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On Robustness of a Sequential Test for Scale Parameter of Gamma and Exponential Distributions

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Abstract

The main aim of the present paper is to study the robustness of the developed sequential probability ratio test (SPRT) for testing the hypothesis about scale parameter of gamma distribution with known shape parameter and exponential distribution with location parameter. The robustness of the SPRT for scale parameter of gamma distribution is studied when the shape parameter has undergone a change. The similar study is conducted for the scale parameter of exponential distribution when the location parameter has undergone a change. The expressions for operating characteristic and average sample number functions are derived. It is found in both the cases that the SPRT is robust only when there is a slight variation in the shape and location parameter in the respective distributions.

1. Introduction

The robustness of sequential probability ratio test (SPRT) has been studied by several authors for various probability distributions when the distribution under consideration has undergone a change. Barlow and Proschan [1], Harter and Moore [2], Montagne and Singpurwalla [3], and others have studied this problem for various probability models.

In this paper, the problem of testing simple hypothesis against simple alternatives for scale parameter of the gamma distribution assuming shape parameter to be known is considered. The gamma distribution plays important role in many areas of the Statistics including areas of life testing and reliability. It is used to make realistic adjustment to exponential distribution in life-testing situations. The fact that a sum of independent exponentially distributed random variables has a gamma distribution, leads to the appearance of gamma distribution in the theory of random counters and other topics associated with precipitation processes.

In Section 2, we state the problem and develop SPRT for testing of hypothesis giving expressions for Operating Characteristic (OC) and Average Sample Number (ASN) functions. In Section 3, robustness of the developed SPRT with respect to OC functions when the distribution considered here has undergone a change in the shape parameter, has been studied. In Section 4 we have studied the robustness of SPRT for the scale parameter of exponential distribution with respect to OC function when the location parameter has undergone a change.

2. Materials and Methods

The set-up of the problem and SPRT
Let X_1, X_2, ….. be a sequence of random variables from a gamma distribution with scale parameter θ (> 0) and shape parameter λ (> 0), whose density function is given by

$$f\left(x:\theta,\lambda\right) = \frac{x^{\lambda-1}e^{-x/\theta}}{\theta^{\lambda}\Gamma\lambda}, x > 0 \qquad (1)$$

where it is assumed that λ is known. Suppose we want to test the null hypothesis H_0: $\theta = \theta_0$ against the alternative H_1: $\theta = \theta_1$ $(> \theta_0)$. For this problem following SPRT is developed.

$$\text{Let } Z = \text{In}\frac{f(x;\theta_1)}{f(x;\theta_0)} = \lambda \text{In}\left(\frac{\theta_0}{\theta_1}\right) + \left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)x \qquad (2)$$

Two numbers A and B $(0 < B < 1 < A)$ are chosen. At

nth stage accept H_0 if $\sum_{i=1}^{n} z_i \leq \ln B$, reject H_0 if

$\sum_{i=1}^{n} z_i \geq \ln A$, otherwise, continue sampling by taking (n + 1) th observation. Here Z_i is obtained by replacing X by X_i in (2). Let (α, β) be the desired strength of SPRT, then according Wald [4], A and B are approximately given by

$$A \approx \frac{1-\beta}{\alpha}, B \approx \frac{\beta}{1-\alpha}, \text{ where } \alpha \in (0,1) \text{ and } \beta \in (0,1).$$

The Operating Characteristic (OC) function $L(\theta)$ is given by

$$L(\theta) \frac{A^{h(\theta)}-1}{A^{h(\theta)}-B^{h(\theta)}} \tag{3}$$

Where $h(\theta)$ is the non-zero solution of

$$E\left(e^{h(\theta)z}\right) = 1 \tag{4}$$

From (1) and (2), since

$$E\left(e^{h(\theta)z}\right) = \left\{1-h(\theta)\theta\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)\right\}^{-\lambda} \left(\frac{\theta_0}{\theta_1}\right)^{h(\theta)\lambda}$$

we get from (4) that

$$\theta = \frac{1-\left(\frac{\theta_0}{\theta_1}\right)^{h(\theta)}}{h(\theta)\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)} \tag{5}$$

The Average Sample Number (ASN) function is given by

$$E(N \mid \theta) = \frac{L(\theta)\ln B + [1-L(\theta)]\ln A}{E(Z)} \tag{6}$$

Provided $E(Z) \neq 0$,
where

$$E(Z) = \lambda\left[\ln\left(\frac{\theta_0}{\theta_1}\right) + \theta\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)\right] \tag{7}$$

The maximum value of ASN occurs in the neighbourhood of $\theta = \tilde{\theta}$, say where $\tilde{\theta}$ is the solution of $E_{\tilde{\theta}}(Z) = 0$ and this value is given by

$$E_{\tilde{\theta}}(N) = \frac{-\ln A . \ln B}{E_{\tilde{\theta}}(Z^2)} \tag{8}$$

It is easy to see that

$$\tilde{\theta} = \frac{\ln(\theta_1/\theta_0)}{\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)} \tag{9}$$

and

$$E_{\tilde{\theta}}(Z^2) = \lambda\left\{\ln\left(\frac{\theta_0}{\theta_1}\right)\right\}^2 \tag{10}$$

Table 1 contains the values of $E_0(N)$, $E_1(N)$ and $E_{\tilde{\theta}}(N)$ for different values of α, β and ratios of and ratios of θ_0 and θ_1.

3. Results and Discussions

3.1. Robustness of the SPRT for Scale Parameter of Gamma Distribution

Let us suppose that there is misspecification for the shape parameter λ in the probability distribution. Then the pdf (1) becomes $f(x; \theta, \lambda^*)$. To study the robustness of the SPRT developed in Section 2 with respect to OC function, consider $h(\theta)$ as the solution of the equation.

$$E_{\lambda^*}\left(e^{h(\theta)z}\right) = 1$$

i.e. $\int_0^\infty \left[\frac{f(x;\theta_1,\lambda)}{f(x;\theta_0,\lambda)}\right]^{h(\theta)} f(x;\theta,\lambda^*)dx = 1$

giving

$$\theta = \frac{1-\left(\frac{\theta_0}{\theta_1}\right)^{\frac{\lambda}{\lambda^*}h(\theta)}}{h(\theta)\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)} \tag{11}$$

For different values of θ, $h(\theta)$ is evaluated and the OC function is obtained. The robustness of SPRT with respect to ASN function can be studied by replacing the denominator of (7) by

$$E_{\lambda^*}(Z) = \int_0^\infty zf(x;\theta,\lambda^*)dx = \lambda \ln\left(\frac{\theta_0}{\theta_1}\right) + \left(\frac{1}{\theta_0}+\frac{1}{\theta_1}\right)\theta\lambda^* \tag{12}$$

We consider the cases $\lambda > \lambda^*$ and $\lambda < \lambda^*$ to study the robustness of the SPRT. In **Table 2**, we present the ASN function for the cases $\lambda > \lambda^*$ for the SPRT of testing the hypothesis $H_0: \theta = 45$ against $H_1: \theta = 50$. The values of OC function for the cases $\lambda > \lambda^*$ and $\lambda < \lambda^*$ respectively are plotted **Figure 1** and **Figure 2**.

3.2. Robustness of SPRT for Exponential Distribution

The random variable X is said to follow exponential distribution with location parameter θ and scale parameter λ

Table 1. ASN values of SPRT for scale parameter of gamma distribution.

| $\lambda = 5, \alpha = 0.01, \beta = 0.01$ | | | | $\lambda = 5, \alpha = 0.05, \beta = 0.05$ | | | |
θ_0/θ_1	$E_0(N)$	$E_1(N)$	$E_\theta(N)$	θ_0/θ_1	$E_0(N)$	$E_1(N)$	$E_\theta(N)$
0.1	0.64	0.13	0.80	0.1	0.38	0.08	0.33
0.2	1.11	0.38	1.63	0.2	0.65	0.22	0.67
0.3	1.79	0.80	2.91	0.3	1.05	0.47	1.20
0.4	2.85	1.54	5.03	0.4	1.68	0.91	2.07
0.5	4.66	2.94	8.79	0.5	2.74	1.73	3.61
0.6	8.13	5.78	16.18	0.6	4.78	3.40	16.64
0.7	15.89	12.53	33.20	0.7	19.35	17.37	13.63
0.8	38.92	33.54	84.81	0.8	22.90	19.73	34.82
0.9	168.02	156.62	380.42	0.9	198.87	192.16	156.20
$\lambda = 5, \alpha = 0.01, \beta = 0.01$				$\lambda = 5, \alpha = 0.05, \beta = 0.05$			
θ_0/θ_1	$E_0(N)$	$E_1(N)$	$E_\theta(N)$	θ_0/θ_1	$E_0(N)$	$E_1(N)$	$E_\theta(N)$
0.1	0.41	0.12	0.51	0.1	0.60	0.09	0.51
0.2	0.72	0.35	1.05	0.2	1.03	0.24	1.05
0.3	1.15	0.74	1.88	0.3	1.66	0.52	1.88
0.4	1.84	1.43	3.24	0.4	2.64	1.00	3.24
0.5	3.01	2.72	5.66	0.5	4.33	1.90	5.66
0.6	5.25	5.36	10.42	0.6	7.54	3.73	10.42
0.7	10.27	11.62	21.38	0.7	14.74	18.10	21.38
0.8	25.15	31.11	54.61	0.8	36.10	21.67	54.61
0.9	108.58	145.27	244.96	0.9	155.84	101.22	244.96

Table 2. ASN function for $\lambda < \lambda^*$ (H$_0$: $\theta = 45$, H$_1$: $\theta = 50$, $\alpha = \beta = 0.05$, $\lambda = 5$.

θ/λ^*	5.0	5.05	5.10	5.15	5.20	5.25
41.00	41.30	44.11	47.32	51.03	55.33	60.39
42.00	48.87	52.92	57.66	63.26	69.93	77.94
43.00	59.62	65.72	73.02	81.81	92.38	104.97
44.00	75.50	84.94	96.24	109.58	124.72	140.68
45.00	99.02	112.80	128.09	143.50	156.49	163.82
46.00	129.74	144.44	155.87	160.87	157.93	147.84
47.00	154.07	157.58	153.15	142.24	127.85	112.80
48.00	149.13	137.97	123.70	109.01	95.54	83.89
49.00	120.96	106.59	93.41	82.03	72.48	64.57
50.00	92.35	81.06	71.59	63.74	57.25	51.83
51.00	71.40	63.49	56.95	51.51	46.96	43.11
52.00	57.12	51.57	46.94	43.03	39.70	36.85
53.00	47.25	43.23	39.82	36.89	34.36	32.16
54.00	40.18	37.15	34.54	32.27	30.28	28.52
55.00	34.91	32.55	30.49	28.68	27.06	25.62

if its probability density function is given by

$$f(x; \theta, \lambda) = \lambda e^{-\lambda(x-\theta)}, \; x > 0, \; \lambda > 0 \qquad (13)$$

where it is assumed that θ is known.

Given a sequence of observations $X_1, X_2, \ldots,$ from (13), we wish to test the null hypothesis H$_0$: $\lambda = \lambda_0$ against the alternative H$_1$: $\lambda = \lambda_1 \, (> \lambda_0)$. We propose the following SPRT.

$$\text{Let } Z = \ln\left\{\frac{f(x; \lambda_1)}{f(x; \lambda_0)}\right\} = \ln\left(\frac{\lambda_1}{\lambda_0}\right) - (\lambda_1 - \lambda_0)(x - \theta) \quad (14)$$

We choose two numbers A and B such that $0 < B < 1 < A$. At n^{th} stage, accept H$_0$ if $\sum_{i=1}^{n} z_i \leq \ln B$, reject H$_0$ if $\sum_{i=1}^{n} Z_i \geq \ln A$, otherwise continue sampling by taking $(n + 1)^{\text{th}}$ observation. Here Z_i is obtained by replacing x by x_i in (3.2). If α and β are the probabilities of type I and type II errors respectively, then $A \cong \frac{1-\beta}{\alpha}, B \cong \frac{\beta}{1-\alpha}$ The OC function $L(\lambda)$ is given by
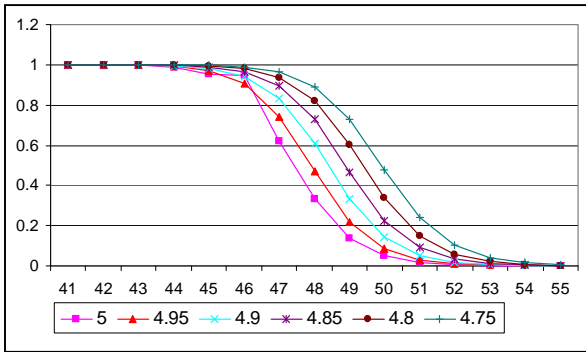
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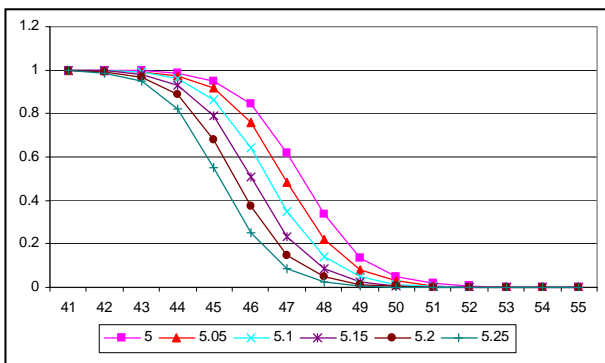**Figure 1. Graph of OC function for gamma distribution $\lambda < \lambda^*$.**

Figure 2. Graph of OC function for gamma distribution $\lambda > \lambda^*$.

Figure 3. Graph of OC function for exponential distribution $\theta < \theta^*$.
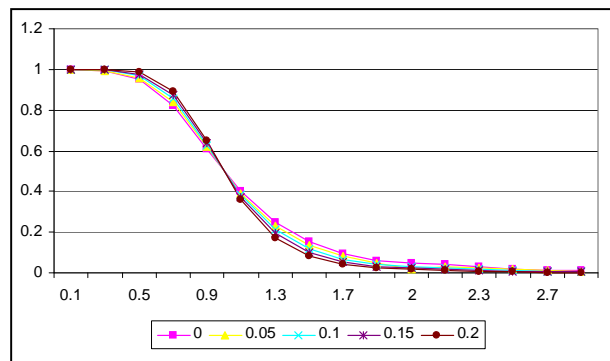
Figure 4. Graph of OC function for exponential distribution $\theta > \theta^*$.

$$L(\lambda) \approx \frac{A^{h(\lambda)} - 1}{A^{h(\lambda)} - B^{h(\lambda)}}, \tag{15}$$

Where $h(\lambda)$ is the solution of $E\left[e^{h(\lambda)Z}\right] = 1$.

Here, $E\left[e^{h(\lambda)Z}\right] = \lambda\left(\dfrac{\lambda_1}{\lambda_0}\right)^{h(\lambda)}$ and hence we obtain

$$\lambda = \left(\frac{\lambda_0}{\lambda_1}\right)^{h(\lambda)}.$$

Let is suppose that the location parameter θ has undergone a change and the pdf (3.1) becomes $f(x;\theta^*;\lambda)$. To study the robustness of the SPRT developed here with respect to OC function, we consider $h(\lambda)$ as the solution of the equation

$$\int_0^{\infty}\left[\frac{f(x;\theta,\lambda_1)}{f(x;\theta,\lambda_0)}\right]^{h(\lambda)} f(x;\theta^*;\lambda)\,dx = 1$$

giving

$$h(\lambda) = \frac{-\ln\lambda}{(\lambda_1 - \lambda_0)(\theta^* - \theta) + \ln\left(\dfrac{\lambda_1}{\lambda_0}\right)} \tag{16}$$
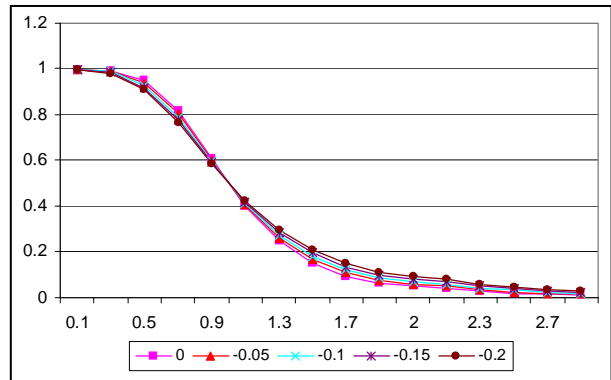
For different values of λ, $h(\lambda)$ is evaluated and OC function is obtained with the help of Equation (15).

The values of OC function for the cases $\lambda < \lambda^*$ and $\lambda > \lambda^*$ are plotted in **Figures 3** & **4** respectively.

4. Conclusions

Some Remarks

1) The values of OC function for the cases $\lambda > \lambda^*$ and $\lambda < \lambda^*$ respectively are plotted **Figure 1** and **Figure 2**. The figures indicate that for $\lambda > \lambda^*$ ($\lambda < \lambda^*$), the OC curve shifts to the right side (left side) of the curve for when $\lambda = \lambda^*$. From the figure it is clear that SPRT is robust for $\lambda^* = \lambda \pm 0.05$ as the deviation in OC function is insignificant. However, for other values of λ^* the test becomes highly non-robust.

2) We have considered the SPRT for testing the hypothesis H_0: $\lambda = 1$ versus H_1: $\lambda = 2$ for $\alpha = \beta = 0.05$. The values of OC function for the cases $\lambda < \lambda^*$ and $\lambda > \lambda^*$ are plotted in **Figures 3** & **4** respectively. It is exhibited from the table that the SPRT is non-robust even for $\lambda^* = \lambda \pm 0.05$. The ASN function for the case $\lambda < \lambda^*$ is tabulated in **Table 2**.

5. References

[1] R. E. Barlow and E. Proschan, "Exponential Life Test Procedures When the Distribution under Test has Monotone Failure Rate," *Journal of American Statistic Association*, Vol. 62, No. 318, 1967, pp. 548-560.

[2] L. Harter and A. H. Moore, "An Evaluation of Exponential and Weibull Test Plans," *IEEE Transactions on Reliability*, Vol. 25, No. 2, 1976, pp. 100-104.

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[4] A. Wald, "Sequential Analysis," John Wiley, New York, 1947.