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On the Gradient-Hamiltonian Systems to the Derivation of Economic Multivariate Total Functions

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Authors' contributions

This work was carried out in collaboration among all authors. Author JAA designed the study, performed the mathematical analysis, handled the mathematical aspect of the literature searches and wrote the first draft of the manuscript. Authors KA and EB managed the analyses of the study in addition to the economic aspect of the literature searches. All authors read and approved the final manuscript.

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ABSTRACT

This paper highlights an application of Gradient or Hamiltonian (Grad-Ham) Systems in deriving multivariate total functions. The objective is to establish a relationship between Gradient or Hamiltonian systems and economic-oriented multivariate marginal functions, and demonstrate how they can significantly be applied to the derivation of economic multivariate total functions. The multivariate marginal functions are represented by the Grad-Ham systems of differential equations whose analytical solutions are based on the partial antiderivative technique. The paper establishes that all economic multivariate marginal functions can respectively be expressed as exact differential equations. It also uncovered that functions that can be optimized are conservative along their optimal paths and that these functions become the first integrals of their respective marginal systems. Finally, it introduces two model examples- one hypothetical and the other based on the Cobb-Douglas Production function- and presents their derivations thereof.

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1. INTRODUCTION

It has become a common objective of all economic agents to optimize such economic decision variables, among others, as prices, profits, output, utility and cost [1,2,3,4]. Marginal analysis has remained one of the indispensable techniques in setting criteria for determining the attainment of optimal values. In Microeconomic analysis, the principle of marginal analysis can mathematically be tackled by applying such key tools as graphs, algebra and calculus. The later forms our main analytical tool of marginal analysis in this paper.

It is an established fact from microeconomic principle that a necessary condition for the attainment of optimality occurs at the point of equality between marginal functions and zero. This means that at every optimal point, the condition $f^{'}(x) = 0$ is necessary, where $f(x)$ represents a total function defined for every $x \in [0, \infty]$ [2]. This condition defines the existence of first integral for $f(x)$ [5]. Thus, given any total function, it is very simple to derive the corresponding marginal function through the calculus of differentiation. Conversely, for a given marginal function, it is possible, through integral techniques, to derive the associated total function. For univariate functions [4] it is very simple to evaluate the derivatives in order to determine the associated marginal functions and conversely. However the situation is different for the case of multivariate functions [2,4,6] especially where the total functions must be derived from a known marginal functions. We reiterate the application of partial derivative [1,2,3,4,6] and partial integral techniques [7] to multivariate cases. To the best of our knowledge, most Economists have used partial derivatives to derive multivariate marginal functions (marginal systems) from multivariate total functions with little or no difficulties in unconstrained optimization. In cases of constrained optimization, the problem is simplified by introducing the Lagrange multiplier prior to the application of partial derivatives [2,3]. What appears to be a gap is the little emphasis placed on the converse case where economic multivariate total functions such as utility, output, cost, and so on, are derived from their respective multivariate marginal derivatives. Making an accomplishment in this direction constitutes the core objective of this paper.

We consider it an important duty to inform that the partial derivatives of the unconstraint multivariate objective total function or the derivatives of the Lagragian (in the case of constrained optimization) constitute the marginal functions. For instance, if we consider a multivariate total function of the form $f(x_1, x_2 \ldots, x_n)$ defined for all $x_i \in [0, \infty]; i = 1,2, \ldots, n$. The partial derivatives [1,2,3,4,6] of $f(x_1, x_2 \ldots, x_n)$ with respect to x_1, x_3 and x_n are respectively

$$\frac{\partial f(x_1, x_2 \ldots, x_n)}{\partial x_1} = \lim_{\tau \to 0} \frac{f(x_1 + \tau, x_2, x_3, \ldots, x_n) - f(x_1, x_2 \ldots, x_n)}{\tau} \quad (1)$$

$$\frac{\partial f(x_1, x_2 \ldots, x_n)}{\partial x_3} = \lim_{\tau \to 0} \frac{f(x_1, x_2, x_3 + \tau \ldots, x_n) - f(x_1, x_2 \ldots, x_n)}{\tau} \quad (2)$$

$$\frac{\partial f(x_1, x_2 \ldots, x_n)}{\partial x_n} = \lim_{\tau \to 0} \frac{f(x_1, x_2, x_3, \ldots, x_n + \tau) - f(x_1, x_2 \ldots, x_n)}{\tau} \quad (3)$$

It is important to note that the above derivatives are defined given that $\tau > 0$; and that the limit exists in (1), (2) and (3). In the field of constrained optimization, we obtain the Lagragian and its associated partial derivatives. We re-emphasize that partial derivatives (1), (2), (3) and that of the Lagragian represent marginal functions. Furthermore, the equivalence of the Hamiltonian system and the Lagragian (or marginal systems) has been established in [8,9], implying the possibility of deriving Hamiltonian functions or their orthogonal gradient functions (economic multivariate total functions) from given sets of marginal systems. As indicated in [10], gradient systems satisfy the conditions necessary for a differential equation to be exact. Since gradient systems are orthogonal to Hamiltonian systems, it also follows that Hamiltonian systems satisfy the conditions for exactness. A necessary condition for optimizing the function $f(x_1, x_2 \ldots, x_n)$ implies the point of equality between each of the three partial derivatives above (including that of the Lagragian) and zero (0). If these derivatives are known, it is the duty of economic analysts to determine the multivariate total functions whose derivatives are these marginal systems.

Our point of contention is that previous economic analysis has been silent on the derivation of multivariate total functions from their respective marginal systems. It is very common to come across materials or literature on univariate total functions being derived from univariate marginal functions but little or none on multivariate cases. In microeconomics, marginal analysis is done both at the univariate (single variable) and the multivariate (equimarginal principle) levels. However, at the multivariate level, emphasis has only been placed on deriving partial marginal functions associated with multivariate total functions. It is therefore our expectation that the equimarginal analysis be extended to deriving multivariate total functions from their respective marginal systems. This is what we intend to address by relating certain properties of marginal systems to gradient or Hamiltonian systems. Once multivariate marginal systems are given, we can use the procedure for finding solution to exact differential equations to derive the corresponding multivariate total functions which represent either gradient or Hamiltonian functions. Thus, the objective of this paper is to derive economic-based multivariate total functions from a given set of multivariate marginal systems leveraging the Grad-Ham systems.

The paper begins with some definitions and properties of gradient and Hamiltonian systems. It also describes the orthogonal relationship between the two systems. Following this is a demonstration of how gradient and Hamiltonian functions can be derived from respective gradient and Hamiltonian systems using an analytical integral technique via separation of variables. Further, it demonstrates a short proof which establishes the existence of exact differential equations for all economic marginal systems. It also proceeds to establish that total functions are conservative along their optimal paths. Finally, two examples of economic marginal systems are analyzed. The first system is hypothetical while the second is a system based on the Cobb-Douglas Production function.

2. MATERIALS AND METHODS

We present under this section the analytical framework that supports the problem we intend to analyze. Our analytical approach rests on the economic marginal systems that share similar properties with gradient or Hamiltonian systems of differential equations. The forgoing discussions is devoted to the review of gradient and Hamiltonian systems.

2.1 Gradient and Hamiltonian Systems

2.1.1 Gradient systems

Let E be an open subset of R^n and $V \in C^2(E)$ such that $V: R^n \to R$, then gradient systems are differential equations of the form

$$\dot{x} = -\nabla V(x) \tag{4}$$

where

$$\nabla V(x) = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n}\right)^T$$

There is a requirement that V be twice continuously differentiable to guarantee that the right hand side of (4) is continuously differentiable function of x [5, 10].

In general, a differential equation $\dot{x} = f(x) = [f_1(x), f_2(x), \dots, f_n(x)]$ is a gradient system if and only if there exists a scalar valued function $V(x)$ such that

$$-[f_1(x), f_2(x), \dots, f_n(x)] = \left[\frac{\partial V(x)}{\partial x_1}, \frac{\partial V(x)}{\partial x_2}, \cdots, \frac{\partial V(x)}{\partial x_n}\right] \tag{5}$$

In dimension $d = 1$, we have

$$\dot{x} = f(x)$$

Here, we always select V of $-f$ in order to guarantee that

$$\frac{dV}{dx} = -f(x)$$

In dimension $d = 2$, a system $\dot{x} = f(x, y), \dot{y} = g(x, y)$ is a gradient system if and only if there exists $V(x)$ subject to the condition that

$$\frac{\partial V(x, y)}{\partial x} = -f(x, y)$$

$$\frac{\partial V(x, y)}{\partial y} = -g(x, y)$$

To find the analytical solution on a ball $\{|x - x* < R \quad R = \infty$ permissible, a necessary and sufficient condition involving equality of the following crossed partial derivatives

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

must be satisfied. Thus for any given gradient system, there exist an exact differential equation [10,12] given by

$$f(x,y)dx + g(x,y)dy = 0$$

such that

$$V(x,y) = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy = 0$$

In general, the necessary and sufficient condition on balls is also represented by the equality of the crossed partial derivatives for all i, j such that

$$\frac{\partial f_i(x)}{\partial x_i} = \frac{\partial f_j(x)}{\partial y_j}; \{1 < i < j \leq n\}$$

We note the following properties of the gradient system as outlined in [10]:

- Equilibrium points of the gradient system correspond to the critical points of $V(\text{i.e.}, \nabla V(x) = 0)$
- At regular points of V (i.e., noncritical points) the gradient vector $\nabla V(x)$ is perpendicular to the level surface of $V(x) = \text{constant}$
- Strict local minimum of V correspond to asymptotically stable equilibrium points of the system

2.1.2 Hamiltonian systems

By definition [10,11], let E be an open subset of R^{2n} and $H \in C^2(E)$, where $H = H(x,y)$ with $x, y \in R^n$. Then, a Hamiltonian system with n degree of freedom on E , is a system of differential equations of the form

$$\dot{x} = \frac{\partial H}{\partial y}$$

$$\dot{y} = -\frac{\partial H}{\partial x}$$

Where

$$\frac{\partial H}{\partial x} = \left(\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, \cdots, \frac{\partial H}{\partial x_n}\right)^T$$

and

$$\frac{\partial H}{\partial y} = \left(\frac{\partial H}{\partial y_1}, \frac{\partial H}{\partial y_2}, \cdots, \frac{\partial H}{\partial y_n}\right)^T$$

We also outline some properties of the Hamiltonian system [5,10].

- The equilibrium points of the Hamiltonian system correspond to the critical points of H. An equilibrium point (a_x, a_y) is called non-degenerate if the determinant of the second derivative of H evaluated at the equilibrium point is nonzero, i.e., $\left|\frac{\partial^2 H(a_x,a_y)}{\partial (x,y)^2}\right| \neq 0$
- Hamiltonian systems are conservative, i.e., $H(x,y)$ remains constant along the trajectories. H is known as the first integral of the system. The trajectories of the system lie on the level surface of $H(x,y) = \text{constant}$
- If the second derivative of H (a symmetric $2n \times 2n$ matrix) evaluated at the equilibrium (a_x, a_y) has eigenvalues with positive real parts (positive definite), the equilibrium point is stable

2.1.3 Orthogonality to planar systems

We note the following from [10,11]:

- Non-degenerate critical points of a planar analytic gradient system (i.e., in R^n) are either a saddle or a node. If (x_0, y_0) is a saddle of V (i.e., the gradient function), it is a saddle of the system. Also, if (x_0, y_0) is a strict local maximum of V , it is an unstable node of the system. Last but not least, if (x_0, y_0) is a strict local minimum of V, it is a stable node of the system
- Given the planar system $\dot{x} = p(x,y)$; $\dot{y} = q(x,y)$, the system orthogonal to this is specified by $\dot{x} = q(x,y); \dot{y} = -p(x,y)$. A gradient system is orthogonal to a Hamiltonian system.

2.2 Derivation of Gradient and Hamiltonian Functions

We demonstrate here how to derive the gradient function V and the Hamiltonian function H from their respective systems. We will limit the demonstration to the case where dimension $d = 2$. We recall that if the system $\dot{x} = f(x,y), \dot{y} = g(x,y)$ is a gradient system, then, there exists $V(x)$ such that

$$\frac{\partial V(x,y)}{\partial x} = -f(x,y)$$

$$\frac{\partial V(x,y)}{\partial y} = -g(x,y)$$

The corresponding Hamiltonian system that is orthogonal to this gradient system is given by

$\dot{x} = g(x,y); \dot{y} = -f(x,y)$, such that

$$\dot{x} = \frac{\partial H}{\partial y} = g(x,y)$$

$$\dot{y} = -\frac{\partial H}{\partial x} = -f(x,y)$$

Since we have already established that these systems can form exact differential equations, we can now apply separation of variables [7] to derive the gradient and Hamiltonian functions $V(x,y)$ and $H(x,y)$ respectively.

To find the solution $V(x,y)$, let us consider for example

$$\frac{\partial V(x,y)}{\partial x} = -f(x,y)$$

Separating variables and integrating partially, we obtain

$$V(x,y) = -\int f(x,y)\partial x + h(y)$$

Where $h(y)$ is a constant in y. Now, we obtain

$$\frac{\partial V(x,y)}{\partial y} = -\frac{\partial[\int f(x,y)\partial x]}{\partial y} + h'(y)$$

Consequently, we have

$$h'(y) = g(x,y) - \frac{\partial[\int f(x,y)\partial x]}{\partial y}$$

$$\Rightarrow \ h(y) = \int \left(g(x,y) - \frac{\partial[\int f(x,y)\partial x]}{\partial y}\right) dy + C\,\{C = \text{constant}\}$$

The solution of the gradient function $V(x,y)$ is given as

$$V(x,y) = \int f(x,y)\partial x + \int \left\{g(x,y) - \frac{\partial[\int f(x,y)\partial x]}{\partial y}\right\} dy + C \quad (6)$$

It is important to note that choosing $\frac{\partial V(x,y)}{\partial y} = -g(x,y)$, will yield the solution

$$V(x,y) = \int g(x,y)\partial y + \int \left\{f(x,y) - \frac{\partial[\int g(x,y)\partial y]}{\partial x}\right\} dx + C \quad (7)$$

The results (6) and (7) are the same irrespective of the initial choice.

If we apply the above procedure to the Hamiltonian system, we obtain the corresponding Hamiltonian function as follows:

$$H(x,y) = \int f(x,y)\partial x + \int \left\{\frac{\partial[\int f(x,y)\partial x]}{\partial y} - g(x,y)\right\} dy + C \quad (8)$$

Or

$$H(x,y) = \int g(x,y)\partial y + \int \left\{\frac{\partial[\int f(x,y)\partial y]}{\partial x} - f(x,y)\right\} dx + C \quad (9)$$

2.3 Exactness of Economic Marginal Systems

Our duty here is to prove that for every economic marginal system there exists a corresponding exact differential equation.

Theorem 1

Giving, for example, a general marginal system for a gradient function $U(x,y)$ as follows:

$$MU_x = \frac{\partial U}{\partial x}$$

$$MU_y = \frac{\partial U}{\partial y}$$

Then, it can be proven that there exists an exact differential equation given by

$$MU_x(x,y)dx + MU_y(x,y)dy = 0 \quad (10)$$

Proof

To prove that (10) is exact, we choose an optimizable function $U(x,y)$ such that

$$\partial U(x,y) = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = 0 \quad (11)$$

Thus, at optimum this function is constant and its total derivative is given by (11) [1,6,10,12]. This property is very common with the conservative property of both Hamiltonian and gradient systems. Thus, economic optimizable functions are conservative along the path to optimality, which implies that $U(x,y)$ is a first integral of the marginal systems.

If we let $MU_x(x,y) = f(x,y)$ and $MU_y(x,y) = g(x,y)$, then we obtain as follows:

$$f(x,y) = \frac{\partial U}{\partial x}$$

and

$$g(x,y) = \frac{\partial U}{\partial y}$$

Now, we observe that

$$\frac{\partial f}{\partial y} = \frac{\partial [MU_x(x,y)]}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial^2 U}{\partial x \partial y}$$

Similarly,

$$\frac{\partial g}{\partial x} = \frac{\partial [MU_y(x,y)]}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right) = \frac{\partial^2 U}{\partial x \partial y}$$

Thus, we have shown that

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = \frac{\partial^2 U}{\partial x \partial y}$$

Or

$$\frac{\partial [MU_x(x,y)]}{\partial y} = \frac{\partial [MU_y(x,y)]}{\partial x}$$

This establishes that every economic marginal system has its corresponding exact differential equation given by (10).

Corollary 1

Define the marginal system for capital (K) and labour (L) by

$$MP_K = \frac{\partial Q}{\partial K} = 2L - 2K + 60$$

$$MP_L = \frac{\partial Q}{\partial L} = -6L + 2K + 380$$

Then from theorem 1, there exists an exact differential equation given by

$$(2L - 2K + 60)dK + (-6L + 2K + 380)dL = 0$$

This guarantees the existence of a first integral which is the total product function $Q(K,L)$.

Corollary 2

The Taylor marginal system for portfolio of an investment for assets A and B can be specified by

$$\dot{\mu_A} = \frac{\partial \mu_P}{\partial \mu_A} = x$$

$$\dot{\mu_B} = \frac{\partial \mu_P}{\partial \mu_B} = 1 - x$$

The variables are defined as follows:

> μ_P is the expected return on portfolio $P = f(A,B)$, μ_A and μ_B are the expected returns on assets A and B respectively, x represents the weight of asset A, while $1 - x$ is the weight of asset B. The consequence of theorem 1 on the above portfolio marginal system is the exact differential equation

$$x d\mu_A + (1 - x) d\mu_B = 0$$

Thus, it is possible to derive the function for the expected return $\mu_P(\mu_A, \mu_B)$ on portfolio for assets A and B.

3. RESULTS AND DISCUSSION

In this section, we present two examples of 2-dimensional marginal systems, one hypothetical and the other based on the Cob-Douglas Production function, which we have termed 'Cob-Douglas Marginal System'.

Our demonstration in establishing the existence of exact differential equations for economic marginal systems has uncovered the hidden properties that economic marginal systems share in common with gradient or Hamiltonian systems. For instance, (10) assists in explaining the fact that economic optimizable functions are conservative along the path to optimality. This means that any optimizable function $U(u,y)$ of the marginal systems becomes the first integral of the system and can be derived using partial antiderivative technique as can be applied to exact differential equations. Another property with respect to non-degenerate equilibrium points can be discovered by making further analysis on $\left|\frac{\partial^2 U}{\partial x \partial y}\right|$ at the equilibrium points. However, since this and other properties have no significant influence on our objective of deriving multivariate total functions from multivariate marginal systems we will not consider them.

Our point of contention is that economic analysis has been silent on the derivation of multivariate total functions from marginal systems. It is very common to come across materials or literature on univariate total functions being derived from univariate marginal functions but little or none on multivariate cases. In microeconomic, marginal analysis is done both at the univariate (single variable) and the multivariate (equimarginal principle) levels. However, at the multivariate level, emphasis has only been placed on deriving partial marginal functions associated with multivariate total functions. It is therefore our expectation that the equimarginal analysis be extended to deriving multivariate total functions from respective marginal systems. This is what we intend to do in this section.

3.1 Illustrative Example 1

Let us consider the marginal utility functions of commodities x and y, respectively specified by

$$MU_x = \frac{\partial U}{\partial x} = -x + y + 20$$

$$MU_y = \frac{\partial U}{\partial y} = -2y + x + 40$$

The task here involves obtaining the total utility function $U(x, y)$ if (for example) $x = 80,\ y = 60$ and $U(x, y) = 2250$. Given this utility system, we can obtain the utility function $U(x, y)$ by establishing an exact differential equation of the form (10). We then solve as illustrated.

$$U(x, y) = \int (-x + y + 20)\, \partial x$$

$$U(x, y) = -\frac{x^2}{2} + xy + 20x + h(y) \tag{12}$$

$$\frac{\partial U}{\partial y} = x + h'(y) \tag{13}$$

It is obvious that

$$-2y + x + 40 = x + h'(y)$$

$$\Rightarrow +h'(y) = -2y + 40$$

$$h(y) = \int (-2y + 40)\, dy$$

$$\Rightarrow h(y) = -y^2 + 40y + K \tag{14}$$

Substitute (14) into (12) to obtain

$$U(x, y) = -\frac{x^2}{2} + xy + 20x - y^2 + 40y + K$$

$$U(x, y) = 2250, x = 80\ and\ y = 60$$

$$2250 = -\frac{80^2}{2} + 80(60) + 20(80) - 60^2 + 40(60) + K$$

$$K = 2250 - 2000 = 250$$

$$U(x, y) = 250 - \frac{x^2}{2} - y^2 + xy + 20x + 40y$$

Thus, we have derived the utility function that satisfies the system in example 1. $U(x, y)$ now becomes the first integral of the above marginal utility system. Same result could be arrived at by applying (6), (7), (8) and (9). It is necessary to verify the equality between the partial derivatives of the derived utility function and that of the marginal utility system.

The derived utility function is an indicative of the analytic endowment of the hypothetical marginal utility system. We now consider a very brief qualitative analysis on the behaviour of the system.

By imposing the first and necessary condition for the attainment of equilibrium points on the marginal utility system, we obtain as thus:

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = 0$$

$$\Rightarrow \left.\begin{aligned} x - y &= 20 \\ -x + 2y &= 40 \end{aligned}\right\} \tag{15}$$

Simultaneous solution of (15) yields $x = 80$ and $y = 60$. The points $x = 80$ and $y = 60$ constitute the equilibrium or optimal solution of the corresponding multivariate total utility function $U(x, y)$ for the marginal utility system. To verify whether this optimal point (x, y) is a maximum or minimum we consider the sufficient condition (also known as the second order condition) as follows:

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=80, y=60} = -1 < 0$$

and

$$\left.\frac{\partial^2 U}{\partial y^2}\right|_{x=80, y=60} = -2 < 0$$

Since the second order test shows negative values for both $\frac{\partial^2 U}{\partial x^2}$ and $\frac{\partial^2 U}{\partial y^2}$, $(80,60)$ is obviously a maximum point. This means that total utility can be maximized by consuming 80 units of x and 60 unit of y. The resulting optimal (maximum) total utility is 2250 utils.

3.2 Illustrative Example 2: The Cob-Douglas Marginal System

In the next example we present a marginal system based on the Cobb-Douglas production function [1,4,6,12,13,14] hence we call it the Cobb-Douglas marginal system. For instance, the Cobb-Douglas marginal system is represented by

$$\dot{k} = \alpha A \left(\frac{k}{l}\right)^{\alpha-1}$$

$$\dot{l} = (1-\alpha) A \left(\frac{k}{l}\right)^{\alpha}$$

Here, k represents the level of capital, l the level of labour, $\alpha, \beta > 0, \alpha+\beta = 1$. A is an efficiency index representing the level of technology, α and β are the respective contributions (partial elasticities) of capital and labour to total product $Q(k,l)$. We intend to derive the total product function $Q(k,l)$that satisfies the above system.

Observe that the Cobb-Douglas system described above can be represented by the exact differential equation

$$\alpha A \left(\frac{k}{l}\right)^{\alpha-1} dk + (1-\alpha) A \left(\frac{k}{l}\right)^{\alpha} dl = 0$$

Where

$$\frac{\partial}{\partial l}\left[\alpha A \left(\frac{k}{l}\right)^{\alpha-1}\right] = \frac{\partial}{\partial k}\left[(1-\alpha) A \left(\frac{k}{l}\right)^{\alpha}\right] = \propto (1-\propto) A k^{\propto-1} l^{-\propto}$$

Equivalently, we have

$$\frac{\partial^2 Q}{\partial k \partial l} = \frac{\partial^2 Q}{\partial l \partial k} = \propto (1-\propto) A k^{\propto-1} l^{-\propto} \neq 0$$

From definition, the following are true:

$$\dot{k} = \frac{\partial Q(k,l)}{\partial k} = \alpha A \left(\frac{k}{l}\right)^{\alpha-1} = \alpha A k^{\alpha-1} l^{1-\alpha}$$

$$\dot{l} = \frac{\partial Q(k,l)}{\partial l} = (1-\alpha) A \left(\frac{k}{l}\right)^{\alpha} = (1-\alpha) A k^{\alpha} l^{-\alpha}$$

From $\dot{l}$, we obtain

$$Q(k,l) = \int (1-\alpha) A k^{\alpha} l^{-\alpha} \partial l$$

$$\Rightarrow Q(k,l) = A k^{\alpha} l^{1-\alpha} + h(k)$$

Here $h(k)$ is a constant in k. To determine this variable constant, we find the partial derivative with respect to k as follows:

$$\frac{\partial Q(k,l)}{\partial k} = \alpha A k^{\alpha-1} l^{1-\alpha} + h'(k)$$

From $\dot{k}$ above, we evaluate $h(k)$ as

$$h(k) = K$$

$$\Rightarrow Q(k,l) = A k^{\alpha} l^{1-\alpha} + k$$

Subject to the initial value condition $Q(0,0) = 0, k = 0$. Therefore, the Cobb-Douglas production function $Q(k,l)$ that satisfies the Cobb-Douglas system is given by

$$Q(k,l) = A k^{\alpha} l^{\beta};\ \beta = 1-\alpha$$

4. CONCLUSION

We have demonstrated how to derive multivariate economic total functions from their respective multivariate marginal systems by relating these multivariate economic marginal functions to Hamiltonian or gradient systems. For instance, we have introduced two multivariate marginal systems, one being hypothetical and the other based on the Cobb-Douglas Production function; and subsequently, derived their respective multivariate total functions. Additionally, we have been able to establish that all economic marginal systems can respectively be expressed as exact differential equations. Last but not least, we have demonstrated that the multivariate total functions derived from the marginal systems are conservative along the path to optimality.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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