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Optimal control model for the outbreak of cholera in Nigeria

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In this work two mathematical models that described the dynamics of cholera in Nigeria were presented. The first model examined the bacteria population using a logistic definition for its growth in the expected habitat and their interaction with the susceptible population. The second model is an optimal control model that includes two time- dependent control functions with one minimizing the contact between the susceptible and the bacteria and the other, the population of the bacteria in the water. The results from the numerical solutions of the models presented showed that increasing the susceptible pool and the infected population above some threshold values were responsible for epidemic cholera. It also showed that the difference between the growth rate (r) and the loss rate (n) of the bacteria plays a huge role in the outbreak as well as the severity of the disease.

Key words: Cholera, mathematical model, optimal control model, numerical solutions.

INTRODUCTION

Cholera has become a worldwide health problem. It is an acute infection caused by the colonization and multiplication of *Vibrio cholerae* 01 or 0139 within the small intestine in humans (Emch et al., 2008). It is a waterborne disease that causes severe diarrhea and vomiting which leads to dehydration of the body and can prove fatal unless treated quickly. Outbreaks result from contaminated food, poor sanitation and dirty drinking water. (Codeco, 2001; Isere et al., 2009)

Until the 20th century, cholera was confined to the Indian sub-continent. From this region, cholera has spread throughout the world seven times since 1817 (Capasso and Paveri–Fontana, 1979; Codeco, 2001; Lawoyin et al., 1999). The disease spread through the Asian continent during the 1960's; reached Africa in 1970 and Latin America in 1991 (Codeco, 2001; Lawoyin et al., 2004; Isere and Osemwenkhae, 2010).

In Nigeria, outbreaks of the disease have been occurring with increasing frequency since the first outbreak in modern times in 1970 (Epstein, 1993 Osemwenkhae et al., 2009). Since then, cholera has continued to cause high mortality in humans, in Nigeria. The year 1999 saw the highest number of reported cases (WHO, 2009). Since then, cholera cases have been

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persistent in the country. Recently, in Kano, on September 22nd 2008, the United Nations office for the coordination of Humanitarian Affairs unit reported that cholera outbreak killed 97 persons in Kano. That report has it that the outbreaks were across Katsina, Zamfara, Bauchi and Kano states in northern Nigeria killing close to 100 people in just two weeks, making it the worst outbreak in the north for several years, according to an official from the National Primary Healthcare Agency (NPHA) in Abuja (UN, 2008). An earlier outbreak of cholera in Kano was reported in the work of Hutin et al. (2003).

More than 60 people have died in Zamfara State in the past two weeks, according to Zamfara's State Commissioner for Religion Affairs. The report says further; "It is quite unusual for northern Nigeria. If up to 100 people have died from cholera in just two weeks, you can imagine how many more are affected by the disease." The commissioner added, the death toll may be higher as reports of new infections are still coming in (UN, 2008). Most recently, cholera outbreak has taken toll in Adamawa State claiming a 52 year-old man, his wife, son and 36 others in the Maiha Local Government area of Adamawa state. (The Guardian Newspaper, Tuesday, August 18, 2009). It becomes significant we carry out a scientific study of this silent killing disease that has become endemic, so as to enhance its control in Nigeria using a mathematical model with a logistic growth term and an optimal control model with time dependent control functions incorporated.

Time dependent control strategies have been applied to various disease models. For instance, in Jung et al. (2002) an optimal control of treatments in a two-strain tuberculosis model was examined. Fister et al. (1998) and Kirshner et al. (1997) both studied time dependent control strategies used for HIV models. In Joshi et al. (2006), two control functions as coefficients in a system of differential equations represented treatment effects in a two- drug regime in an HIV immunology model. The goal was to maximize the concentration of T cells while minimizing the toxic effects of the drugs. However, time dependent control strategies for cholera is an innovation as few mathematical models on cholera did not include time dependent controls [Compare Capasso and Paveri–Fontana (1979) and Codeco (2001)].

In this work, we will examine two formulated models one without the control term and the other with control terms that are time-dependent. Simulations of these models are carried out to see the effect of these controls in the population dynamics of the disease.

FORMULATION OF MATHEMATICAL MODEL USING THE LOGISTIC TERM WITHOUT CONTROL

The logistic equation was first used in modeling human population by Verhulst in 1838. He followed a suggestion from his mentor Quetelet, that the resistance to growth should be quadratic and not linear (Britton, 2003). In 1940, this idea was confirmed when human population data of the USA was plotted overtime. The curve took a bell shape indicating that human population is non-linear.

The logistic equation was revived by Pearl and Reed in 1920. Pearl and others thought that fitting such a curve to a population time series would provide realistic short term forecasts as well as estimates of the ultimate steady state population *K*.

A careful observation however, shows that bacteria population growth is fitted excellently with a logistic growth equation (Britton, 2003). Since the bacteria population grows at a density-dependent rate and the probability of catching cholera depends on the concentration of V. cholerae in aquatic environment, then it behooves us to know the carrying capacity of the organism. The logistic growth term does that and is non-linear with respect to the bacteria population and makes the model more realistic since the population growth of bacteria is non-linear. Hence, we proposed a model using the logistic growth approach. In formulation, the dynamics of the infected is extended to include demographic factors, thus removing the assumption inherent in a closed population model. The mathematical model is proposed below:

$$\frac{dS}{dt} = \Lambda - a\theta(B)S - \mu S \tag{1}$$

$$\frac{dI}{dt} = a\theta(B)S - (\mu + d + \beta)I \tag{2}$$

$$\frac{dB}{dt} = rB(1 - B/K) - nB + eI \tag{3}$$

$$S(0) = N, I(0) \geq 0,\ B(0) = 0 \tag{4a}$$

However, the parameters for the solution of the model are presented below:

The symbols used:

S: Number of susceptibles
I: Number of infected
B: Concentration of toxigenic V. cholerae in water (cells/m) (cells/ml)
N: Total human population
μ: Natural Human death rate (day^{-1})
d: Disease related death rate (day^{-1})
a: Rate of exposure to contaminated water (day^{-1})
k: Concentration of V. cholerae in water that yields 50% chance of catching cholera (cells/ml)
K: The carrying capacity of V. cholerae
β: Rate at which people recover from cholera (day^{-1}).
r: Growth rate of V. cholerae in the aquatic environment (day^{-1}).
n: Loss rate of V cholerae in the aquatic environment (day^{-1}).
e: Contribution of each infected person to the population of V. cholera ($cell/ml\ day^{-1}\ person^{-1}$)
α: Net mortality rate of V. cholerae in the aquatic environment
Λ: Recruitment rate into the susceptible class.
θ(B): Probability of susceptible to catch cholera

$$\theta(B) = \frac{B}{k + B} \tag{4b}$$

where k and B are as defined above.

Figure 1 shows the schematic representation of the flow between the different classes of the state variables (S.I.B).

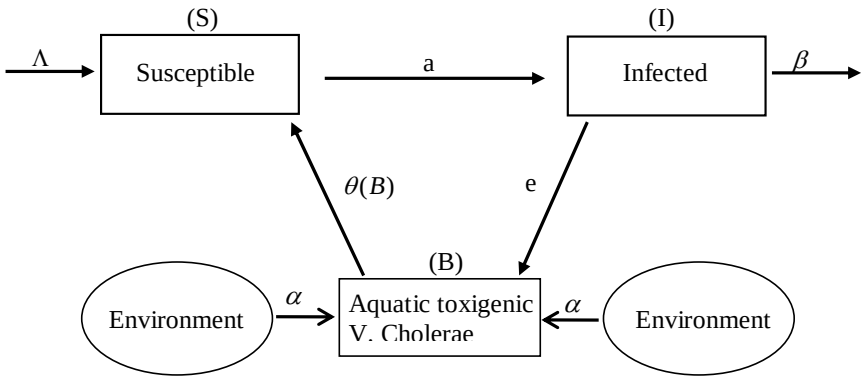

Figure 1. The relationship between the state variables: the Susceptible, Infected and the pathogenic bacteria (SIB).

Table 1. Parameters for the model.

Parameters	Description	Values A	Values B
N	Total human populations	1,000	100,000
μ	Natural human death rate	0.000559	0.00559
Λ	Recruitment rate	0.559	559
d	Cholera related death rate	0.000156	0.00156
a	Exposure rate to contaminated food and water	0.5	0.5
k	Concentration of V. cholerae in water to yield 50% chance of catching cholera	10^6	10^6
K	The carrying capacity of V. cholerae	10^8	10^8
β	The recovery rate from cholera	0.2	0.2
r	Growth rate of V. cholerae	0.2497	0.40
n	Loss rate of V. cholerae	0.4	0.64
e	Contribution rate from the infected	10	10

Source: parameters customized from Codeco (2001) for demonstration purpose.

The basic reproductive number R_0 and parameters for the model

The basic reproductive number R_0 is a non-dimensional quantity that measures the secondary infection caused by a typically infected person. If $R_0 > 1$ then there would be an epidemic since one infected is able to infect more than one person in a system. Otherwise, there would be a disease free state, that is, whenever $R_0 < 1$. What happens when $R_0 = 1$; We have the threshold condition. But cholera becomes endemic whenever $R_0 > 1$ over a long period of time. Therefore, it becomes imperative to determine this quantity for any dynamical system. For our model it is given below:

$$R_0 = \frac{ea}{k(n-r)(d+\beta+\mu)} S_0 \quad (5)$$

Table 1 below gives a brief description of the parameters.

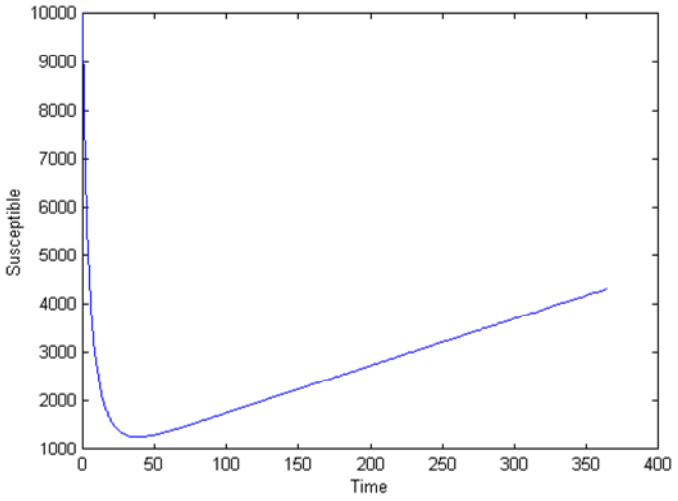

Figure 2. The susceptible population.

Numerical solutions of the model without control

Figures 2 to 4 represents the plots from values A, while Figures 5 to 7 represents those from values B. The state variables (SIB) are plotted in the vertical axis against time in the horizontal axis for the entire Figures 2 to 7.

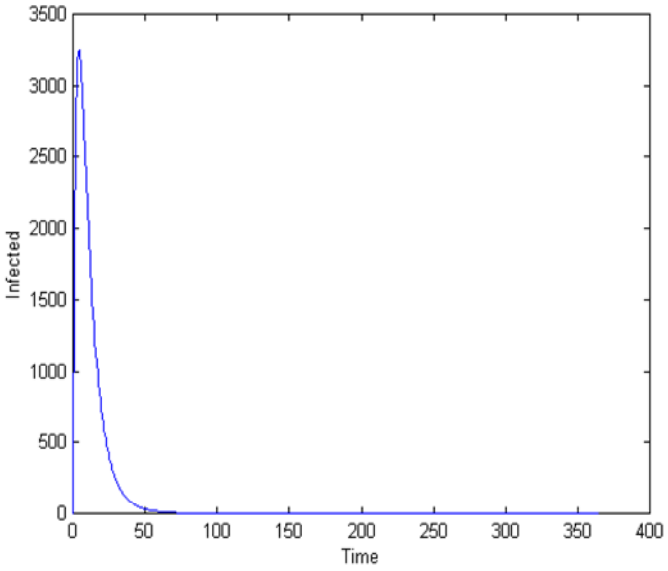

Figure 3. The infected population.

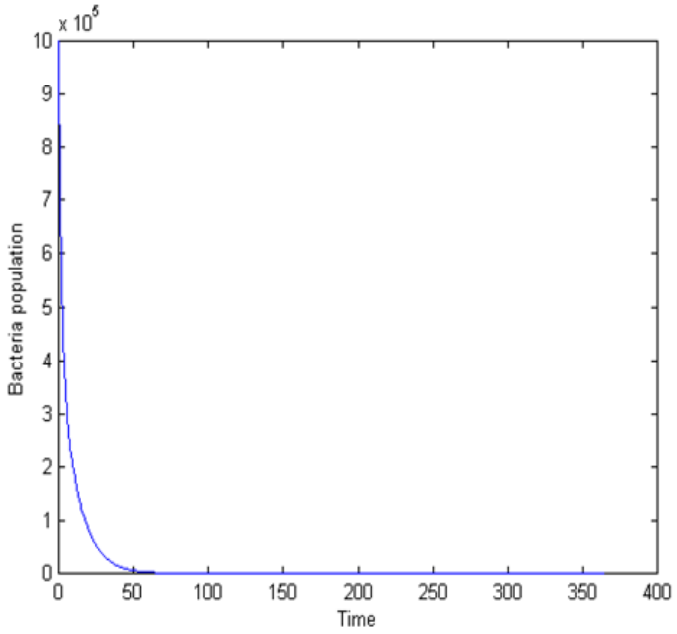

Figure 4. The bacteria population.

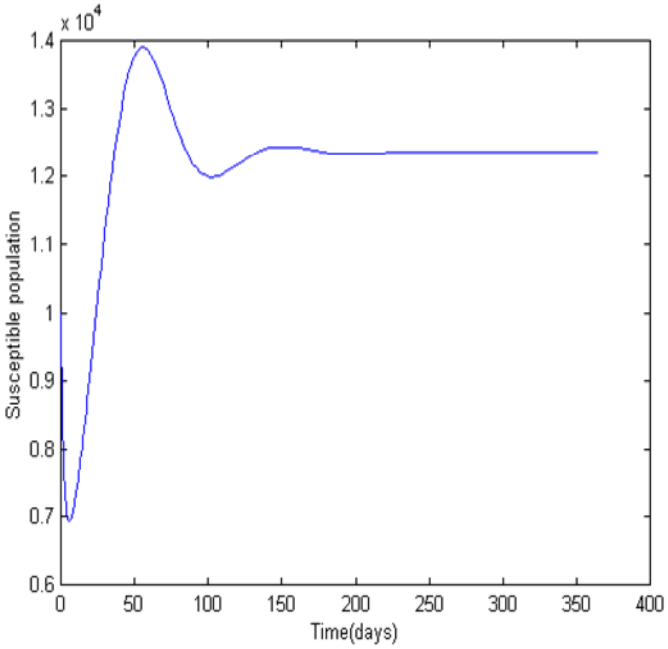

Figure 5. The susceptible population.

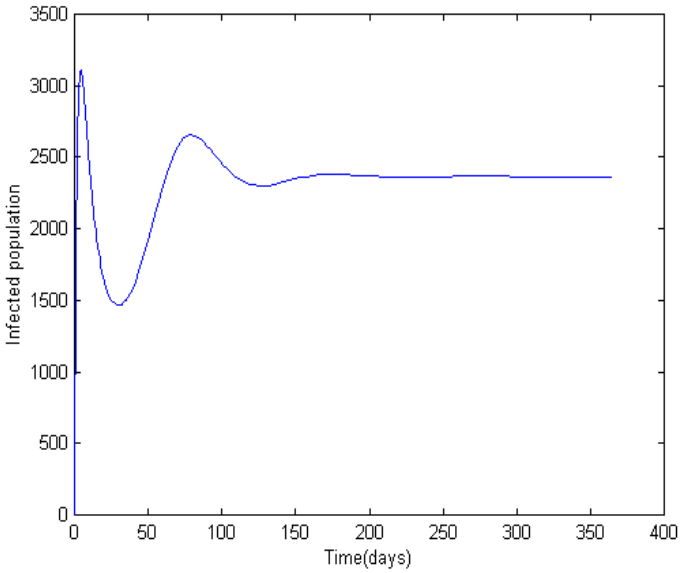

Figure 6. The iInfected population.

The plot in Figure 2 shows a short decline and after which there was an exponential growth. This result is due to an interplay between the natural death rate (μ) and the exposure rate to contaminated food and water (α), both are relatively small.

The result in Figure 3 is a reverse of the Figure 2. Whenever the infected population is low the susceptible soars up and vice versa.

The obvious consequence of Figures 2 and 3 is what we see in Figure.4. The population of the bacteria is at low ebb.

The results of the set of values B from the parameters in Table 1 are given below:

The result of Figure 5 is reacting to the slight increase in the net loss rate (the difference between the loss rate (n) and the growth rate (r)). The contribution from the infected is also factor.

The factors mentioned in Figure 5 are responsible for the plot in Figure 6. Interestingly, this result is again the mirror image of the result of Figure 5.

This mild increase in the bacteria population is predicated by the infected population which could be effect of immigrants and contribution of the infected to the aquatic environment as well as the natural growth rate of the bacteria.

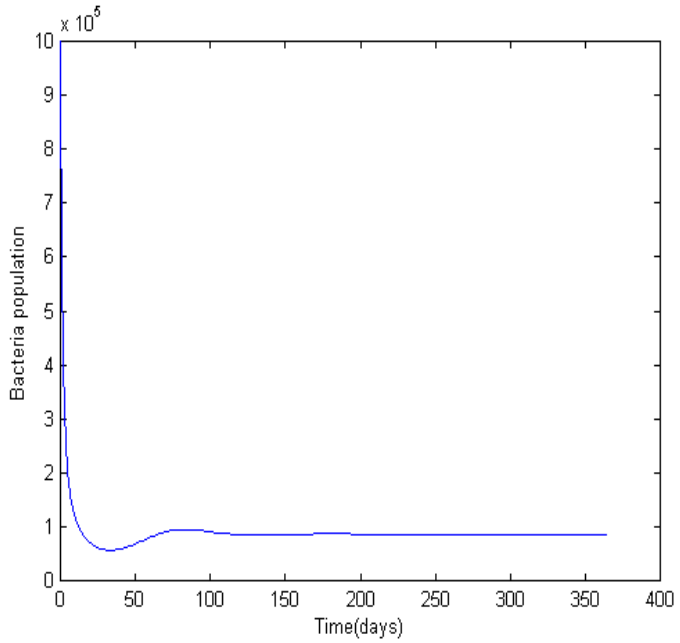

Figure 7. The bacteria population.

Optimal control model for the dynamics of cholera

Optimal control is the standard method for solving dynamic optimization problems, when those problems are expressed in continuous time (Lenhart and Workman, 2006). In this work, we use this method as part of control measures for cholera epidemics in Nigeria.

The proposed model that incorporates time dependent controls is presented below:

$$\frac{dS}{dt} = \Lambda - (1-u_1(t))\frac{aB}{k+B}S - \mu S \tag{6}$$

$$\frac{dI}{dt} = (1-u_1(t))\frac{aB}{k+B}S - (d+\mu+\beta)I \tag{7}$$

$$\frac{\mathrm{dB}}{dt} = rB\left(1-\frac{B}{K}\right) + eI - u_2(t)nB \tag{8}$$

$$S(0) = S, \qquad I(0) \geq 0, \qquad B(0) = 0 \tag{9}$$

where the objective functional to be optimized is

$$J(u_1,u_2) = \int_0^T \left(B(t) + \frac{A_1}{2}u_1^2(t) + \frac{A_2}{2}u_2^2(t) + I(t) \right) dt \tag{10}$$

with u_1^2 and u_2^2 being the systemic combination of the organism population and the cost of reducing contact of infected and the susceptible. The parameters here are as defined in Equations (1) - (3).

The control functions u_1 and u_2 represent the reduction of contact between infected persons and the susceptible, and the treatment of water to reduce the growth of the organism respectively. In Equations (6) to (8) above, the organism population grows at a density - dependent rate $rB(1-B/K)$. The control coefficients u_2 and $(1-u_1)$ reduce the organism growth and the contacts between infected and susceptible accordingly.

Our interest is to find a pair of control u_1 and u_2 in appropriately chosen class, and associated state variables SIB to minimize the objective functional Equation (10) above.

Again we minimize the contact of the infected with the susceptible and the associated cost of doing so. The population of the V. cholerae in an aquatic environment was also minimized. This was done by minimizing the objective functional above. Here, it was assumed that the cost associated with reducing contact of the infected and the cost of reducing the V. cholerae population in an aquatic environment are non-linear and take a quadratic form. This agrees with the non-linear nature of the model used. The coefficients A_1 and A_2 are balancing cost factors due to the size and importance of the parts making up the objective functional.

Hence we are interested in finding an optimal control pair u_1^* and u_2^*, such that:

$$J(u_1^*,u_2^*) = \min_U J(u_1,u_2) \tag{11}$$

We assume that the controls u_1 *and* u_2 are Lebesgue measurable.

That is, $U = \{u_i(t) : a_i \leq u_i(t) \leq b_i$ for $0 \leq t \leq T$, $u_i(t)$ are lebesgue measurable, $i = 1, 2\}$ where $0 \leq a_i \leq b_i < 1$, $i = 1, 2$ and $0 \leq u_2(t) \leq 1/n$. The goal is to find an optimal control pair $u_i \in U$, $i = 1, 2$ and associated state variables SIB to minimize the objective functional and a_i b_i, $i = 1, 2,$ are fixed non-negative constants.

Next, applying the Pontryagin's Maximum Principle (Kirschner et al., 1997), we derive necessary conditions for our optimal control and corresponding state variables, including constraints on the controls. Since we have three state variables, SIB, we shall have three corresponding adjoint variables where λ_1 corresponds to S, and λ_2 corresponds to I and λ_3 corresponds to B.

The Hamiltonian adjoint equations

The Hamiltonian equation is formed by allowing each of the adjoint variables to correspond to each of the state variables accordingly and combining the result with the objective functional as below:

$$H = B + \frac{A_1}{2}u_1^2 + \frac{A_2}{2}u_2^2 + I + \lambda_1\left[\Lambda - (1-u_1(t))\frac{aBS}{k+B} - \mu S\right]$$
$$+ \lambda_2\left[(1-u_1(t))\frac{aBS}{k+B} - (\mu+d+\beta)I\right] + \lambda_3\left[rB\left(1-\frac{B}{K}\right) + el - u_2(t)nB\right] \tag{12}$$

The adjoint equations are formed by taking the derivative of the Hamiltonian with respect to each of the state variables (SIB). Hence adjoint equations are:

$$\lambda_1' = -\frac{\partial H}{\partial S} = \lambda_1(1-u_1(t))\frac{aB}{k+B} + \lambda_1\mu - \lambda_2(1-u_1(t))\frac{aB}{k+B} \quad (13)$$

$$\lambda_2' = -\frac{\partial H}{\partial I} = \lambda_2(d+\mu+\beta) - \lambda_3 e - 1 \quad (14)$$

$$\lambda_3' = -\frac{\partial H}{\partial B} = \lambda_1\frac{(1-u_1(t))aKS}{(K+B)^2} - \lambda_2\frac{(1-u_1(t))aKS}{(K+B)^2} - \lambda_3\frac{r(K-2B)}{K} + \lambda_3 u_2(t)n - 1 \quad (15)$$

The Optimality Equations

The equations are obtained by finding the derivative of the Hamiltonian equation with respect to the control variables, equating to zero and solving the resulting equation. Hence the optimal equations are:

$$\frac{\partial H}{\partial u_1} = A_1 u_1 + \lambda_1\frac{aBS}{k+B} - \lambda_2\frac{aBS}{k+B}$$

Then the optimal value for u_1 is:

$$A_1 u_1 + \lambda_1\frac{aBS}{k+B} - \lambda_2\frac{aBS}{k+B} = 0$$

That is $$u_1^* = \frac{aB^*S^*}{A_1(k+B^*)}[\lambda_2 - \lambda_1] \quad (16)$$

Similarly, $$\frac{\partial H}{\partial u_2} = A_2 u_2 - \lambda_3 nB$$

Hence the optimal value for u_2, is

$$A_2 u_2 - \lambda_3 nB = 0$$

$$u_2^* = \frac{\lambda_3 nB^*}{A_2}$$

As our control u_1 is bounded below by a_1 and bounded above by b_1 and u_2 bounded below by a_2 and above by b_2, we must constrain the values of the control and obtain the characterization

$$u_1^* = \begin{cases} a_1, & when \frac{aBS}{A_1(k+B)}[\lambda_2-\lambda_1] \le a_1 \\ \frac{aBS}{A_1(k+B)}[\lambda_2-\lambda_1], & when\, a_1 < \frac{aBS}{A_1(k+B)}(\lambda_2-\lambda_1) < b_1 \\ b_1, & when \frac{aBS}{A_1(k+B)}[\lambda_2-\lambda_1] \ge b_1 \end{cases} \quad (18)$$

$$u_2^* = \begin{cases} a_2, & when \frac{\lambda_3 nB}{A_2} \le a_2 \\ \frac{\lambda_3 nB}{A_2}, & when\, a_2 < \frac{\lambda_3 nB}{A_2} < b_2 \\ b_2, & when \frac{\lambda_3 nB}{A_2} \ge b_2 \end{cases} \quad (19)$$

The foregoing shows that the optimality conditions (taking derivatives of the Hamiltonian with respect to the controls) only hold in the interior of the control set. Next, we need to resolve the optimal control model numerically.

NUMERICAL RESULTS OF THE OPTIMAL CONTROL MODEL

Here, we are going to examine the effect of $u_1(t)$ and $u_2(t)$ - control 1 and control 2 respectively vis-à-vis the epidemiological classes: the susceptible, the infected and the bacteria population classes (SIB) as shown in Figure 10. It is to be noted that u_1 (control 1) is actually controlling the contact between the infected and susceptible while u_2 (control 2) is minimizing the concentration of the bacteria population in the water. Three sets of results are presented using the parameters from Table 2. Figures 8 and 9 correspond with the values on column A and column B respectively. Using values C, we varied k to see its effect on the system.

The differences in these parameters lie in the growth and loss rates of the bacteria, that is, $'r'$ and $'n'$ respectively.

Consequently, as shown in Figure 11, keeping $k = 10^5$ will be easier to control. The system remains controlled for more than four days before it goes out of hand. The conclusion is that, the system would effectively be controlled when the concentration of bacteria that yields 50% chance of catching cholera (k) is not too high. The value of 10^5 is adequate for an effective control of the system. Here the first value has $k = 10^5$ and the second value has $k = 10^7$.

DISCUSSION

In this work, two models were examined. We presented a cholera model with a logistic term for its growth in the expected habitat and their interaction with the susceptible population. The second model was an optimal control model that included two time-dependent control functions with one minimizing the contact between the susceptible and the bacteria and the other, the population of the bacteria in the water.

The mathematical model formulated really provided insight into the dynamics of cholera in Nigeria. The

Table 2. Parameters description

Parameters	Description	Values A	Values B	Values C
N	Total human populations	1,000	100,000	10,000
μ	Natural human death rate	0.000559	0.000551	0.000569
Λ	Recruitment rate	0.559	0.00551	0.0569
d	Cholera related death rate	0.000156	0.00156	0.000156
a	Exposure rate to contaminated food and water	0.5	0.5	1
k	Concentration of V. cholera in water to yield 50% chance of catching cholera	10^6	10^6	10^6, 10^5
K	The carrying capacity of V. cholera	10^8	10^8	10^8
β	The recovery rate from cholera	0.2	0.2	0.2
r	Growth rate of V. cholerae	0.2497	0.10	0.07
n	Loss rate of V.cholerae	0.4	0.043	0.4
e	Contribution rate from infected	10	10	10

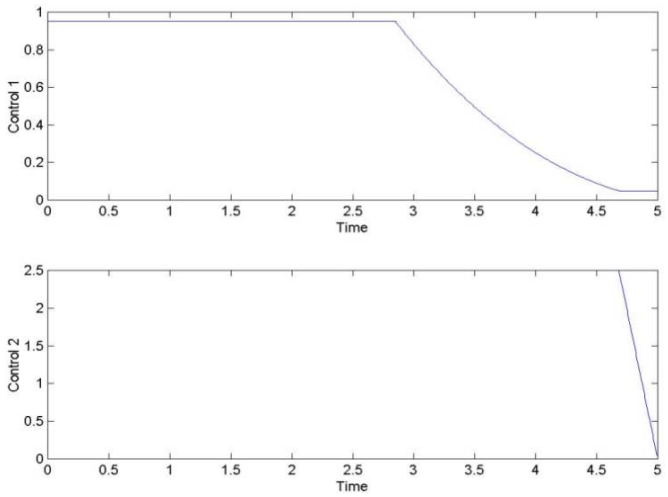

Figure 8. The controls (I[st] Entries).

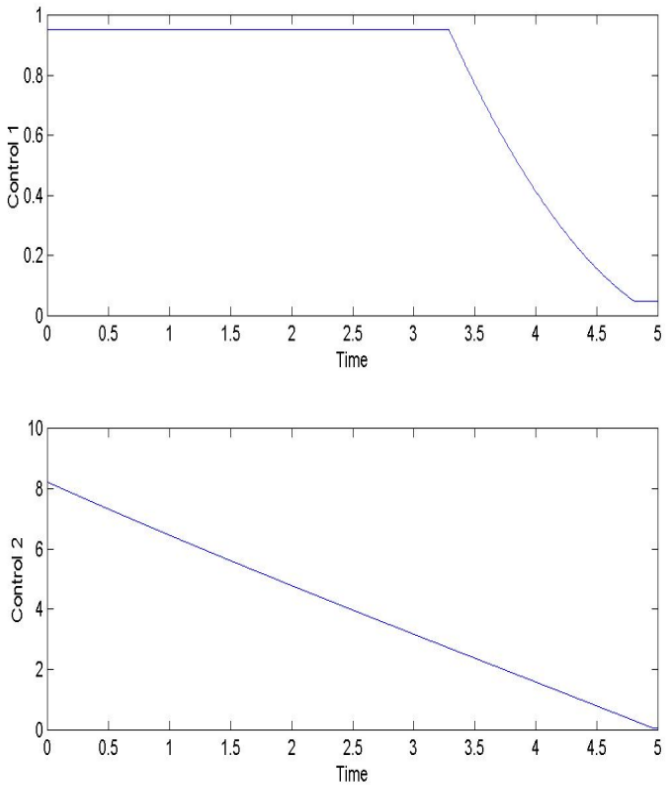

Figure 9. The Controls (2[nd] Entries).

results actually substantiated the minimum condition for the development of epidemic and endemic cholera stated: that if the rate of exposure to contaminated food and drink (a) and the contribution of the infected (e) to the bacteria population is less than the concentration of the bacteria that yields 50% chance of catching cholera (k), R_0 will be asymptotically stable. Therefore, if we would attain a cholera-free community, we should keep the susceptible population minimal, always below the threshold condition. Λ can be controlled if immigration is checked. In Nigeria, prior to 1970, the susceptible could have been below the threshold. However, in the late 1960's immigrants from the Asian continent came into the country, and that increased the susceptible pool and few infective that came in triggered an epidemic in 1970 (Lawoyin et al., 1999). Now that cholera has become endemic in Nigeria, we should avoid situations that will encourage the growth of *V. cholerae* in our community by ensuring sanitary condition perhaps through environmental sanitation exercise, and also providing portable drinking water.

Conflict of Interests

The author(s) have not declared any conflict of interests.

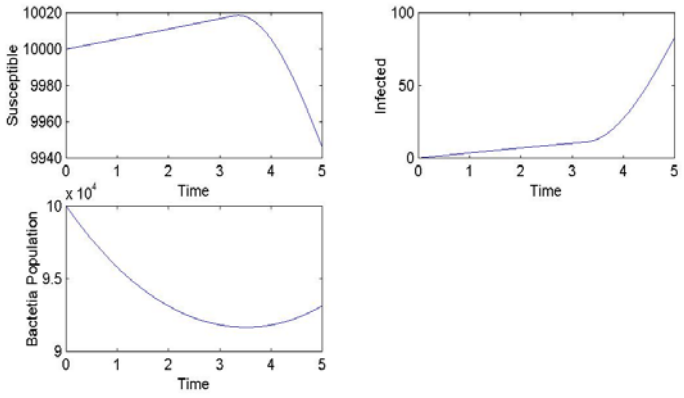

Figure 10. The effect on The SIB-Populations.

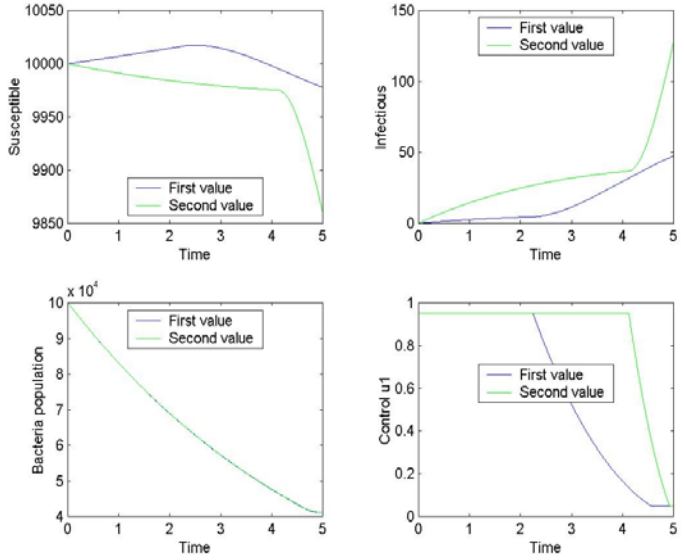

Figure 11. The effect from varying K on SIB(where k is the conc. of V. cholerae that yields 50% of catching cholera).

REFERENCES

Britton KN (2003). The Essential Mathematical Biology. Springer-Verlag, London .

Capasso V, Paveri–Fontana SL (1979). A mathematical model for the 1973 cholera epidemic in the European Mediterranean region. Rev Epidem et Sante 27:121-132.

Codeco TC (2001). Endemic and epidemic dynamics of cholera: The role of the aquatic reservoir. BMC Infectious Diseases – available from http://link.springer.com/article/10.1186%2F1471-2334-1-1

Emch M, Caryl F, Islam MS, Mohammed A (2008). Seasonality of cholera from 1974 – 2005: A review of global patterns. Int. J. Health Geogr. 7:31. Available online at: http://www.ij-healthgeographics.com/content/7/1/31

Epstein PR (1993). Algal blooms in the spread and persistence of cholera. Biosystems 31:209-221

Fister KR, Lenhart S, McNally JS (1998). Optimizing chemotherapy in an HIV model. Electron J. Diff. Equ. 32:1–12.

Hutin Y, Luby S, Paquet C (2003). A large cholera outbreak in Kano City, Nigeria: The importance of hand washing with soap and the danger of street-vended water. J. Water Health 1(1):45-52.

Isere AO, Osemwenkhae JE, Erimafa JT (2009). Investigation of the trend of cholera outbreaks in Nigeria using Exponentially Weighted Moving Average control chart. J. Eng. Sci. App. (JESA). 6(1&2):51-58.

Isere AO, Osemwenkhae JE (2010). VIBRIO CHOLERAE IN NIGERIA: A logistic investigation of Its stability in its habitat. J. Nig. Ann. Nat. Sci. 10(1):71-76.

Joshi HR, Lenhart S, Li MY, Wang L (2006). Optimal Control methods applied to disease models, in Gumel AB et al. (eds), Mathematical Studies on Human Disease Dynamics: Emerging Paradigms and Challenges. AMS Contemp. Math. Ser. 410: 187-207.

Jung E, Lenhart S, Feng Z (2002). Optimal control of treatment in a two-strain tuberculosis model. Discrete Cont. Dyn-B 2(4):473-482.

Kirschner D, Lenhart S, Serbin S (1997). Optimal Control of the Chemotherapy of HIV. J. Math. Biol. 35:775-792.

Lawoyin TO, Oguntodede NA, Olumide EA, Ohadeko MO (1999). Outbreak of Cholera in Ibadan, Nigeria. Eur. J. Epidemiol. 15:367-370, 365-368.

Lenhart S, Workman JT (2006). An Introduction to Optimal control Applied to Immunology problems. Proc. Symposia Appl. Math. 64:85-89.

Osemwenkhae JE, Isere AO, Okuonghae DU (2009). Endemicity of cholera in Nigeria: A Mathematical model to investigate its nature. J. Nig. Assoc. Math. Phy. 14:165-170.

UN (2008). Office for the coordination of humanitarian affairs integrated regional information networks (IFIN).

WHO (2009). Weekly Epidemiological Record. 84(31):309-324.. Available at http://www.who.int/wer