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Sloshing of Heterogeneous Liquid in Partially Filled Tanks: Example of a Vibration System without Compactness

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Abstract

The effect of the density of the liquid on the sloshing in partially filled tanks is studied. The liquid is assumed to be almost-homogeneous (i.e. a liquid whose density in equilibrium is practically a linear function of the height, which differs very little from a constant). In this case the linearized Euler's equation of the liquid is presented and analyzed, the relevant operators are studied. The Weyl's criterion is used for computing the spectrum of the fundamental operator A_{11}. We obtain nonclassically spectrum with continuous part filling an interval.

Keywords: Heterogeneous liquid; sloshing; Weyl's criterion; essential spectrum.

Mathematics Subject Classification: 76B03, 35Q35.

Notations

In equilibrium position:

Ω	*Domain occupied by the liquid*
Ω'	*Domain occupied by the (elastic or rigid) body*
γ	*Horizontal free surface*
Σ	*Wall of the tank wetted by the liquid*
σ	*Wall of the tank wetted by the gas*

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$\vec{n}$	*Unit vector normal to* $\Sigma \cup \sigma$ *(and* γ *) and directed to the exterior of* Ω
g	*Acceleration of the gravity*
h	*The maximum height of the liquid*
ρ_0	*Density of the liquid*
P^0	*Constant pressure above the free surface*
$Ox_1x_2x_3$	*Orthogonal coordinate system (* Ox_1x_2 *being of the plan of* γ *and* Ox_3 *directed upwards)*

At the instant t:

$\vec{u}(x,t)$	*Displacement of a particle of the liquid from its equilibrium position*
$p(x,\mathrm{t})$	*Dynamic pressure of a particle of the liquid that occupies the position* $x(x_1,x_2,x_3)$
$\vec{u}'(x,t)$	*Displacement of a particle of the body from its equilibrium position*

1 Introduction

Liquid sloshing constitutes a broad class of problems of great practical importance with regard to the safety of liquid transportation systems, such as tank trucks on highways, liquid tank carriages on rail roads, ocean going vessels and propellant tanks in liquid rocket engines.

The overall observation from the experimental results presented in [1] gives an idea that sloshing in a tank is a function of various parameters such as liquid depth, the dimensions of the tank, the amplitude, and frequency of excitation and density of the liquid.

For sloshing of an inviscid or viscous homogeneous liquid in a rigid tank we refer to the pioneering book by Moïseyev and Rumiantsev [2], For the computational mechanics point of view, the reader may find in [3] appropriate variational formulations and associated finite element analysis for the linear liquid sloshing in elastic tanks. The case of a viscous homogeneous liquid has been studied by using computational methods in [4,5] and for a theoretical study of a viscous heterogeneous liquid we refer to the reference [6].

The general case of an heterogeneous inviscid liquid has not been studied yet, most of the researchers preferred a homogeneous liquid in their studies because the problem in the case of heterogeneous liquid is more complicated, in this case we obtain nonclassically spectrum (an essential spectrum appears).

In this work, we propose to investigate the three-dimensional linear sloshing problem of an incompressible inviscid liquid in partially filled tanks, taking into account the effects of the density of the liquid, which has usually neglected in practice [1], and has been the object of limeted specific studies before: Capodanno and its collaborators in the planar case [7,8,9].

Considering the particular case, introduced by Capodanno [7,8,9], of an *almost-homogeneous* liquid, i.e. whose density in equilibrium position is practically a linear function of the height, differing a little bit from a constant. This hypothesis modifies significantly the spectrum of the problem, and the main difficulty consists in studying and computing the spectrum of the fundamental operator A_{11} which appears in the linearized Euler's equation of the liquid.

Using Weyl's criterion [10], we show that $\sigma(A_{11}) = \sigma_{ess}(A_{11}) = [0, \beta g]$ and we argue that the presence of the essential part of the spectrum is due to the hypothesis of almost-homogeneity, in contrast to the classical case in which the fluid is homogeneous and the spectrum is entirely discrete [11].

2 Problem Statement

Let consider an elastic (or rigid) body that occupies a domain Ω' bounded by a regular closed fixed surface Γ and an regular closed internal surface (Fig. 1). The domain bounded by this surface is partially filled by an heavy incompressible inviscid liquid, that occupies in equilibrium position, a domain Ω bounded by a surface Σ and the horizontal free surface γ ; we denote by σ the part of the internal surface of the body that is above Ω and is wetted by a gas with constant pressure P^0 .

We use an orthogonal coordinate system $Ox_1x_2x_3$, Ox_1x_2 being of the plan of γ and Ox_3 directed upwards. The system is supposed at the constant temperature and in a constant gravity field $\vec{g} = -g\vec{x}_3$.

We study the small oscillations of the liquid about its equilibrium position in the framework of the linear theory.

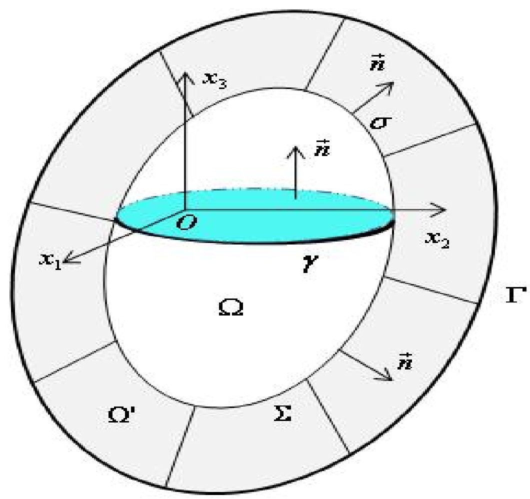

Fig. 1. Model of the system

3 The Case of the Almost Homogeneous Liquid

Let be h the maximum height of the liquid in the equilibrium position.

We suppose that density in equilibrium position in the form

$$\rho_0(x_3) = \rho(1 - \beta x_3) + o(\beta h),$$

where ρ and β are positive constants, β being sufficiently small so that $(\beta h)^2$, $(\beta h)^3$,…are negligible with respect to βh .

Then, the liquid is called **"*almost-homogeneous in* Ω "**.

Like in the Boussinesq approximation of the convective motions of the viscous liquids, the linearized equation of the liquid takes the form ([7,8]).

$$\rho\ddot{\vec{u}} = -\overrightarrow{\text{grad}}p - \rho\beta g u_3 \vec{x}_3 \quad \text{in} \quad \Omega \tag{1}$$

where $\vec{u}(x,t)$, $p(x,\text{t})$ are the displacement from the equilibrium of the particle that occupies the position $x(x_1,x_2,x_3)$ at the instant t, the dynamic pressure in this point.

4 Weyl's Decompositions

We suppose that $\vec{u}$ belong to the spaces

$$\vec{u} \in J(\Omega) = \left\{ \vec{u} \in \mathscr{L}^2(\Omega) \overset{\text{def}}{=} \left[L^2(\Omega)\right]^3; \quad \text{div}\,\vec{u} = 0 \right\},$$

we seek it in the form

$$\vec{u} = \vec{v} + \vec{U}$$

with

$$\vec{v} \in J_0(\Omega) = \left\{ \vec{v} \in \mathscr{L}^2(\Omega); \quad \text{div}\vec{u} = 0 \; ; u_{n|_{\Sigma\cup\gamma}} = 0 \right\}$$

$$G_h(\Omega) = \left\{ \vec{U} = \overrightarrow{\text{grad}}\Phi \; ; \quad \Phi \in H^1(\Omega) \; ; \int_\Omega \Phi d\Omega = 0; \; \text{div}\vec{U} = \Delta\Phi = 0 \right\}$$

In accordance to the orthogonal decomposition in $\mathscr{L}^2(\Omega)$ [11]

$$J(\Omega) = J_0(\Omega) \oplus G_h(\Omega)$$

Let us recall [11,13] that

$$\mathscr{L}^2(\Omega) = J_0(\Omega) \oplus G(\Omega),$$

where $G(\Omega)$ is the space of the potential fields and that

$$G(\Omega) = G_h(\Omega) \oplus G_0(\Omega)$$

where

$$G_0(\Omega) = \left\{ \overrightarrow{\text{grad}}\; q \; , \; q \in H_0^1(\Omega) \right\}.$$

5 Transformation of Euler's Equation

Let $\vec{u}'(x_1,x_2,x_3,t)$ the displacement of a particle of the body from its equilibrium position to its position at the instant t.

We introduce the space [11,12].

$$V=\left\{\begin{array}{l} W=\begin{pmatrix}\vec{u}'\\ \vec{U}\end{pmatrix}\ ;\ \vec{u}'\in\hat{\Xi}^1(\Omega')=\left\{\vec{u}'\in\Xi^1(\Omega');\ \vec{u}'_{|\Gamma}=0\right\};\\ \vec{U}=\overrightarrow{\mathrm{grad}}\Phi\ ;\ \Phi\in\tilde{H}^1(\Omega)=\left\{\Phi\in H^1(\Omega);\ \int_{\Sigma\cup\gamma}\Phi d(\partial\Omega)=0\right\};\\ \mathrm{div}\vec{U}=0\ ;\ U_{n|\gamma}\in L^2(\Gamma);\ U_{n|_\Sigma}=u'_{n|_\Sigma}\in H^{1/2}(\Sigma)\end{array}\right\},$$

equipped with the hilbertian norm defined by

$$\|W\|_V^2=\|\vec{u}'\|_1^2+\int_\Omega\left|\vec{U}\right|^2 d\Omega+\left\|U_{n|\gamma}\right\|_{L^2(\gamma)}^2+\left\|U_{n|_\Sigma}\right\|_{H^{1/2}(\Sigma)}^2,$$

and, setting

$$\tilde{W}=\begin{pmatrix}\vec{w}'\\ \tilde{\vec{U}}\end{pmatrix},$$

The space χ, completion of V for the norm associated to the scalar product

$$\left(W,\tilde{W}\right)_\chi=\int_{\Omega'}\rho'\vec{u}'\cdot\bar{\vec{w}}'d\Omega'+\int_\Omega\rho\vec{U}\cdot\bar{\tilde{\vec{U}}}\,d\Omega$$

The Euler's equation (1) can be written

$$\ddot{\vec{v}}+\ddot{\vec{U}}=-\frac{1}{\rho}\overrightarrow{\mathrm{grad}}p-\beta g v_3\vec{x}_3-\beta g U_3\vec{x}_3$$

Consequently, if P_0 is the orthogonal projector from $\mathscr{L}^2(\Omega)$ into $J_0(\Omega)$, we have

$$\ddot{\vec{v}}=-\beta g P_0\left(v_3\vec{x}_3\right)-\beta g P_0\left(U_3\vec{x}_3\right) \tag{2}$$

In order to obtain a definitive form of this equation, we introduce a few operators.

We set

$$\beta g P_0\left(v_3\vec{x}_3\right)=A_{11}\vec{v}\ \ ;\ \beta g P_0\left(U_3\vec{x}_3\right)=A_{12}W\,.$$

A_{11} (resp A_{12})is bounded from $J_0(\Omega)$ (resp χ) into $J_0(\Omega)$.

Then, the equation (2) can be written

$$\ddot{\vec{v}}+A_{11}\vec{v}+A_{12}W=0 \tag{3}$$

and we have

$$\int_\Omega \beta g\ v_3\bar{\vec{v}}_3 d\Omega = \left(A_{11}\vec{v},\vec{\bar{v}}\right)_{J_0(\Omega)} \quad ; \quad \int_\Omega \beta g\ U_3\bar{\vec{v}}_3 d\Omega = \left(A_{12}W,\vec{\bar{v}}\right)_{J_0(\Omega)}$$

A_{11} is self-adjoint and not negative. Its spectrum will be studied in the following paragraph.

On the other hand, we have for $\vec{v}\in J_0(\Omega)$, $\tilde{W}\in\chi$:

$$\left|\int_\Omega \beta g\ v_3\bar{\tilde{U}}_3 d\Omega\right| \le c_0 \|\vec{v}\|_{J_0(\Omega)} \left\|\vec{\tilde{U}}\right\|_{L^2(\Omega)} \le c_0' \|\vec{v}\|_{J_0(\Omega)} \left\|\tilde{W}\right\|_\chi ,$$

where c_0 and c_0' are suitable positive constants.

Therefore , we can write

$$\int_\Omega \beta g\ v_3\bar{\tilde{U}}_3 d\Omega = \left(A_{21}\vec{v},\tilde{W}\right)_\chi \tag{4}$$

A_{21} being bounded from $J_0(\Omega)$ into χ .

It is easy to see that A_{21} and A_{21} are mutually adjoint.

Indeed, we have

$$\left(A_{21}\vec{v},\tilde{W}\right)_\chi = \overline{\int_\Omega \beta g\ \tilde{U}_3\bar{v}_3 d\Omega} = \overline{\left(A_{12}\tilde{W},\vec{v}\right)}_{J_0(\Omega)} = \left(\vec{v},A_{12}\tilde{W}\right)_{J_0(\Omega)} \tag{5}$$

6 The Spectrum of the Operator A_{11}

In order to study the spectrum of the problem, it is necessary to study the spectrum of the self- adjoint operator A_{11} .

Since

$$0 \le \left(A_{11}\vec{v},\vec{v}\right)_{J_0(\Omega)} = \int_\Omega \beta g\ \left|v_3\right|^2 d\Omega \le \beta g \|\vec{v}\|^2_{J_0(\Omega)} ,$$

we have

$$\|A_{11}\| \le \beta g$$

and

$$\sigma\left(A_{11}\right) \subset \left[0,\beta g\right] .$$

We have the following

Theorem

Let $\sigma\left(A_{11}\right)$ the spectrum of the operator A_{11} and $\sigma_{ess}\left(A_{11}\right)$ its essential spectrum.

We have

$$\sigma\left(A_{11}\right)=\sigma_{ess}\left(A_{11}\right)=\left[0,\beta g\right] .$$

Proof:

We are inspired the proof given in the book [11] for the Coriolis operator.

Using Weyl's criterion [10], for each μ , $0<\mu<1$, we are going to construct a sequence $\left\{\vec{v}_k\right\}\in J_0\left(\Omega\right)$ such that

$$\frac{\left\|\frac{1}{\beta g}A_{11}\vec{v}_k-\mu\vec{v}_k\right\|_{J_0(\Omega)}}{\left\|\vec{v}_k\right\|_{J_0(\Omega)}}\to 0 \quad \text{when } k\to+\infty .$$

i) We must construct the sequence $\left\{\vec{v}_k\right\}$, so that we can calculate $A_{11}\vec{v}_k$.

We can set

$$A_{11}\vec{v}=P_0\left(\beta g v_3\vec{x}_3\right)=\beta g v_3\vec{x}_3-\frac{1}{\rho}\overrightarrow{\text{grad}}\varphi \text{ , since } \overrightarrow{\text{grad}}\varphi\in G\left(\Omega\right) .$$

Since $A_{11}\vec{v}\in J_0\left(\Omega\right)$, we have

$$\text{div}\left(A_{11}\vec{v}\right)=0 \text{ and consequently } \Delta\varphi=\rho\beta g\,\text{div}\left(v_3\vec{x}_3\right) ;$$

$$A_{11}\vec{v}\cdot\vec{n}=0 \text{ on } \partial\Omega \text{ and consequently } \frac{\partial\varphi}{\partial n}=\rho\beta g v_3\vec{x}_3\cdot\vec{n} \text{ on } \partial\Omega .$$

Then, if v_3 is known, φ is solution of a Neumann problem and we have $A_{11}\vec{v}$. We must choose $\vec{v}$ so that we can calculate explicitly φ.

ii) In order to construct Weyl's sequence, we take

$$\vec{v}=\begin{pmatrix} v_1=\dfrac{\partial\Delta q}{\partial x_3}+b\dfrac{\partial\Delta q}{\partial x_2} \\ v_2=\dfrac{\partial\Delta q}{\partial x_3}-b\dfrac{\partial\Delta q}{\partial x_1} \\ v_3=-\dfrac{\partial\Delta q}{\partial x_1}-\dfrac{\partial\Delta q}{\partial x_2} \end{pmatrix}, \tag{6}$$

where $q \in \mathscr{D}(\Omega)$ and b is a constant that will be determined in the following.

We have easily $\operatorname{div}\vec{v} = 0$ and $v_{n|_{\partial\Omega}} = 0$, and $\vec{v} \in J_0(\Omega)$.

On the other hand, we have

$$\operatorname{div}(v_3\vec{x}_3) = \Delta\left(-\frac{\partial^2 \Delta q}{\partial x_1 \partial x_3} - \frac{\partial^2 \Delta q}{\partial x_2 \partial x_3}\right).$$

Consequently, the Neumann problem for φ is

$$\begin{cases} \Delta\varphi = -\rho\beta g \Delta\left(\dfrac{\partial^2 \Delta q}{\partial x_1 \partial x_3} + \dfrac{\partial^2 \Delta q}{\partial x_2 \partial x_3}\right) & \text{in } \Omega, \\ \dfrac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{7}$$

The problem has the obvious solution

$$\varphi = -\rho\beta g\left(\frac{\partial^2 q}{\partial x_1 \partial x_3} + \frac{\partial^2 q}{\partial x_2 \partial x_3}\right),$$

so that we have

$$\frac{1}{\beta g} A_{11}\vec{v} = v_3\vec{x}_3 + \overrightarrow{\text{grad}}\left(\frac{\partial^2 q}{\partial x_1 \partial x_3} + \frac{\partial^2 q}{\partial x_2 \partial x_3}\right),$$

i.e

$$\frac{1}{\beta g} A_{11}\vec{v} = \begin{pmatrix} \dfrac{\partial^3 q}{\partial x_1^2 \partial x_2} + \dfrac{\partial^3 q}{\partial x_1 \partial x_2 \partial x_3} \\ \dfrac{\partial^3 q}{\partial x_1 \partial x_2 \partial x_3} + \dfrac{\partial^3 q}{\partial x_2^2 \partial x_3} \\ -\left(\dfrac{\partial \Delta q}{\partial x_1} + \dfrac{\partial^3 q}{\partial x_1 \partial x_2^2}\right) - \left(\dfrac{\partial \Delta q}{\partial x_2} - \dfrac{\partial^3 q}{\partial x_2 \partial x_3^2}\right) \end{pmatrix} \tag{8}$$

Now, we are going to construct Weyl's sequence $\{\vec{v}_k\}$.

At first, we consider a sequence $\{\vec{v}_{n\,m}\}$ with

$$q = q_{n\,m}(x) = e^{i(nx_1 + mx_3)}\psi(x), \tag{9}$$

with $\psi \in \mathscr{D}(\Omega)$ does not depend on n and m and is equal to1in a sphere $|x - x_0| < r \ \ (x_0 \in \Omega)$, r being sufficiently small so that this sphere is interior to Ω.

Calculating the first derivatives of $q_{n\,m}$ and the first derivatives of $\Delta q_{n\,m}$ and noting that the derivatives of ψ are bounded in Ω and equal to zero in the sphere $|x - x_0| < r$, where $\psi = 1$, by virtue of (6) we find

$$\begin{cases} v_{n\,m,1} = -im\left(m^2 + n^2\right)\psi e^{i(nx_1 + mx_3)} + O\left(n^2 + m^2\right) \\ v_{n\,m,2} = -im\left(m^2 + n^2\right)\psi e^{i(nx_1 + mx_3)} + bin\left(n^2 + m^2\right)\psi e^{i(nx_1 + mx_3)} + O\left(n^2 + m^2\right) \\ v_{n\,m,3} = in\left(m^2 + n^2\right)\psi e^{i(nx_1 + mx_3)} + O\left(n^2 + m^2\right) \end{cases} \tag{10}$$

where $\dfrac{O\left(n^2 + m^2\right)}{n^2 + m^2}$ is uniformly bounded in Ω and equal to zero in the sphere $|x - x_0| < 0$.

iii) Calculating the third derivatives of $q_{n\,m}$ and using (8), (9) we find

$$\frac{1}{\beta g} A_{11}\vec{v} = \begin{pmatrix} -in^2 m\psi e^{i(nx_1 + mx_3)} + O\left(n^2 + m^2\right) \\ O\left(n^2 + m^2\right) \\ in^3\psi e^{i(nx_1 + mx_3)} + O\left(n^2 + m^2\right) \end{pmatrix}$$

Then, it is easy to calculate the components of

$$\frac{1}{\beta g} A_{11}\vec{v}_{n\,m} - \lambda \vec{v}_{n\,m}$$

and to prove that they are $O\left(n^2 + m^2\right)$ if we choose

$$b = \frac{m}{n} \ ; \quad \lambda = \frac{n^2}{n^2 + m^2}$$

So, we have

$$v_{n\,m,2} = O\left(n^2 + m^2\right) ,$$

the other components are unchanged and

$$\frac{1}{\beta g} A_{11}\vec{v}_{n\,m} - \frac{n^2}{n^2 + m^2}\vec{v}_{n\,m} = O\left(n^2 + m^2\right) \tag{11}$$

iv) For applying Weyl's criterion, we must estimate $\left\|\vec{v}_{n\,m}\right\|_{J_0(\Omega)}$.

We have

$$\left|\vec{v}_{n\,m}\right|^2 = \left(n^2+m^2\right)^3 \left|\psi\right|^2 + O\left(\left(n^2+m^2\right)^{5/2}\right) \tag{12}$$

and consequently

$$\left|\vec{v}_{n\,m}\right|^2 \le c_1'\left(n^2+m^2\right)^3 \quad , \quad \left(c_1' > 0\right),$$

so that

$$\left\|\vec{v}_{n\,m}\right\|^2_{J_0(\Omega)} \le c_2'\left(n^2+m^2\right)^3 \quad , \quad \left(c_2' = c_1'\left(meas\Omega\right)\right) .$$

In the sphere , where $\psi = 1$, $O\left(n^2+m^2\right) = 0$, and by virtue of (12) we have

$$\left|\vec{v}_{n\,m}\right|^2 = \left(n^2+m^2\right)^3 \tag{13}$$

and then

$$\left\|\vec{v}_{n\,m}\right\|^2_{J_0(\Omega)} \ge \int_{|x-x_0|\le r} \left|\vec{v}_{n\,m}\right|^2 d\Omega = c_0'\left(n^2+m^2\right)^3 \ , \left(c_0' = \frac{4\pi r^3}{3}\right)$$

Finally, we are obtain the double-sided estimate

$$c_0'\left(n^2+m^2\right)^3 \le \left\|\vec{v}_{n\,m}\right\|^2_{J_0(\Omega)} \le c_2'\left(n^2+m^2\right)^3$$

Let $\mu \in \left]0,1\right[$. For every $\varepsilon > 0$, we can find a rational number $\dfrac{\overline{m}}{\overline{n}}$ such that

$$\mu < \frac{\overline{n}^2}{\overline{n}^2+\overline{m}^2} = \frac{1}{1+\left(\dfrac{\overline{m}}{\overline{n}}\right)^2} < \mu+\varepsilon$$

$\overline{m}$ and $\overline{n}$ are defined by μ and ε .

Choosing $m = k\overline{m}$, k integer, we have

$$\mu < \frac{n^2}{n^2+m^2} < \mu+\varepsilon$$

We have

$$\left\|\frac{1}{\beta g}A_{11}\vec{v}_{n\,m} - \mu\vec{v}_{n\,m}\right\|_{J_0(\Omega)} \le \left\|\frac{1}{\beta g}A_{11}\vec{v}_{n\,m} - \frac{n^2}{n^2+m^2}\vec{v}_{n\,m}\right\|_{J_0(\Omega)} + \left(\frac{n^2}{n^2+m^2} - \mu\right)\left\|\vec{v}_{n\,m}\right\|^2_{J_0(\Omega)}$$

Since

$$\frac{1}{\beta g}A_{11}\vec{v}_{n\,m}-\frac{n^2}{n^2+m^2}\vec{v}_{n\,m}=O\left(n^2+m^2\right),$$

we can write

$$\left\|\frac{1}{\beta g}A_{11}\vec{v}_{n\,m}-\frac{n^2}{n^2+m^2}\vec{v}_{n\,m}\right\|_{J_0(\Omega)}\leq c_3'\left(n^2+m^2\right),\quad \left(c_3'>0\right)$$

so that

$$\left\|\frac{1}{\beta g}A_{11}\vec{v}_{n\,m}-\mu\vec{v}_{n\,m}\right\|_{J_0(\Omega)}\leq c_3'\left(n^2+m^2\right)+\varepsilon\sqrt{c_2'}\left(n^2+m^2\right)^{3/2}$$

Using the inequality

$$\left\|\vec{v}_{n\,m}\right\|_{J_0(\Omega)}\geq\sqrt{c_0'}\left(n^2+m^2\right)^{3/2},$$

we obtain

$$\frac{\left\|\frac{1}{\beta g}A_{11}\vec{v}_{n\,m}-\mu\vec{v}_{n\,m}\right\|_{J_0(\Omega)}}{\left\|\vec{v}_{n\,m}\right\|_{J_0(\Omega)}}\leq\left(\frac{c_3'}{\sqrt{c_0'}}\frac{1}{\sqrt{\bar{n}^2+\bar{m}^2}}\right)\frac{1}{k}+\varepsilon\sqrt{\frac{c_2'}{c_0'}}$$

The first term of the right-hand side tends to zero when $k\to+\infty$, so that, for k sufficiently large, we have

$$\frac{\left\|\frac{1}{\beta g}A_{11}\vec{v}_{k\bar{n},\,k\bar{m}}-\mu\vec{v}_{k\bar{n},\,k\bar{m}}\right\|_{J_0(\Omega)}}{\left\|\vec{v}_{k\bar{n},\,k\bar{m}}\right\|_{J_0(\Omega)}}\leq 2\varepsilon\sqrt{\frac{c_2'}{c_0'}} \tag{14}$$

Consequently, the sequences $\left\{\vec{v}_{k\bar{n},\,k\bar{m}}\right\}$ is Weyl's sequences.

Finally we have $\mu\beta g\in\left]0,\beta g\right[\Rightarrow\mu\beta g\in\sigma\left(A_{11}\right)$

Then, since $\sigma\left(A_{11}\right)$ is closed, we have $\left[0,\beta g\right]\subset\sigma\left(A_{11}\right)$, and then $\sigma\left(A_{11}\right)=\left[0,\beta g\right]$.

Consequently, there is not a discrete spectrum, so that the spectrum of A_{11} coincides with its essential spectrum $\sigma_{ess}\left(A_{11}\right)$: $\sigma\left(A_{11}\right)=\sigma_{ess}\left(A_{11}\right)=\left[0,\beta g\right]$ and, obviously $\left\|A_{11}\right\|=\beta g$.

Remark

In another work, using variational equation of the coupled system (liquid – tank) and Euler's equation in the three dimensional case, we argue that the problem have a discrete spectrum comprised of a countable set of positive real eigenvalues, whose accumulation point is the infinity in the domain $\left]\boldsymbol{\beta g},+\infty\right[$: [7,8].

7 Conclusions

i) The spectrum of the problem is composed by an essential part, which fills the closed interval $[0, \beta g]$, and a discrete part that lies outside this interval and is comprised of a countable set of positive real eigenvalues, whose accumulation point is the infinity.
ii) *Physically, the interval* $[0, \beta g]$ *is a domain of resonance.* and a system studied present high risk of instability.

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Competing Interests

Authors have declared that no competing interests exist.

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