Journal of Advances in Mathematics and Computer Science

36(5): 52-65, 2021; Article no.JAMCS.69313
ISSN: 2456-9968
(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

The Analysis for the Recovery Cases of COVID-19 in Egypt using Odd Generalized Exponential Lomax Distribution

Hanem Mohamed[1*], Amina E. Abo-Hussien[1], Salwa A. Mousa[1] and Magda M. Ismail[1]

[1]Department of Statistics, Faculty of Commerce, AL-Azhar University (Girls' Branch), Cairo, Egypt.

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2021/v36i530363
Editor(s):
(1) Dr. Francisco Welington de Sousa Lima, Universidade Federal do Piauí, Teresina, Brazil.
(2) Dr. Dariusz Jacek Jakóbczak, Koszalin University of Technology, Poland.
(3) Dr. Metin Basarir, Sakarya University, Turkey.
Reviewers:
(1) Abdelmadjid Maireche, University of M'sila, Algeria.
(2) Jubran Labban, Al-Qadisiya University, Iraq.
Complete Peer review History: http://www.sdiarticle4.com/review-history/69313

Original Research Article

Received 03 April 2021
Accepted 09 June 2021
Published 19 June 2021

Abstract

In this paper, an odd generalized exponential Lomax (OGEL, in short) distribution has been considered. Some mathematical properties of the distribution are studied. The methods of maximum likelihood and maximum product of spacing are used for estimating the model parameters. Moreover, 95% asymptotic confidence intervals for the estimates of the parameters are derived. The Monte Carlo simulation is conducted for the two proposed methods of estimation to evaluate the performance of the various proposed estimators. The proposed methods are utilized to find estimates of the parameters of OGEL distribution for the daily recovery cases of COIVD-19 in Egypt from 12 May to 30 September 2020.The practical applications show that the proposed model provides better fits than the other models.

Keywords: Odd generalized exponential Lomax distribution; maximum likelihood estimation; maximum product of spacing; asymptotic confidence interval; COVID-19.

**Corresponding author: Email: nonaeslam97@gmail.com, eng.hanimmohammeda.team@azhar.edu.eg;*

1 Introduction

The coronavirus disease 2019 (COVID-19) firstly appears in China and it's become an important problem affecting countries around the world, one of these countries is Egypt. It has spread throughout the province. In order to analyze the real data for the COVID-19 this requires the choice of statistical models based on probability distributions Bantan et al. [1]. In this paper, the reported COVID-19 number of recovery cases in Egypt, for the period 20May 2020 to 20 September 2020, is modeled using statistical distributions. There are several new families of probability distributions that are proposed by several authors. Such families have great flexibility and generalize many well-known distributions. Therefore, several classes have been proposed, in the statistical literature, by adding one or more parameters to generate new distributions. Among this literature, exponential Lomax El-Bassiouny et al. [2], exponentiated Weibull- Lomax Hassan and Abd-Allah [3], the odd Lomax generator Cordeiro et al. [4], the generalized odd inverted exponential-G family Chesneau et al. [5]. The odd log-logistic Lindley-G Alizadeh et al. [6] and the odd Dagum family of distributions Afify et al. [7]. The generalized exponential and Lomax distributions are two important distributions in studies and practice. These distributions have several important statistical properties. Gupta and Kundu [8] introduced the generalized exponential distribution and derived some properties of this distribution, Gupta and Kundu [9].

The Lomax distribution is introduced as an important model for lifetime analysis, it is also called Pareto type II distribution. The distribution is widely used in several fields such as business and econometrics Lomax [10].

Alzaatreh, et al. [11] proposed a new generalized family of distributions called the T –X family. The T –X family consists of many sub-families of distributions. Based on this technique, one can develop new distributions that may be very general and flexible or for fitting specific types of data distributions such as highly left-tailed, right-tailed, thin-tailed, or heavy-tailed distributions as well as bimodal distributions.

In this article an odd distribution is considered from the generalized exponential and Lomax distribution, called the odd generalized exponential distribution (OGEL)based on the *T-X* family.

A random variable T has the generalized exponential distribution with two parameters λ and γ if it has the pdf and CDF are given by:

$$f(t;\lambda,\gamma) = f_\lambda(t) = \gamma\lambda e^{-\lambda t}(1 - e^{-\lambda t})^{\gamma-1}, t > 0, \lambda, \gamma > 0. \quad (1)$$

$$F(t;\lambda,\gamma) = (1 - e^{-\lambda t})^{\gamma}, t > 0, \lambda, \gamma > 0. \quad (2)$$

A random variable X has the Lomax distribution with two parameters β and θ if it has the pdf and cdf given by:

$$f(x;\beta,\theta) = \frac{\beta}{\theta}(1 + \frac{x}{\theta})^{-(\beta+1)}, x > 0, \beta, \theta > 0. \quad (3)$$

$$F(x;\beta,\theta) = F_\theta(x) = 1 - \left(1 + \frac{x}{\theta}\right)^{-\beta}, x > 0, \beta, \theta > 0. \quad (4)$$

Based on the T –X family, the OGEL distribution using (1) and (4) is derived as follows:

$$F(x;\lambda,\gamma,\beta,\theta) = \int_0^{W(F_\theta(x))} f_\lambda(t)dt,$$

Where: $W\big(F_\theta(x)\big) = \frac{F_\theta(x)}{1-F_\theta(x)} = \frac{1-(1+\frac{x}{\theta})^{-\beta}}{(1+\frac{x}{\theta})^{-\beta}} = (1 + \frac{x}{\theta})^{\beta} - 1.$

Then

$$F(x;\lambda,\gamma,\beta,\theta) = \left[1 - e^{-\lambda\left[(1+\frac{x}{\theta})^{\beta}-1\right]}\right]^{\gamma}. \quad (5)$$

This paper is organized as follows. The density function of the Odd Generalized Exponential-Lomax (OGEL) distribution is derived. The main descriptive properties are introduced. The maximum likelihood estimation (ML) and maximum product spacing (MPS) of the parameters are discussed. A simulation study is conducted. An application with real data is analyzed.

2 The Odd Generalized Exponential Lomax Distribution

In this section, four parameters of the OGEL probability distribution are derived.the pdf Corresponding to (5) is given by:

$$f(x;\lambda,\gamma,\beta,\theta)=\frac{\lambda\gamma\beta}{\theta}\left(1+\frac{x}{\theta}\right)^{\beta-1} e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\left[1-e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right]^{\gamma-1},x>0,\lambda,\gamma,\beta>0,\theta>1. \quad (6)$$

$x\sim OGEL(\lambda,\gamma,\beta,\theta)$, denotes arandom variable with the pdf of (6).

3 Some Descriptive Properties of the OGEL Distribution

This section provides some properties of the OGEL distribution.

3.1 Main properties of the OGEL

a. The survival function denoted by $S(x)$, is given by:

$$S(x)=1-F(x)=1-\left[1-e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right]^{\gamma}. \quad (7)$$

b. The hazard rate function, $h(x)$, is given by:

$$h(x)=\frac{\lambda\gamma\beta(1+\frac{x}{\theta})^{\beta-1}e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\left[1-e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right]^{\gamma-1}}{\theta\left[1-\left[1-e^{-\lambda\left[(1+\frac{x}{\theta})^{\beta}-1\right]}\right]^{\gamma}\right]}. \quad (8)$$

c. The reversed hazard rate function,$r(x)$,is given by:

$$r(x)=\frac{\lambda\gamma\beta(1+\frac{x}{\theta})^{\beta-1}e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}}{\theta\left[1-e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right]}. \quad (9)$$

d. The cumulative hazard rate function, $H(x)$, is given by:

$$H(x)=-\ln\left\{1-\left[1-e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right]^{\gamma}\right\}. \quad (10)$$

e. The Quantile function and the median are given respectively by:

The quantile function (x_q), is given by:

$$x_q=F(x_q)^{-1},\ \ 0<q<1 \quad (11)$$

Substitute (5) in Eq. (11), the quantile function is

$$x_q = \theta\left\{\left[1-\frac{1}{\lambda}ln\left[1-(q)^{\frac{1}{\gamma}}\right]^{\frac{1}{\beta}}\right]-1\right\}. \tag{12}$$

In particular, when $q = 0.5$ the median can be defined as:

$$\text{median} = x_{0.5} = \theta\left\{\left[1-\frac{1}{\lambda}ln\left[1-(0.5)^{\frac{1}{\gamma}}\right]^{\frac{1}{\beta}}\right]-1\right\}. \tag{13}$$

f. The r^{th} non-central moment is given by:

$$\acute{\mu}_r = E(x^r) = \frac{\lambda\gamma\beta}{\theta}\int_0^\infty x^r\,(1+\frac{x}{\theta})^{\beta-1}e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\left[1-e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right]^{\gamma-1} dx \tag{14}$$

Since $0<\left[1-e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right] < 1\ for\ x > 0,$ by using binomial expansion terms,and by using the definition of

the special function Beta in the form, $\beta^*(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\,dy$

Ther^{th} non –central moment of OGEL distribution is as follows:

$$\mu_r` = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{\gamma-1}\binom{\gamma-1}{i}\binom{j}{k}(-1)^{i+j+k}\frac{\lambda^{i+j}\gamma\beta(i+1)^j\theta^r}{(j)!}\beta^*[r+1,\beta(1+k-j)-(r+1)]. \tag{15}$$

Under the condition $(\beta j + r + 1) < \beta(k+1)$.Depending on (15), the basic statistical properties are given as follows:

(i) The mean and the variance, of the OGEL distribution are, respectively, given by

$$\acute{\mu}_1 = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{\gamma-1}\binom{\gamma-1}{i}\binom{j}{k}(-1)^{i+j+k}\frac{\lambda^{i+j}\gamma\beta(i+1)^j\theta}{(j)!}\beta^*[2,\beta(1+k-j)-2] \tag{16}$$

And

$$V(x) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{\gamma-1}\binom{\gamma-1}{i}\binom{j}{k}(-1)^{i+j+k}\frac{\lambda^{i+j}\gamma\beta(i+1)^j\theta^2}{(j)!}\beta^*[3,\beta(1+k-j)-3] - \left\{\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{\gamma-1}\binom{\gamma-1}{i}\binom{j}{k}(-1)^{i+j+k}\frac{\lambda^{i+j}\gamma\beta(i+1)^j\theta}{(j)!}\beta^*[2,\beta(1+k-j)-2]\right\}^2. \tag{17}$$

(ii) The $\mu_r$the central moment of OGEL can be obtained easily from the r^{th} moments through the relation

$$\mu_r = \sum_{r=0}^{n}\binom{n}{r}(-\mu)^{n-r}\acute{\mu}_r,$$

Then the $\mu_r$the central moment of OGEL is given by

$$\mu_r = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{\gamma-1}\sum_{r=0}^{n}\binom{\gamma-1}{i}\binom{j}{k}\binom{n}{r}(-\mu)^{n-r}(-1)^{i+j+k}\frac{\lambda^{i+j}\gamma\beta(i+1)^j\theta^r}{(j)!}\beta^*[r+1,\beta(1+k-j)-(r+1)]. \tag{18}$$

g. The moment generating function is obtained as follows:

$$M_X(t)=\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{\gamma-1}\binom{\gamma-1}{i}\binom{j}{k}(-1)^{i+j+k}\frac{\lambda^{i+j}\gamma\beta(i+1)^{j}\theta^{r}t^{r}}{(j)!(r)!}\beta^{*}[r+1,\beta(1+k-j)-(r+1)]\ . \qquad (19)$$

Under the condition $(\beta j+r+1)<\beta(k+1)$.

h. Order statistics

Let $X_1, X_2, \dots, X_n$ be a simple random sample of size n from OGEL with cumulative distribution function $F(x)$ and probability density function $f(x)$ given by (5) and (6) respectively. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the order statistics obtained from this sample. The probability density function and the cumulative distribution function of the k^{th}order statistic, say $y = X_{(k)}$ are given by:

$$f_Y(y)=\frac{n!}{(k-1)!\,(n-k)!}[F(y)]^{k-1}[1-F(y)]^{n-k}f(y),$$

$$=\frac{n!}{(k-1)!(n-k)!}\frac{\lambda\gamma\beta}{\theta}(1+\frac{y}{\theta})^{\beta-1}e^{-\lambda\left[\left(1+\frac{y}{\theta}\right)^{\beta}-1\right]}\left[1-e^{-\lambda\left[(1+\frac{y}{\theta})^{\beta}-1\right]}\right]^{\gamma k-1}\left\{1-\left[1-e^{-\lambda\left[(1+\frac{y}{\theta})^{\beta}-1\right]}\right]^{\gamma}\right\}^{n-k} \qquad (20)$$

And

$$F_Y(y)=\sum_{j=k}^{n}\binom{n}{j}[F(y)]^{j}[1-F(y)]^{n-j}$$

$$=\sum_{j=k}^{n}\binom{n}{j}\left[1-e^{-\lambda\left[(1+\frac{y}{\theta})^{\beta}-1\right]}\right]^{\gamma j}\left\{1-\left[1-e^{-\lambda\left[(1+\frac{y}{\theta})^{\beta}-1\right]}\right]^{\gamma}\right\}^{n-j}. \qquad (21)$$

3.2 Graphical description

The pdf curves of the OGEL distribution are displayed in Fig. 1 for some selected values of the parameters.

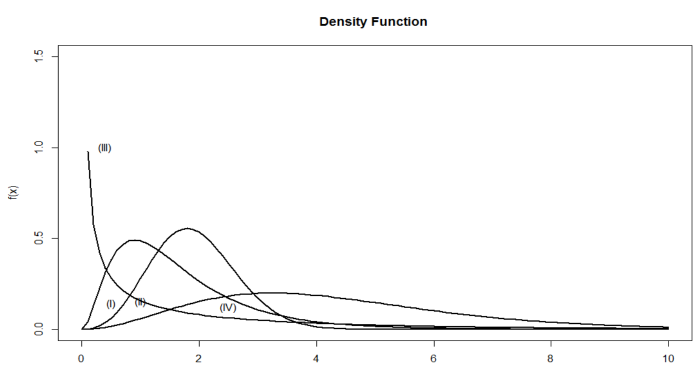

Fig. 1. I: $(\boldsymbol{\lambda=0.25,\gamma=3,\beta=3,\theta=2})$, **Fig. 1. II:** $(\boldsymbol{\lambda=0.5,\gamma=3,\beta=1.5,\theta=2.5})$,
Fig. 1. III: $(\boldsymbol{\lambda=0.3,\gamma=0.25,\beta=1.2,\theta=1.5})$, **Fig. 1. IV:** $(\boldsymbol{\lambda=0.6,\gamma=3,\beta=2,\theta=1.5})$.
Fig. 1 pdf of OGEL distribution

Fig. 1 shows that:

- The density curves of the OGEL are more flexible for changing values of the parameters.
- The density curves take various shapes such as symmetrical, right-skewed, reversed J-shaped, and unimodal.

The h(x) curves of two OGEL populations are displayed in Fig. 2 The first population is when $\gamma<1$, $(\lambda=0.25,\gamma=0.25,\beta=3,\theta=2)$, $(\lambda=0.5,\gamma=0.25,\beta=1.5,\theta=2.5)$, . The second population is when $\gamma>1$,$(\lambda=2,\gamma=3,\beta=1,\theta=1.5)$, $(\lambda=2,\gamma=3,\beta=0.9,\theta=3)$.

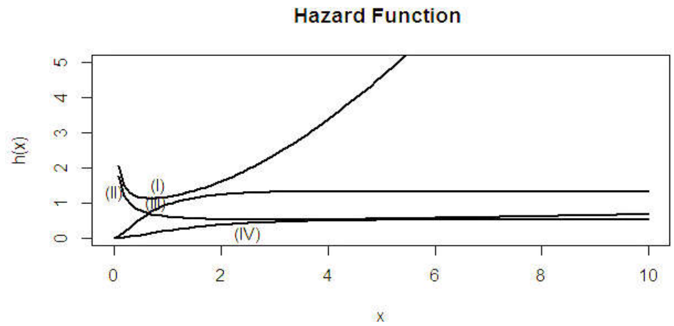

Fig. 2. I: $(\boldsymbol{\lambda = 0.25, \gamma = .25, \beta = 3, \theta = 2})$, **Fig. 2. II:** $(\boldsymbol{\lambda = 0.5, \gamma = 0.25, \beta = 1.5, \theta = 2.5})$,
Fig. 2.III: $(\boldsymbol{\lambda = 2, \gamma = 3, \beta = 1, \theta = 1.5})$, **Fig. 2. IV:** $(\boldsymbol{\lambda = 2, \gamma = 3, \beta = 0.9, \theta = 3})$,
Fig. 2. h(x) of the OGEL distribution

Fig. 2 shows that:

- The h(x) curves of the OGEL are more flexible for changing values of the parameters.
- The h(x) curves take different shapes such as constant, increasing, decreasing, and reversed J shape.

This fact implies that the OGEL can be very useful for fitting data sets with various shapes.

4 Parameters Estimation

In this section, ML and MPS methods are discussed to obtain the estimators of the parameters of the OGEL distribution based on complete samples.

4.1 Maximum likelihood estimation

The maximum likelihood method is used to estimate the unknown parameters of the OGEL distribution based on complete samples Casella and Berger [12].

Let $x_1, \ldots, x_n$ be a random sample of size n from OGEL, with a set of parameters $\underline{\varphi} = (\lambda, \gamma, \beta, \theta)$, the likelihood function based on a complete sample is given by:

$$L\left(\underline{\varphi}; \underline{x}\right) \propto \left(\frac{\lambda\gamma\beta}{\theta}\right)^n \prod_{i=1}^{n} \left(1 + \frac{x}{\theta}\right)^{\beta-1} e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]} \left\{1 - e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right\}^{\gamma-1}. \tag{22}$$

Then, the log likelihood function is given by:

$$l = nLn(\lambda) + nLn(\gamma) + nLn(\beta) - nLn(\theta) + (\beta - 1)\sum_{i=1}^{n} Ln\left(1 + \frac{x}{\theta}\right) - \lambda \sum_{i=1}^{n}\left[\left(1 + \frac{x}{\theta}\right)^{\beta} - 1\right] + (\gamma - 1)\sum_{i=1}^{n} Ln\left\{1 - e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right\}. \tag{23}$$

The first –order partial derivatives of l concerning λ, γ, β and θ and equating them to zero are as follows:

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\left[\left(1 + \frac{x}{\theta}\right)^{\beta} - 1\right] + (\gamma - 1)\sum_{i=1}^{n} \frac{e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}{\left\{1-e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right\}} = 0 \tag{24}$$

$$\frac{\partial l}{\partial \gamma}=\frac{n}{\gamma}+\sum_{i=1}^{n}Ln\left\{1-e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right\}=0, \tag{25}$$

$$\frac{\partial l}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}Ln\left(1+\frac{x}{\theta}\right)-\lambda\sum_{i=1}^{n}\left\{\left(1+\frac{x}{\theta}\right)^{\beta}\ Ln\left(1+\frac{x}{\theta}\right)\right\}+(\gamma-1)\sum_{i=1}^{n}\frac{\lambda e^{-\frac{\lambda}{\theta^{\beta}}\left[(x+\theta)^{\beta}-\theta^{\beta}\right]}\left(1+\frac{x}{\theta}\right)^{\beta}Ln\left(1+\frac{x}{\theta}\right)}{1-e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}}=0 \tag{26}$$

$$\frac{\partial l}{\partial \theta}=\frac{-n}{\theta}-(\beta-1)\sum_{i=1}^{n}\frac{x}{\theta^{2}\left(1+\frac{x}{\theta}\right)}+\lambda\sum_{i=1}^{n}\beta\left(1+\frac{x}{\theta}\right)^{\beta-1}\left(\frac{x}{\theta^{2}}\right)-(\gamma-1)\sum_{i=1}^{n}\frac{\lambda\beta x\left(1+\frac{x}{\theta}\right)^{\beta-1}}{\theta^{2}\left\{1-e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right\}}=0 \tag{27}$$

The normal equations (24), (26) and (27) do not have an explicit solution and they have to be solved numerically. The ML estimate of γ is obtained as a solution of (23) in a closed form as follows:

$$\hat{\gamma}=\frac{-n}{\sum_{i=1}^{n}Ln\left\{1-e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right\}}. \tag{28}$$

For interval estimation of the parameters the observed variance covariance matrix of the ML estimators are derived in by taking the negative expectation of the second derivatives of the natural logarithm of the likelihood function with respect to$\underline{\varphi}$.

$$\left[I\left(\underline{\varphi}\right)\right]^{-1}=-\begin{bmatrix}\left(\frac{\partial^{2}\mathscr{L}}{\partial\lambda^{2}}\right) & \left(\frac{\partial^{2}\mathscr{L}}{\partial\lambda\partial\gamma}\right) & \left(\frac{\partial^{2}\mathscr{L}}{\partial\lambda\partial\beta}\right) & \left(\frac{\partial^{2}\mathscr{L}}{\partial\lambda\partial\theta}\right)\\ \left(\frac{\partial^{2}\mathscr{L}}{\partial\gamma\partial\lambda}\right) & \left(\frac{\partial^{2}\mathscr{L}}{\partial\gamma^{2}}\right) & \left(\frac{\partial^{2}\mathscr{L}}{\partial\gamma\partial\beta}\right) & \left(\frac{\partial^{2}\mathscr{L}}{\partial\gamma\partial\theta}\right)\\ \left(\frac{\partial^{2}\mathscr{L}}{\partial\beta\partial\lambda}\right) & \left(\frac{\partial^{2}\mathscr{L}}{\partial\beta\partial\gamma}\right) & \left(\frac{\partial^{2}\mathscr{L}}{\partial\beta^{2}}\right) & \left(\frac{\partial^{2}\mathscr{L}}{\partial\beta\partial\theta}\right)\\ \left(\frac{\partial^{2}\mathscr{L}}{\partial\theta\partial\lambda}\right) & \left(\frac{\partial^{2}\mathscr{L}}{\partial\theta\partial\gamma}\right) & \left(\frac{\partial^{2}\mathscr{L}}{\partial\theta\partial\beta}\right) & \left(\frac{\partial^{2}\mathscr{L}}{\partial\theta^{2}}\right)\end{bmatrix}^{-1}_{(\lambda=\hat{\lambda},\gamma=\hat{\gamma},\beta=\hat{\beta},\theta=\hat{\theta})} \tag{29}$$

According to the estimate asymptotic multivariate normal $N_m(\varphi)(0,\ (\varphi)^{-1})$ distribution of φ can be used to constructan approximate confidence interval for the parameters.

4.2 Maximum product spacing estimation

One of the most common methods for estimating the parameters of a distribution is the ML method. Although this method is consistent, asymptotically efficient, it was found to be unbounded and inefficient in the estimation in various cases, such as involving certain mixtures of a continuous distribution, heavy- tailed distributions and J- shaped distributions Thongkairat, et al. [13].

The maximum product of spacings (MPS) method was introduced by Chang and Amin [14] as an alternative to ML for the estimation of parameters of continuous univariate distributions. The MPS estimators are consistent, asymptotically normal and efficient.

Suppose that an ordered random sample $x_1,\ldots,x_n$ drawn from OGEL distribution with parameters $\underline{\varphi}=(\lambda,\gamma,\beta,\theta)$and CDF (5) the 1-step spacing is constructed as:

$$D_i(\varphi)=\ F_{\varphi}\ (x_{i:n})-\ F_{\varphi}(x_{i-1:n}),\ \ i=1,\ldots,n. \tag{30}$$

Where$\sum_{i=1}^{n}D_i=1$. To estimate the unknown parameters, the product spacings are defined and the geometric mean of spacings is maximized as follows:

$$G=\left[\prod_{i=1}^{n+1}D_i(\varphi)\right]^{\frac{1}{n+1}}=\left\{\prod_{i=1}^{n+1}F_{\varphi}\ (x_{i:n})\ -\ F_{\varphi}(x_{i-1:n})\right\}^{\frac{1}{n+1}}. \tag{31}$$

Then, by taking the logarithm of G

$$\ln G = \frac{1}{n+1} \sum_{i=1}^{n+1} ln\left\{F_{\underline{\varphi}}\,(x_{i:n}) \;-\; F_{\underline{\varphi}}(x_{i-1:n})\right\}\;, let\; H = \ln G \tag{32}$$

In this study the maximization of the quantity in (32) is defined as:

$$\hat{\varphi}_{MPS} = \arg max \sum_{i=1}^{n+1} ln\{F_{\varphi}\,(x_{i:n}) \;-\; F_{\varphi}(x_{i-1:n})\} \tag{33}$$

Substitute (5) in (32) the function H is given by:

$$H = \frac{1}{n+1}\left\{lnD_1 + \sum_{i=2}^{n} lnD_i + lnD_{n+1}\right\}$$

$$= \frac{1}{n+1}\left\{ln\left[1 - e^{-\lambda\left[(1+\frac{x_1}{\theta})^{\beta}-1\right]}\right]^{\gamma} + \sum_{i=2}^{n} ln\left[\left[1 - e^{-\lambda\left[\left(1+\frac{x_i}{\theta}\right)^{\beta}-1\right]}\right]^{\gamma} - \left[1 - e^{-\lambda\left[(1+\frac{x_{(i-1)}}{\theta})^{\beta}-1\right]}\right]^{\gamma}\right]\right\} + \frac{1}{n+1}\left\{ln\left[1 - \right.\right.$$

e−λ(1+x𝑛θ)β−1γ

(34)

Taking the partial derivative of (34) with respect to $\underline{\varphi} = (\lambda, \gamma, \beta, \theta)$ and equating tozero;

$$\frac{\partial}{\partial\lambda} H\left(\underline{\varphi}\right) = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\varphi)}\left[\Delta_1\left(x_{i,n}\middle|\underline{\varphi}\right) - \Delta_1\left(x_{i-1,n}\middle|\underline{\varphi}\right)\right] = 0 \tag{35}$$

$$\frac{\partial}{\partial\gamma} H\left(\underline{\varphi}\right) = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\varphi)}\left[\Delta_2\left(x_{i,n}\middle|\underline{\varphi}\right) - \Delta_2\left(x_{i-1,n}\middle|\underline{\varphi}\right)\right] = 0 \tag{36}$$

$$\frac{\partial}{\partial\beta} H\left(\underline{\varphi}\right) = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\varphi)}\left[\Delta_3\left(x_{i,n}\middle|\underline{\varphi}\right) - \Delta_3\left(x_{i-1,n}\middle|\underline{\varphi}\right)\right] = 0 \tag{37}$$

$$\frac{\partial}{\partial\theta} H\left(\underline{\varphi}\right) = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\varphi)}\left[\Delta_4\left(x_{i,n}\middle|\underline{\varphi}\right) - \Delta_4\left(x_{i-1,n}\middle|\underline{\varphi}\right)\right] = 0 \tag{38}$$

Where,

$$\Delta_1 = \left(x_{i,n}\middle|\underline{\varphi}\right) = -e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\gamma\left[-\left(\frac{x_{i,n}}{\theta}+1\right)^{\beta}+1\right]\left[1 - e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right]^{\gamma-1} \tag{39}$$

$$\Delta_2 = \left(x_{i,n}\middle|\underline{\varphi}\right) = Ln\left\{1 - e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right\}\left[1 - e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right]^{\gamma} \tag{40}$$

$$\Delta_3 = \left(x_{i,n}\middle|\underline{\varphi}\right) = \left[1 - e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right]^{\gamma-1}\left[\lambda\gamma e^{-\lambda\left[\left(1+\frac{x}{\theta}\right)^{\beta}-1\right]}\right]\left[Ln\left(\frac{x_{i,n}}{\theta}+1\right)(\frac{x_{i,n}}{\theta}+1)^{\beta}\right] \tag{41}$$

$$\Delta_4 = \left(x_{i,n}\middle|\underline{\varphi}\right) = \gamma\left[1 - e^{-\lambda\left[\left(1+\frac{x_{i,n}}{\theta}\right)^{\beta}-1\right]}\right]^{\gamma-1}\lambda e^{-\lambda\left[\left(1+\frac{x_{i,n}}{\theta}\right)^{\beta}-1\right]}\left[\frac{\beta x_{i,n}}{\theta^2}(1+\frac{x_{i,n}}{\theta})^{\beta-1}\right] \tag{42}$$

Cheng and Amin [14] showed that maximizing *H* as a method of parameter estimation is as efficient as ML estimation, and the MPS estimators are consistent under more general conditions than the ML estimators.

The MPS method shows asymptotic properties like the Maximum likelihood estimators. Cheng and Amin [14] introduced the variance covariance matrix of the MPS estimators. Therefore, the asymptotic properties of MPS can be used to construct the asymptotic confidence intervals for the parameters Anatolyev et al. [15]. Let $I\left(\underline{\hat{\varphi}}\right)$ is the observed Fishers information matrix it can be defined as:

$$I\left(\underline{\hat{\varphi}}\right) = \begin{bmatrix} \left(\frac{\partial^2 H}{\partial\lambda^2}\right) & \left(\frac{\partial^2 H}{\partial\lambda\partial\gamma}\right) & \left(\frac{\partial^2 H}{\partial\lambda\partial\beta}\right) & \left(\frac{\partial^2 H}{\partial\lambda\partial\theta}\right) \\ \left(\frac{\partial^2 H}{\partial\gamma\partial\lambda}\right) & \left(\frac{\partial^2 H}{\partial\gamma^2}\right) & \left(\frac{\partial^2 H}{\partial\gamma\partial\beta}\right) & \left(\frac{\partial^2 H}{\partial\gamma\partial\theta}\right) \\ \left(\frac{\partial^2 H}{\partial\beta\partial\lambda}\right) & \left(\frac{\partial^2 H}{\partial\beta\partial\gamma}\right) & \left(\frac{\partial^2 H}{\partial\beta^2}\right) & \left(\frac{\partial^2 H}{\partial\beta\partial\theta}\right) \\ \left(\frac{\partial^2 H}{\partial\theta\partial\lambda}\right) & \left(\frac{\partial^2 H}{\partial\theta\partial\gamma}\right) & \left(\frac{\partial^2 H}{\partial\theta\partial\beta}\right) & \left(\frac{\partial^2 H}{\partial\theta^2}\right) \end{bmatrix}^{-1} = -\begin{bmatrix} H''_{\lambda\lambda} & H''_{\lambda\gamma} & H''_{\lambda\beta} & H''_{\lambda\theta} \\ H''_{\gamma\lambda} & H''_{\gamma\gamma} & H''_{\gamma\beta} & H''_{\gamma\theta} \\ H''_{\beta\lambda} & H''_{\beta\gamma} & H''_{\beta\beta} & H''_{\beta\theta} \\ H''_{\theta\lambda} & H''_{\theta\gamma} & H''_{\theta\beta} & H''_{\theta\theta} \end{bmatrix}, \quad (43)$$

Therefore, based on these derivatives, the information matrix$I\left(\underline{\hat{\varphi}}\right)$ can be obtained. The approximate (1−b) 100% confidence intervals for the parameters λ, γ, β and θ are given as, $\hat{\lambda} \pm \alpha_{\frac{b}{2}}(\sqrt{V(\hat{\lambda})}$,$\hat{\gamma} \pm \alpha_{\frac{b}{2}}(\sqrt{V(\hat{\gamma})}$,$\hat{\beta} \pm \alpha_{\frac{b}{2}}(\sqrt{V(\hat{\beta})}$ $and \hat{\theta} \pm \alpha_{\frac{b}{2}}(\sqrt{V(\hat{\theta})}$ respectively, where $\alpha_{\frac{b}{2}}$ is the upper $\frac{b}{2}$ percentile of standard normal distribution, $(\hat{\lambda}, \hat{\gamma}, \hat{\beta}, \hat{\theta})$ are the MPS estimates of the parameters (λ, γ, β, θ).

5 Simulation Study

In this section a simulation study is introduced to illustrate the theoretical results considering ML and MPS methods based on generated data from OGEL distribution by taking the parameter θ as known for all methods of estimation.

For each method of estimation, the initial parameter values and sample sizes, the *averages*, *mean square error (MSE), relative biases (RB)*and *asymptotic confidence interval (ACI)* is calculated using the following formulae:

1) $Bias^2 = (average\ of\ the\ parameter - true\ value\ of\ the\ parameter)^2$
2) $MSE = mean[(estimator - true\ value)^2]$. (44)
3) $Relative\ Bias = \frac{Bias}{true\ value}$. (45)
4) The asymptotic multivariate normal of ML estimation for the parameters $\hat{\omega} = (\hat{\lambda}, \hat{\gamma}, \hat{\beta})$ can be used to compute the asymptotic$100(1-\vartheta)\%, 0 < \vartheta < 1$, for the parameters as follows:

$$\hat{\lambda} \pm z_{(1-\frac{\vartheta}{2})}\sqrt{Var(\hat{\lambda})}, \hat{\gamma} \pm z_{(1-\frac{\vartheta}{2})}\sqrt{Var(\hat{\gamma})}, \hat{\beta} \pm z_{(1-\frac{\vartheta}{2})}\sqrt{Var(\hat{\beta})}, \quad (46)$$

5) The asymptotic confidence intervals of MPS estimation for the parameters $\hat{\omega} = (\hat{\lambda}, \hat{\gamma}, \hat{\beta})$ can be used to compute the asymptotic $100(1-b)\%, 0 < b < 1$, for the parameters as follows

$$\hat{\lambda} \pm z_{(\frac{\vartheta}{2})}\sqrt{V(\hat{\lambda})}\ , \hat{\gamma} \pm z_{(\frac{\vartheta}{2})}\sqrt{V(\hat{\gamma})},\ \hat{\beta} \pm z_{(\frac{\vartheta}{2})}\sqrt{V(\hat{\beta})} \quad (47)$$

Where $z_{(\frac{\vartheta}{2})}$ is the upper$(\frac{\vartheta}{2})$percentile of the standard normal distribution.

The following steps are used to compute the ML and MPS estimates for OGEL distribution for different sample sizes [n=100, 150, 200, 300].

1. Generate random samples of size n from OGEL distribution [n=100, 150, 200, 300] by using (12).
2. Obtain the ML estimates by solving equations (24-27).
3. Obtain the MPS estimates by solving equations (35-38).
4. Compute the MSE, RB and ACI for each estimate and for the ML and MPS methods using equations (44), (45), (46) and (47).
5. Repeat the above steps for all methods of estimation and different sample sizes with repetition 1000 using the R Studio program version (1.1.463).

   The results of the simulation study are illustrated in Tables (1-2). From these tables, it is noticed that:
   - As expected the MSE decreased when n increased.
   - The MSE of the MPS estimates is less than the MSE of the ML estimates for all parameters and sample sizes.

- The RB of the MPS estimates is less than the RB of the ML estimates for all parameters and sample sizes.
- As expected, the performance of the MPS estimates is appropriate than the ML estimates.

Table 1. The Average, MSE, RB and ACI for MLE and MPS for the parameters (λ=0.8, γ=0.8, β=0.5, θ=1.5) for OGEL distribution for 1000 repetitions and different sample sizes

Sample size	Methods	Parameters	Averages	MSE	RB	ACI LI	UI
		λ	0.47	0.26	0.41	0.10	1.35
	ML	γ	0.82	0.03	0.02	0.56	1.19
		β	1.04	0.37	1.08	0.57	1.63
100		λ	0.50	0.24	0.37	0.10	1.46
	MPS	γ	0.79	0.02	0.01	0.54	1.15
		β	0.99	0.32	0.99	0.52	1.60
		λ	0.44	0.19	0.44	0.13	1.12
	ML	γ	0.80	0.01	0.01	0.58	1.09
		β	1.03	0.33	1.06	0.62	1.52
		λ	0.47	0.19	0.41	0.12	1.23
150	MPS	γ	0.78	0.01	0.01	0.57	1.07
		β	0.99	0.30	0.99	0.57	1.47
		λ	0.43	0.18	0.46	0.16	0.95
	ML	γ	0.81	0.01	0.01	0.63	1.06
		β	1.01	0.30	1.04	0.68	1.42
200		λ	0.45	0.16	0.43	0.16	0.99
	MPS	γ	0.79	0.01	0.01	0.61	1.04
		β	0.98	0.27	0.97	0.65	1.38
		λ	0.41	0.16	0.47	0.19	0.78
	ML	γ	0.80	0.01	0.01	0.65	1.00
		β	1.01	0.28	1.03	0.74	1.31
300		λ	0.43	0.16	0.45	0.19	0.81
	MPS	γ	0.78	0.01	0.00	0.64	0.98
		β	0.99	0.26	0.98	0.72	1.28

Table 2. The Average, MSE, RB and ACI for MLE and MPS for the parameters (λ=0.5, γ=0.8, β=0.8, θ=1.5) for OGEL distribution for 1000 repetitions and different sample sizes

Sample size	Methods	Parameters	Averages	MSE	RB	ACI LI	UI
		λ	0.94	0.98	0.88	0.19	**2.89**
	ML	γ	1.56	0.16	0.04	0.98	**2.52**
		β	1.04	0.31	0.31	0.47	**1.72**
100		λ	1.01	1.46	1.01	0.18	**3.13**
	MPS	γ	1.50	0.14	.003	0.92	**2.40**
		β	.99	0.35	0.33	0.44	**1.71**
		λ	0.84	0.36	0.68	0.23	**2.11**
	ML	γ	1.54	0.09	0.03	1.02	**2.22**
		β	1.01	0.28	0.30	0.59	**1.60**
		λ	0.88	0.45	0.76	0.23	**2.29**
150	MPS	γ	1.49	0.09	0.00	0.99	**2.17**
		β	0.99	0.32	0.33	0.54	**1.56**
		λ	0.82	0.28	0.65	0.29	**1.90**
	ML	γ	1.53	0.07	0.02	1.10	**2.14**
		β	1.02	0.27	0.32	0.62	**1.47**

Sample size	Methods	Parameters	Averages	MSE	RB	ACI LI	UI
200		λ	.86	0.33	0.72	0.29	**2.01**
	MPS	γ	1.49	0.06	.002	1.07	**2.10**
		β	.99	0.31	0.34	0.58	**1.46**
		λ	0.81	0.21	0.62	0.34	**1.61**
	ML	γ	1.52	0.05	0.01	1.16	**2.33**
		β	1.01	0.27	0.32	0.68	**1.38**
300		λ	0.84	0.24	0.68	0.35	**1.72**
	MPS	γ	1.50	0.04	0.00	1.14	**2.00**
		β	0.99	0.29	0.34	0.66	**1.37**

6 An Application

In this section, a real data set is provided to show how the OGEL distribution can be applied in practice.This data will be studied in two ways, firstly comparing the two methods of estimation. Secondly illustrate the importance and flexibility of the OGEL distribution with its sub-models [generalized Exponential (GE), odd Exponential Lomax (OEL)]. The estimation of the unknown parameters for OGEL distribution will be obtained by the ML and MPS methods. The estimation of the unknown parameters for each distribution will be obtained by the maximum-likelihood method. The values for some models of the following statistics: the -2ln L statistic, Akaike information criterion (AIC), Bayesian information criterion (BIC) and the consistent Akaike information criterion (CAIC) are used to compare the candidate distributions. In general, the smaller values of these statistics, the appropriate fit to real data.

The COVID-19 data set is analyzed to illustrate the flexibility of OGEL distribution. The data set is referred to daily recovery cases of a random sample of 142 days in the interval of (12May to 30 September 2020) of coronavirus patients in the ARAB REPUBLIC of EGYPT these data are given as,

154 160 140 173 151 222 268 302 252 223 157 254 179 93 127 178 154 152 182 344 410 380 523 406 402 380 423 414 411 503 402 417 421 402 398 401 411 387 400 399 409 397 402 400 403 399 402 400 412 509 421 407 402 413 623 512 480 523 512 521 602 543 556 569 591 556 611 566 512 544 549 602 991 904 933 928 1007 1121 1066 1211 1402 1499 1318 1611 1503 1613 1716 1655 1119 1006 1101 1109 1013 989 968 977 908 908 911 991 909 856 800 809 890 900 996 899 807 883 818 809 899 798 789 890 903 900 900 878 803 908 876 788 900 808 776 804 609 800 708 866 800 811 700 801 887 843 706 506 508 400

Table 3. Descriptive Statistics for the COVID-19 data set

Mean	Median	Mode	St. D	Variance	Skewness	Kurtosis	25^{th} P.	75^{th} P.
663.5	596.5	402	349	121832.9	0.70	0.41	402	900

It can be noticed from Table 3 that the data set is moderately skewed and platykurtic.

Before analyzing the data, Anderson darling(AD) goodness of fit test with its (p- value=0.4007) is calculated to check the validity of the fitted model with its sub-models, for the considered models the values of AD are [for OGEL=1.74, OEL=1.78 and GE=1.98], also the scaled- TTT plot can be used to verify our distribution validity Arrest [16]. It allows identifying the shape of the h(t) graphically. The empirical scaled-TTT plot of the COVID-19 data set is shown in Fig. 3. This Figure indicates that the TTT plot is concave than convex which indicates an upside down bathtub hazard rate. It verifies our distribution validity. The plot of the empirical CDF of the fitted COVID-19 data set is displayed in Fig. 3 for the ML and MPS methods of estimation. The ML and MPS estimates are calculated with their standard errors (SEs), in Table. 4.

Table 4. ML and MPS estimates, SEs of the model parameters $(\lambda, \gamma, \beta, \theta)$ for the 142 recovery cases of Covid-19 patient's data

Methods	Parameters	Estimates	SEs	LI	UI
	λ	1.63	2.86	1.16	2.10
	γ	2.47	0.62	2.36	2.57
ML	β	3.96	9.07	2.47	5.45
	θ	36.35	134.97	14.15	58.55
	λ	0.33	1.12	0.12	0.52
	γ	2.12	0.86	1.97	2.26
MPS	β	1.77	1.01	1.61	1.94
	θ	4.27	14.55	1.88	6.67

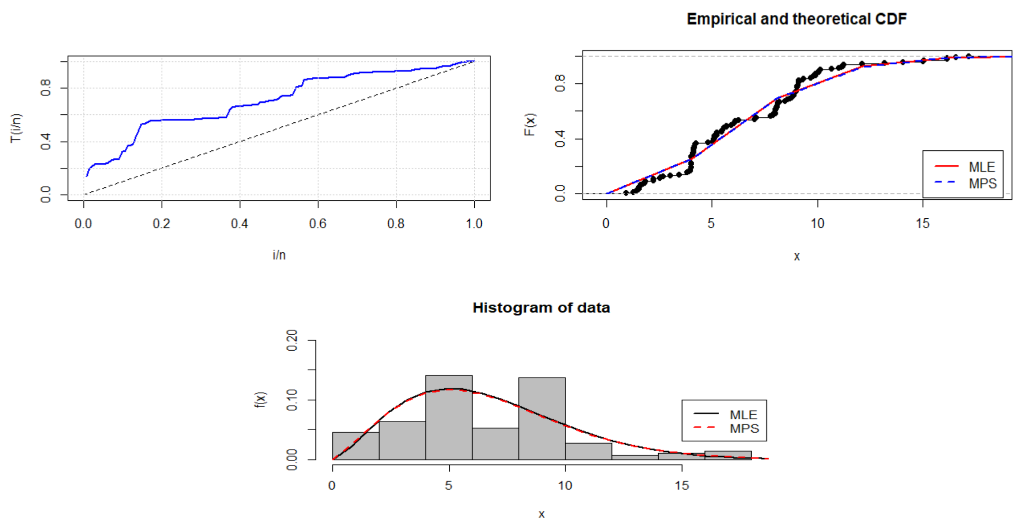

Fig. 3. The TTT plot, the Empirical CDF and histogram of the OGEL distribution for the COVID- 19data sets

Table 5 lists the numerical values AIC, BIC and CAIC for OGEL and its sub models OEL and GE, the ML estimates, and the corresponding standard errors (SEs) in parentheses of the parameters for all fitted models. For the Covid-19 data the OGEL distribution has the lowest values among all fitted models, it could be chosen as the best model. Fig. 4 supported this result very well.

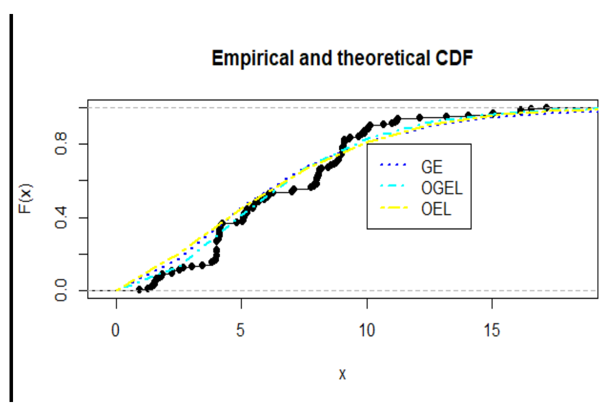

Fig. 4. The CDFs of the OGEL model and other fitted models for the COVID-19 data set

Table 5. Numerical values AIC, BIC and CAIC for OGEL and its sub- models OEL and GE for Covid-19 data

Models	Estimates					Goodness-of-fit		
	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\theta}$	-2ln L	AIC	CAIC	BIC
OGEL	0.93 (1.54)	2.44 (0.76)	2.26 (1.40)	12.33 (21.9)	738.12	750.85	751.15	762.67
OEL	0.001 (0.002)		2.01 (0.02)	0.09 (0.04)	739.07	765.68	765.86	774.54
GE	0.30 (0.02)	3.77 (0.53)			742.88	760.74	760.82	766.65

7 Conclusions

In this paper, a probability distribution called Odd Generalized Exponential-Lomax distribution has been derived. This is based on the T-X family of distributions proposed by Alzaatreh et al. [11]. The OGEL distribution generalizes the exponential Lomax distribution presented by El-Bassiouny et al. [2]. The main properties of the proposed distribution are derived. ML and MPS methods of estimation are used to estimate the model parameters. A simulation study is provided to evaluate the performance of the estimates. The distribution has been applied to analyze the recovery cases of COIVD- 19 in Egypt from 12 May to 30 September 2020, by comparing the OGEL with its sub models the fitting results show that the OGEL could be chosen as the best model. That is, the OGEL model could be used for similar analyses in another period.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Bantan RAR, Jamal F, Elgarhy M. On the analysis of new COVID-19 cases in Pakistan using an exponentiated version of the M family of distributions. Mathematics. 2020;8(6):1–20.

[2] El-Bassiouny AH, Abdo NF, Shahen HS. Exponential lomax distribution. International Journal Computer Applications. 2015;121(13):24-29.

[3] Hassan AS, Abd-Alla M. Exponentiated weibull lomax: Properties and estimation. J. Data Sci. 2018;16:277-298.

[4] Cordeiro GM, Afify AZ, Ortega EM, Suzuki AK, Mead ME. The odd Lomax generator of distributions: Properties, estimation and applications. J. Comput. Appl. Math. 2019;347:222–237.

[5] Chesneau, C, Djlbrila, S. The generalized odd inverted exponential-G family of distributions: Properties and applications. Eurasian Bulletin of Mathematics (ISSN: 2687-5632). 2019;2(3):86-110. ISSN 2687-5632.

[6] Alizadeh M, Afify AZ, Eliwa M, Ali S. The odd log-logistic Lindley-G family of distributions: Properties, Bayesian and non-Bayesian estimation with applications. Comput. Stat. 2020;35:281–308.

[7] Afify A, Alizadeh M. The Odd dagum family of distributions: Properties and applications. J. Appl. Probab. Stat. 2020;15:45–72.

[8] Gupta RD, Kundu D. Generalized exponential distributions. Australian and New Zealand Journal of Statistics. 1999a;41(2):173 - 188.

[9] Gupta RD, Kundu D. Generalized exponential distributions: Statistical inferences. Technical Report, the University of New Brunswick, Saint John; 1999b.

[10] Lomax KS. Business failures; Another example of the analysis of failure data. Journal of the American Statistical Association. 1954;49: 847–852.
DOI: 10.1080/01621451-1954-10501239.

[11] Alzaatreh A, Lee C, Famoye F. A new method for generating families of continous distributions. Metron. 2013;71:63-79.
DOI: 10-1007/s40300-013-0007-y

[12] Casella G, Berger RL. Statistical inference. Brooks/Cole Publishing Company; 1990.

[13] Thongkairat S, Yamaka W, Sriboonchitta S. Maximum product spacings method for the estimation of parameters of linear regression. Journal of Physics: Conference Series. 2018;1053:012110.
doi: 10.108811742- 659611053/1/012110.

[14] Cheng RCH, Amin NAK. Estimating parameters in continuous univariate distributions with a shifted origin. Journal of the Royal Statistical Society. Series B Methodological. 1983;45(3):394–403.
DOI: 10.1111/j.2517-6161.1983.tb01268.x.

[15] Anatolyev S, Kosenok G. An alternative to maximum likelihood based on spacings. Econometric Theory. 2005;21(2): 472-476.
doi: 10.1017/s0266466605050255.

[16] Aarset MV. How to identify a bathtub hazard rate. IEEE Transactions on Reliability. 1987;36(1):106-108.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/69313