)^{-1} = \mathcal{N}_{-1}\mathcal{N}_{-k} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} N_{-k+1} & N_{-k-1} & N_{-k} \\ N_{-k} & N_{-k-2} & N_{-k-1} \\ N_{-k-1} & N_{-k-3} & N_{-k-2} \end{pmatrix} \\
&= \begin{pmatrix} N_{-k} & N_{-k-2} & N_{-k-1} \\ N_{-k-1} & N_{-k-3} & N_{-k-2} \\ N_{1-k} - N_{-k} & N_{-k-1} - N_{-k-2} & N_{-k} - N_{-k-1} \end{pmatrix} \\
&= \begin{pmatrix} N_{-k} & N_{-k-2} & N_{-k-1} \\ N_{-k-1} & N_{-k-3} & N_{-k-2} \\ N_{-k-2} & N_{-k-4} & N_{-k-3} \end{pmatrix} \\
&= \begin{pmatrix} N_{-(k+1)+1} & N_{-(k+1)-1} & N_{-(k+1)} \\ N_{-(k+1)} & N_{-(k+1)-2} & N_{-(k+1)-1} \\ N_{-(k+1)-1} & N_{-(k+1)-3} & N_{-(k+1)-2} \end{pmatrix} \\
&= \mathcal{N}_{-(k+1)}
\end{aligned}$$

Thus, by induction on n, this proves (g) for $n \geq 0$. Suppose now that $n \leq 0$. Say $m = -n$. Then (g) can be written as

$$\mathcal{N}_m = (\mathcal{N}_{-m})^{-1}$$

and we prove this. Since $m \geq 0$, from the first part of the proof, we have

$$\mathcal{N}_{-m} = (\mathcal{N}_m)^{-1}$$

and so

$$(\mathcal{N}_{-m})^{-1} = ((\mathcal{N}_m)^{-1})^{-1} = \mathcal{N}_m$$

which completes the proof.

(j) Taking $-n+1$ for m and 1 for n in $\mathcal{V}_0\mathcal{V}_{m+n} = \mathcal{V}_m\mathcal{V}_n$ which is given in (h), we obtain that

$$\mathcal{V}_0\mathcal{V}_{-n} = \mathcal{V}_{-n+1}\mathcal{V}_{-1}. \tag{4.1}$$

If we multiply both side of the equation (4.1) with $\mathcal{V}_0$ we have the relation

$$\begin{aligned}\mathcal{V}_0\mathcal{V}_0\mathcal{V}_{-n} &= \mathcal{V}_0\mathcal{V}_{-n+1}\mathcal{V}_{-1}\\ &= \mathcal{V}_{-n+2}\mathcal{V}_{-1}\mathcal{V}_{-1}.\end{aligned}$$

Repeating this process we then obtain

$$\mathcal{V}_0^{n-1}\mathcal{V}_{-n} = \mathcal{V}_{-1}^{n}.$$

Thus, it follows that

$$\mathcal{V}_{-n} = \mathcal{V}_0^{1-n}\mathcal{V}_{-1}^{n}.$$

This completes the proof. □

Note that using Lemma 4.5 (j) and (d), we obtain

$$\mathcal{V}_{-n} = (\mathcal{V}_0)^{1-n}(\mathcal{V}_{-1})^{n} = (\mathcal{V}_n\mathcal{N}_{-n})^{1-n}\mathcal{V}_{-1}^{n} = \mathcal{N}_{-n}^{1-n}\mathcal{V}_n^{1-n}\mathcal{V}_{-1}^{n}$$

and then by Lemma (i), we get

$$\mathcal{V}_{-n} = \mathcal{N}_n^{n-1}\mathcal{V}_n^{1-n}\mathcal{V}_{-1}^{n}.$$

Using Lemma 4.5 and comparing matrix entries, we have next result.

Corollary 4.6. *For generalized Narayana, Narayana, Narayana-Lucas and Narayana-Perrin numbers, we have the following identities:*

(a) $V_{m+n} = N_mV_{n+1} + N_{m-2}V_n + N_{m-1}V_{n-1} = N_{m+1}V_n + N_{m-1}V_{n-1} + N_mV_{n-2}.$

(b) $N_{m+n} = N_mN_{n+1} + N_{m-2}N_n + N_{m-1}N_{n-1} = N_{m+1}N_n + N_{m-1}N_{n-1} + N_mN_{n-2}.$

(c) $U_{m+n} = N_mU_{n+1} + N_{m-2}U_n + N_{m-1}U_{n-1} = N_{m+1}U_n + N_{m-1}U_{n-1} + N_mU_{n-2}.$

(d) $H_{m+n} = N_mH_{n+1} + N_{m-2}H_n + N_{m-1}H_{n-1} = N_{m+1}H_n + N_{m-1}H_{n-1} + N_mH_{n-2}.$

(e) $V_{m+1}V_n+V_{m-1}V_{n-1}+V_mV_{n-2} = V_mV_{n+1}+V_{m-2}V_n+V_{m-1}V_{n-1} = V_0V_{m+n+1}+(V_1-V_0)V_{m+n}+(V_2-V_1)V_{m+n-1}.$

(f) $N_{m+1}N_n + N_{m-1}N_{n-1} + N_mN_{n-2} = N_mN_{n+1} + N_{m-2}N_n + N_{m-1}N_{n-1} = N_{m+n}.$

(g) $U_{m+1}U_n + U_{m-1}U_{n-1} + U_mU_{n-2} = U_mU_{n+1} + U_{m-2}U_n + U_{m-1}U_{n-1} = 3U_{m+n+1} - 2U_{m+n}.$

(h) $H_{m+1}H_n+H_{m-1}H_{n-1}+H_mH_{n-2} = H_mH_{n+1}+H_{m-2}H_n+H_{m-1}H_{n-1} = 3H_{m+n+1}-3H_{m+n}+2H_{m+n-1}.$

Proof. We prove (a) and (e) by using Lemma 4.5 (d) and (h). The others are special cases of (a) and (e). Lemma 4.5 (d), i.e., $\mathcal{N}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{N}_m = \mathcal{V}_{m+n}$, can be writtten as

$$\begin{aligned}\begin{pmatrix} V_{m+n+1} & V_{m+n-1} & V_{m+n}\\ V_{m+n} & V_{m+n-2} & V_{m+n-1}\\ V_{m+n-1} & V_{m+n-3} & V_{m+n-2}\end{pmatrix} &= \begin{pmatrix} N_{m+1} & N_{m-1} & N_m\\ N_m & N_{m-2} & N_{m-1}\\ N_{m-1} & N_{m-3} & N_{m-2}\end{pmatrix}\begin{pmatrix} V_{n+1} & V_{n-1} & V_n\\ V_n & V_{n-2} & V_{n-1}\\ V_{n-1} & V_{n-3} & V_{n-2}\end{pmatrix}\\ &= \begin{pmatrix} V_{n+1} & V_{n-1} & V_n\\ V_n & V_{n-2} & V_{n-1}\\ V_{n-1} & V_{n-3} & V_{n-2}\end{pmatrix}\begin{pmatrix} N_{m+1} & N_{m-1} & N_m\\ N_m & N_{m-2} & N_{m-1}\\ N_{m-1} & N_{m-3} & N_{m-2}\end{pmatrix}.\end{aligned}$$

Now, by multiplying the matrices and then by comparing the 2nd rows and 1st columns entries, we get the required identities in (a).

Lemma 4.5 (h), i.e., $\mathcal{V}_n\mathcal{V}_m = \mathcal{V}_m\mathcal{V}_n = \mathcal{V}_0\mathcal{V}_{m+n}$, can be writtten as

$$\begin{aligned} & \begin{pmatrix} V_{n+1} & V_{n-1} & V_n \\ V_n & V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-3} & V_{n-2} \end{pmatrix} \begin{pmatrix} V_{m+1} & V_{m-1} & V_m \\ V_m & V_{m-2} & V_{m-1} \\ V_{m-1} & V_{m-3} & V_{m-2} \end{pmatrix} \\ = & \begin{pmatrix} V_{m+1} & V_{m-1} & V_m \\ V_m & V_{m-2} & V_{m-1} \\ V_{m-1} & V_{m-3} & V_{m-2} \end{pmatrix} \begin{pmatrix} V_{n+1} & V_{n-1} & V_n \\ V_n & V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-3} & V_{n-2} \end{pmatrix} \\ = & \begin{pmatrix} V_1 & V_2 - V_1 & V_0 \\ V_0 & V_1 - V_0 & V_2 - V_1 \\ V_2 - V_1 & V_0 + V_1 - V_2 & V_1 - V_0 \end{pmatrix} \begin{pmatrix} V_{m+n+1} & V_{m+n-1} & V_{m+n} \\ V_{m+n} & V_{m+n-2} & V_{m+n-1} \\ V_{m+n-1} & V_{m+n-3} & V_{m+n-2} \end{pmatrix} \end{aligned}$$

Now, by multiplying the matrices and then by comparing the 2nd rows and 1st columns entries, we get the required identities in (e). □

As an application of Lemma 4.5 (i) and Corollary 4.6 (b), we present the following example.

Example 4.7. *For all integers n, we have the following identities.*

$$N_{-n} = N_{n-1}^2 - N_n N_{n-2}$$

and

$$N_{n+1}^3 + N_n^3 + N_{n-1}^3 - 2N_n N_{n+1}^2 + (N_{n-1} + N_{n+1})N_n^2 - (3N_n - N_{n-1})N_{n+1}N_{n-1} = 1.$$

Solution. By comparing the 2nd rows and 1st columns entries of both sides of the relation $\mathcal{N}_{-n} = (\mathcal{N}_n)^{-1}$ which is given in Lemma 4.5 (i), we get

$$\begin{aligned} N_{-n} &= \frac{N_{n-1}^2 - N_n N_{n-2}}{N_{n-3}N_n^2 - 2N_n N_{n-1} N_{n-2} + N_{n-1}^3 - N_{n+1}N_{n-3}N_{n-1} + N_{n+1}N_{n-2}^2} \\ &= \frac{N_{n-1}^2 - N_n N_{n-2}}{N_{n+1}^3 + N_n^3 + N_{n-1}^3 - 2N_n N_{n+1}^2 + (N_{n-1} + N_{n+1})N_n^2 - (3N_n - N_{n-1})N_{n+1}N_{n-1}} \end{aligned} \tag{4.2}$$

where we used the identities

$$\begin{aligned} N_n &= N_{n-1} + N_{n-3} \Rightarrow N_n - N_{n-1} = N_{n-3}, \\ N_{n+1} &= N_n + N_{n-2} \Rightarrow N_{n+1} - N_n = N_{n-2}. \end{aligned}$$

Using (taking $m = n$ in) Corollary 4.6 (b), we get

$$N_{2n} = N_{n-1}^2 + N_n N_{n+1} + N_n N_{n-2}. \tag{4.3}$$

In [30, Corollary 12 (a)], the following formula is presented for N_{-n} :

$$N_{-n} = 2N_n^2 + N_{2n} - 3N_{n+1}N_n.$$

which (using (4.3)) can be written as

$$N_{-n} = N_n N_{n-2} - 2N_n N_{n+1} + 2N_n^2 + N_{n-1}^2. \tag{4.4}$$

Note that

$$\begin{aligned} N_{n-1}^2 - N_n N_{n-2} &= (N_n N_{n-2} - 2N_n N_{n+1} + 2N_n^2 + N_{n-1}^2) - 2N_n (N_n - N_{n+1} + N_{n-2}) \\ &= N_n N_{n-2} - 2N_n N_{n+1} + 2N_n^2 + N_{n-1}^2 \end{aligned}$$

because $N_n - N_{n+1} + N_{n-2} = 0$. So the rights sides of the equations (4.2) and (4.4) must be equal. This completes the solution. □

Theorem 4.8. *For all integers m and n, we have the following identities.*

(a) $\mathcal{V}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{V}_m = (V_1 - V_2)^2\mathcal{N}_{m+n+4} - 2(V_1 - V_2)(V_0 + V_1 - V_2)\mathcal{N}_{m+n+3} + (V_2^2 - V_1^2 + V_0^2 + 4V_0V_1 - 4V_0V_2)\mathcal{N}_{m+n+2} + 2(V_1 - V_0)(V_0 + V_1 - V_2)\mathcal{N}_{m+n+1} + (V_0 - V_1)^2\mathcal{N}_{m+n}.$

(b) $\mathcal{V}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{V}_m = (V_2 - V_1)\mathcal{V}_{m+n+2} + (V_0 + V_1 - V_2)\mathcal{V}_{m+n+1} + (V_1 - V_0)\mathcal{V}_{m+n}.$

(c) $\mathcal{N}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{N}_m = (V_2 - V_1)\mathcal{N}_{m+n+2} + (V_0 + V_1 - V_2)\mathcal{N}_{m+n+1} + (V_1 - V_0)\mathcal{N}_{m+n}.$

(d) $31\mathcal{N}_m\mathcal{V}_n = 31\mathcal{V}_n\mathcal{N}_m = (11V_1 - 3V_0 - 2V_2)\mathcal{U}_{m+n+2} + (V_0 - 14V_1 + 11V_2)\mathcal{U}_{m+n+1} + (11V_0 + V_1 - 3V_2)\mathcal{U}_{m+n}.$

(e) $53\mathcal{N}_m\mathcal{V}_n = 53\mathcal{V}_n\mathcal{N}_m = (4V_0 + 15V_1 - 6V_2)\mathcal{H}_{m+n+2} + (15V_2 - 11V_1 - 10V_0)\mathcal{H}_{m+n+1} + (15V_0 - 10V_1 + 4V_2)\mathcal{H}_{m+n}.$

Proof.

(a) It follows from Theorem 4.1 (c) and Lemma 4.5 (c).

(b) It follows from Theorem 4.1 (c) and Lemma 4.5 (d).

(c) It follows from Theorem 4.1 (c) and Lemma 4.5 (c).

(d) It follows from Theorem 4.2 (c) and Lemma 4.5 (e).

(e) It follows from Theorem 4.3 (c) and Lemma 4.5 (f). □

Note that in Theorem 4.8 we use (c)'s of Theorems 4.1, 4.2 and 4.3. Using (a),(b),(d),(e),(f),(g)(h),(i),(j)'s of Theorems 4.1, 4.2 and 4.3 we can establish other recurence relations.

Using Theorem 4.8 and comparing matrix entries, we have next result.

Theorem 4.9. *For generalized Narayana, Narayana, Narayana-Lucas and Narayana-Perrin numbers, we have the following identities:*

(a) $V_mV_{n+1} + V_{m-2}V_n + V_{m-1}V_{n-1} = V_{m+1}V_n + V_{m-1}V_{n-1} + V_mV_{n-2} = (V_1 - V_2)^2N_{m+n+4} + 2(V_2 - V_1)(V_0 + V_1 - V_2)N_{m+n+3} + (V_0^2 + 4V_0V_1 - 4V_0V_2 - V_1^2 + V_2^2)N_{m+n+2} + (2V_1 - 2V_0)(V_0 + V_1 - V_2)N_{m+n+1} + (V_0 - V_1)^2N_{m+n}.$

(b) $V_mV_{n+1} + V_{m-2}V_n + V_{m-1}V_{n-1} = V_{m+1}V_n + V_{m-1}V_{n-1} + V_mV_{n-2} = (V_2 - V_1)V_{m+n+2} + (V_0 + V_1 - V_2)V_{m+n+1} + (V_1 - V_0)V_{m+n}.$

(c) $N_mV_{n+1} + N_{m-2}V_n + N_{m-1}V_{n-1} = N_{m+1}V_n + N_{m-1}V_{n-1} + N_mV_{n-2} = (V_2 - V_1)N_{m+n+2} + (V_0 + V_1 - V_2)N_{m+n+1} + (V_1 - V_0)N_{m+n}.$

(d) $31(N_mV_{n+1} + N_{m-2}V_n + N_{m-1}V_{n-1}) = 31(N_{m+1}V_n + N_{m-1}V_{n-1} + N_mV_{n-2}) = (11V_1 - 3V_0 - 2V_2)U_{m+n+2} + (V_0 - 14V_1 + 11V_2)U_{m+n+1} + (11V_0 + V_1 - 3V_2)U_{m+n}.$

(e) $53(N_mV_{n+1} + N_{m-2}V_n + N_{m-1}V_{n-1}) = 53(N_{m+1}V_n + N_{m-1}V_{n-1} + N_mV_{n-2}) = (4V_0 + 15V_1 - 6V_2)H_{m+n+2} + (15V_0 - 10V_1 + 4V_2)H_{m+n} + (15V_2 - 11V_1 - 10V_0)H_{m+n+1}.$

Proof. By multiplying matrices and then by comparing the 2nd rows and 1st columns entries in Theorem 4.8 (a), we get the required identities in (a). The remaining of identities can be proved by considering again Theorem 4.8. □

Taking $V_n = N_n$ in Theorem 4.9, we obtain the following corollary.

Corollary 4.10. *For Narayana numbers, we have the following identities:*

(a) $N_mN_{n+1} + N_{m-2}N_n + N_{m-1}N_{n-1} = N_{m+1}N_n + N_{m-1}N_{n-1} + N_mN_{n-2} = N_{m+n}.$

(b) $31(N_mN_{n+1} + N_{m-2}N_n + N_{m-1}N_{n-1}) = 31(N_{m+1}N_n + N_{m-1}N_{n-1} + N_mN_{n-2}) = 9U_{m+n+2} - 3U_{m+n+1} - 2U_{m+n}.$

(c) $53(N_mN_{n+1}+N_{m-2}N_n+N_{m-1}N_{n-1}) = 53(N_{m+1}N_n+N_{m-1}N_{n-1}+N_mN_{n-2}) = 9H_{m+n+2}+4H_{m+n+1}-6H_{m+n}$.

Taking $V_n = U_n$ in Theorem 4.9, we get the following corollary.

Corollary 4.11. *For Narayana-Lucas numbers, we have the following identities:*

(a) $U_mU_{n+1}+U_{m-2}U_n+U_{m-1}U_{n-1} = U_{m+1}U_n+U_{m-1}U_{n-1}+U_mU_{n-2} = 9N_{m+n+2}-12N_{m+n+1}+4N_{m+n}$.

(b) $U_mU_{n+1}+U_{m-2}U_n+U_{m-1}U_{n-1} = U_{m+1}U_n+U_{m-1}U_{n-1}+U_mU_{n-2} = 3U_{m+n+1}-2U_{m+n}$.

(c) $N_mU_{n+1}+N_{m-2}U_n+N_{m-1}U_{n-1} = N_{m+1}U_n+N_{m-1}U_{n-1}+N_mU_{n-2} = 3N_{m+n+1}-2N_{m+n}$.

(d) $N_mU_{n+1}+N_{m-2}U_n+N_{m-1}U_{n-1} = N_{m+1}U_n+N_{m-1}U_{n-1}+N_mU_{n-2} = U_{m+n}$.

(e) $53(N_mU_{n+1}+N_{m-2}U_n+N_{m-1}U_{n-1}) = 53(N_{m+1}U_n+N_{m-1}U_{n-1}+N_mU_{n-2}) = 21H_{m+n+2}-26H_{m+n+1}+39H_{m+n}$.

Taking $V_n = H_n$ in Theorem 4.9, we obtain the following corollary.

Corollary 4.12. *For Narayana-Perrin numbers, we have the following identities:*

(a) $H_mH_{n+1}+H_{m-2}H_n+H_{m-1}H_{n-1} = H_{m+1}H_n+H_{m-1}H_{n-1}+H_mH_{n-2} = 4N_{m+n+4}+4N_{m+n+3}-11N_{m+n+2}-6N_{m+n+1}+9N_{m+n}$.

(b) $H_mH_{n+1}+H_{m-2}H_n+H_{m-1}H_{n-1} = H_{m+1}H_n+H_{m-1}H_{n-1}+H_mH_{n-2} = 2H_{m+n+2}+H_{m+n+1}-3H_{m+n}$.

(c) $N_mH_{n+1}+N_{m-2}H_n+N_{m-1}H_{n-1} = N_{m+1}H_n+N_{m-1}H_{n-1}+N_mH_{n-2} = 2N_{m+n+2}+N_{m+n+1}-3N_{m+n}$.

(d) $31(N_mH_{n+1}+N_{m-2}H_n+N_{m-1}H_{n-1}) = 31(N_{m+1}H_n+N_{m-1}H_{n-1}+N_mH_{n-2}) = -13U_{m+n+2}+25U_{m+n+1}+27U_{m+n}$.

(e) $N_mH_{n+1}+N_{m-2}H_n+N_{m-1}H_{n-1} = N_{m+1}H_n+N_{m-1}H_{n-1}+N_mH_{n-2} = H_{m+n}$.

The next two theorems provide us the convenience to obtain the powers of generalized Narayana, Narayana, Narayana-Lucas and Naraya-perrin matrix sequences.

Theorem 4.13. *For all integers m, n and r, the following identities hold:*

(a) $\mathcal{N}_n^m = \mathcal{N}_{mn}$,

(b) $\mathcal{N}_{n+1}^m = \mathcal{N}_1^m\mathcal{N}_{mn}$,

(c) $\mathcal{N}_{n-r}\mathcal{N}_{n+r} = \mathcal{N}_n^2 = \mathcal{N}_2^n$.

Proof. We prove for $m, n, r \geq 0$. The other cases can be proved similarly.

(a) We can write $\mathcal{N}_n^m$ as

$$\mathcal{N}_n^m = \mathcal{N}_n\mathcal{N}_n...\mathcal{N}_n \text{ } (m \text{ times}).$$

Using Theorem 4.5 (c) iteratively, we obtain the required result:

$$\begin{aligned}
\mathcal{N}_n^m &= \underbrace{\mathcal{N}_n\mathcal{N}_n...\mathcal{N}_n}_{m \text{ times}} \\
&= \mathcal{N}_{2n}\underbrace{\mathcal{N}_n\mathcal{N}_n...\mathcal{N}_n}_{m-1 \text{ times}} \\
&= \mathcal{N}_{3n}\underbrace{\mathcal{N}_n\mathcal{N}_n...\mathcal{N}_n}_{m-2 \text{ times}} \\
&\vdots \\
&= \mathcal{N}_{(m-1)n}\mathcal{N}_n \\
&= \mathcal{N}_{mn}.
\end{aligned}$$

(b) As a similar approach in (a) we have

$$\mathcal{N}_{n+1}^m = \mathcal{N}_{n+1}.\mathcal{N}_{n+1}...\mathcal{N}_{n+1} = \mathcal{N}_{m(n+1)} = \mathcal{N}_m\mathcal{N}_{mn} = \mathcal{N}_1\mathcal{N}_{m-1}\mathcal{N}_{mn}.$$

Using Theorem 4.5 (c), we can write iteratively $\mathcal{N}_m = \mathcal{N}_1\mathcal{N}_{m-1}$, $\mathcal{N}_{m-1} = \mathcal{N}_1\mathcal{N}_{m-2}$, ..., $\mathcal{N}_2 = \mathcal{N}_1\mathcal{N}_1$. Now it follows that

$$\mathcal{N}_{n+1}^m = \underbrace{\mathcal{N}_1\mathcal{N}_1...\mathcal{N}_1}_{m \text{ times}}\mathcal{N}_{mn} = \mathcal{N}_1^m\mathcal{N}_{mn}.$$

(c) Theorem 4.5 (c) gives

$$\mathcal{N}_{n-r}\mathcal{N}_{n+r} = \mathcal{N}_{2n} = \mathcal{N}_n\mathcal{N}_n = \mathcal{N}_n^2$$

and also

$$\mathcal{N}_{n-r}\mathcal{N}_{n+r} = \mathcal{N}_{2n} = \underbrace{\mathcal{N}_2\mathcal{N}_2...\mathcal{N}_2}_{n \text{ times}} = \mathcal{N}_2^n.$$

We have analogues results for the matrix sequence $\mathcal{V}_n$.

Theorem 4.14. *For all integers m, n and r, the following identities hold:*

(a) $\mathcal{V}_{n-r}\mathcal{V}_{n+r} = \mathcal{V}_n^2$,

(b) $\mathcal{V}_n^m = \mathcal{V}_0^m\mathcal{N}_{mn}$.

Proof.

(a) We use Binet's formula of generalized Narayana sequence which is given in Theorem 2.3. So

$$\begin{aligned}
& \mathcal{V}_{n-r}\mathcal{V}_{n+r} - \mathcal{V}_n^2 \\
= & (A\alpha^{n-r} + B\beta^{n-r} + C\gamma^{n-r})(A\alpha^{n+r} + B\beta^{n+r} + C\gamma^{n+r}) - (A\alpha^n + B\beta^n + C\gamma^n)^2 \\
= & AB\alpha^{n-r}\beta^{n-r}(\alpha^r - \beta^r)^2 + AC\alpha^{n-r}\gamma^{n-r}(\alpha^r - \gamma^r)^2 + BC\beta^{n-r}\gamma^{n-r}(\beta^r - \gamma^r)^2 \\
= & 0
\end{aligned}$$

since $AB = AC = BC = 0$ (see Lemma 4.4). Now we get the result as required.

(b) By Theorem 4.13, we have

$$\mathcal{V}_0^m\mathcal{N}_{mn} = \underbrace{\mathcal{V}_0\mathcal{V}_0...\mathcal{V}_0}_{m \text{ times}}\underbrace{\mathcal{N}_n\mathcal{N}_n...\mathcal{N}_n}_{m \text{ times}}.$$

When we apply Lemma 4.5 (b) iteratively, it follows that

$$\begin{aligned}
\mathcal{V}_0^m\mathcal{N}_{mn} &= (\mathcal{V}_0\mathcal{N}_n)(\mathcal{V}_0\mathcal{N}_n)...(\mathcal{V}_0\mathcal{N}_n) \\
&= \mathcal{V}_n\mathcal{V}_n...\mathcal{V}_n = \mathcal{V}_n^m.
\end{aligned}$$

This completes the proof. □

5 CONCLUSION

There have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas. Many authors use matrix methods in their work. On the other hand, the matrix sequences have taken so much interest for different type of numbers. See, for example, [29,18,19,27]. In this paper, we define the matrix sequence of generalized Narayana numbers. The method used in this paper can be used for the other linear recurrence sequences, too. It is our intention to continue the study and explore some properties of some type of matrix sequences of special numbers, such as matrix sequences of Hexanacci and Hexanacci-Lucas numbers.

In this paper, we obtain some fundamental properties on matrix sequence of generalized Narayana numbers. We can summarize the sections as follows:

- In section 1, we present some background about generalized Narayana numbers.
- In section 2, we define generalized Narayana matrix sequence and then the generating functions, the Binet formulas, and summation formulas over these new matrix sequence have been presented. We have written sum identities in terms of the generalized Narayana matrix sequence, and then we have presented the formulas as special cases the corresponding identity for the generalized Narayana sequence. All the listed identities in the theorem and corollary may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered. We can mention some applications of sum formulas. Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the numbers of the sequences.
- In section 3, we obtain some identities of generalized Narayana and Narayana, Narayana-Lucas and Narayana-Perrin numbers.
- In section 4, we show that there always exist interrelation between generalized Narayana, Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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