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Complete Einstein's Equations of Motion for Test Particles Exterior to Spherical Massive Bodies using a Varying Potential

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Authors' contributions

This work was carried out in collaboration among all authors. Author AUM designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors AUM and MMA managed the analyses of the study. Authors AUM and IIE managed the literature searches. All authors read and approved the final manuscript.

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ABSTRACT

In this article, a generalized varying gravitational scalar potential was used to completely define the metric tensors and coefficients of affine connections for spherical massive bodies whose tensor field varies with time, radial distance and polar angle. The completely defined metric tensors and coefficients of affine connections were used to study Einstein's equations of motion for test particles within this field. The results obtained to the limit of c^0 reduced to the corresponding Schwarzchild equations and to the limit of c^{-2}, it contained additional terms not found in Schwarzchild equations which can be used in the study of blackhole and gravitational wave in this field and other astrophysical phenomena.

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1. INTRODUCTION

General Relativity is the geometrical theory of gravitation published by Albert Einstein in 1915/1916. It unifies Special Relativity and Sir Isaac Newton's law of universal gravitation with the insight that gravitation is not due to a force but rather a manifestation of curved space and time, with the curvature being produced by the mass-energy and momentum content of the space-time. General Relativity is the most widely accepted theory of gravitation. In actual fact, the mathematical material (namely, differential geometry) needed to attain a deep understanding of general relativity is not particularly difficult and requires a background no greater than that provided by standard courses in advanced calculus and linear algebra [1].

After the publication of Einstein's geometrical gravitational field equations (EGGFE) in 1915, the search for their exact and analytical solutions for all the gravitational fields in nature began [1-3]. Schwarzchild first constructed the exact solution to this field equation in static and pure radial spherical polar coordinates in 1916 by considering astrophysical bodies such as the sun and the stars [4]. In Schwarzchild's metric, the tensor field varies with radial distance only. Research has shown that spherical systems doesn't depend on radial distance only, therefore in this article we study the complete Einstein's equations of motion, for test particle exterior to spherical distribution of mass whose tensor field varies with time, radial distance and polar angle using our recent scalar potential constructed by [5].

2. THEORETICAL FRAME WORK

The covariant metric tensors for this distribution of mass or pressure in spherical polar coordinates $f(t,r,\theta)$ constructed by [4,6-7] are given as:

$$g_{00} = \left[1 + \frac{2f(t,r,\theta)}{c^2}\right]. \tag{2.1}$$

$$g_{11} = -\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1}. \tag{2.2}$$

$$g_{22} = -r^2. \tag{2.3}$$

$$g_{33} = -r^2 \sin^2\theta. \tag{2.4}$$

$$g_{\mu\nu} = 0 \text{, otherwise} \tag{2.5}$$

The contravariant metric tensors for this field were obtained using Quotient Theorem [4,6-7] are given as

$$g^{00} = \left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1}. \tag{2.6}$$

$$g^{11} = -\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]. \tag{2.7}$$

$$g^{22} = -\frac{1}{r^2}. \tag{2.8}$$

$$g^{33} = -\frac{1}{r^2\sin^2\theta}. \tag{2.9}$$

$$g^{\mu\nu} = 0\text{, otherwise} \tag{2.10}$$

In our recent article [5], the gravitational scalar potential $f(t,r,\theta)$ is given as

$$f(t,r,\theta) = -\frac{k}{r}\exp\left(t - \frac{r\theta}{c}\right), \tag{2.11}$$

where $k = GM$.

Put equation (2.11) into (2.1), (2.2),(2.6) and (2.7) gives,

$$g_{00} = \left[1 - \frac{2k}{c^2 r}\exp\left(t - \frac{r\theta}{c}\right)\right], \tag{2.12}$$

Expanding (2.12) binomially to order 2, gives

$$g_{00} = \left[1 - \frac{2k}{c^2 r}\left\{1 + \left(t - \frac{r\theta}{c}\right) + \frac{\left(t - \frac{r\theta}{c}\right)^2}{2!} + \ldots\right\}\right], \tag{2.13}$$

The term $\left(t - \frac{r\theta}{c}\right)$ converge to zero

to the limit of c^{-2} (2.13) reduces to

$$g_{00} = \left[1 - \frac{2k}{c^2 r} - \frac{2kt}{c^2 r} - \frac{kt^2}{c^2 r}\right]. \tag{2.14}$$

$$g_{11} = -\left[1 - \frac{2k}{c^2 r}\exp\left(t - \frac{r\theta}{c}\right)\right]^{-1}. \quad (2.15)$$

Similarly expanding (2.15) binomially and taking to the limit of c^{-2} gives

$$g_{11} = -\left[1 + \frac{2k}{c^2 r} + \frac{2kt}{c^2 r} + \frac{kt^2}{c^2 r}\right]. \quad (2.16)$$

$$g^{00} = \left[1 - \frac{2k}{c^2 r}\exp\left(t - \frac{r\theta}{c}\right)\right]^{-1}. \quad (2.17)$$

Similarly expanding (2.17) binomially and taking to the limit of c^{-2} gives

$$g^{00} = \left[1 + \frac{2k}{c^2 r} + \frac{2kt}{c^2 r} + \frac{kt^2}{c^2 r}\right]. \quad (2.18)$$

$$g^{11} = -\left[1 - \frac{2k}{c^2 r}\exp\left(t - \frac{r\theta}{c}\right)\right]. \quad (2.19)$$

Similarly expanding (2.19) binomially and taking to the limit of c^{-2} gives

$$g^{11} = -\left[1 - \frac{2k}{c^2 r} - \frac{2kt}{c^2 r} - \frac{kt^2}{c^2 r}\right]. \quad (2.20)$$

The above results give the complete gravitational metric tensors for this field. Remarkably our metric tensors are similar to that obtained by [6,8].

The coefficients of affine connections, in [4,7,9-10] by the metric tensors of space-time are determined using equations (2.1)-(2.10),

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\xi}\left(g_{\alpha\xi,\beta} + g_{\beta\xi,\alpha} - g_{\alpha\beta,\xi}\right). \quad (2.21)$$

where $g^{\mu\xi}$ is the covariant metric tensors and $g_{\alpha\xi,\beta}$ the contravariant metric tensors.

Explicitly within this region, the coefficients of affine connections in terms of (t, r, θ) are given by

$$\Gamma^{0}{}_{00} = \frac{1}{c^2}\left[g^{00}\right]\frac{\partial f(t,r,\theta)}{\partial t}. \quad (2.22)$$

$$\Gamma^{0}{}_{00} = \frac{1}{c^2}\left[1 - \frac{2k}{c^2 r} - \frac{2kt}{c^2 r} - \frac{kt^2}{c^2 r}\right]\left[-\frac{k}{r}\exp\left(t - \frac{r\theta}{c}\right)\right]. \quad (2.23)$$

To the limit of c^{-2}, (2.23) reduces to

$$\Gamma^{0}{}_{00} = -\frac{k}{c^2 r} - \frac{kt}{c^2 r} - \frac{kt^2}{2c^2 r}. \quad (2.24)$$

$$\Gamma^{0}{}_{10} = \Gamma^{0}{}_{01} = \frac{1}{c^2}\left[g^{00}\right]\frac{\partial f(t,r,\theta)}{\partial r}. \quad (2.25)$$

$$\Gamma^{0}{}_{10} = \Gamma^{0}{}_{01} = \frac{1}{c^2}\left[1 - \frac{2k}{c^2 r} - \frac{2kt}{c^2 r} - \frac{kt^2}{c^2 r}\right]\left[\frac{k}{r^2}\exp\left(t - \frac{r\theta}{c}\right) + \frac{k\theta}{cr}\exp\left(t - \frac{r\theta}{c}\right)\right]. \quad (2.26)$$

To the limit of c^{-2}, (2.26) reduces to

$$\Gamma^{0}{}_{10} = \Gamma^{0}{}_{01} = \frac{k}{c^2 r^2} + \frac{kt}{c^2 r^2} + \frac{kt^2}{2c^2 r^2}. \quad (2.27)$$

$$\Gamma^{0}{}_{11} = \frac{1}{c^2}\left[g^{11}\right]\frac{\partial f(t,r,\theta)}{\partial t}. \tag{2.28}$$

$$\Gamma^{0}{}_{11} = \frac{1}{c^2}\left[1-\frac{2k}{c^2 r}-\frac{2kt}{c^2 r}-\frac{kt^2}{c^2 r}\right]\left[\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]. \tag{2.29}$$

To the limit of c^{-2}, (2.29) reduces to

$$\Gamma^{0}{}_{11} = \frac{k}{c^2 r}+\frac{kt}{c^2 r}+\frac{kt^2}{2c^2 r}. \tag{2.30}$$

$$\Gamma^{0}{}_{02} = \Gamma^{0}{}_{20} = \frac{1}{c^2}\left[g^{00}\right]\frac{\partial f(t,r,\theta)}{\partial \theta}. \tag{2.31}$$

$$\Gamma^{0}{}_{02} = \Gamma^{0}{}_{20} = \frac{1}{c^2}\left[1+\frac{2k}{c^2 r}+\frac{2kt}{c^2 r}+\frac{kt^2}{c^2 r}\right]\left[\frac{k}{c}\exp\left(t-\frac{r\theta}{c}\right)\right]. \tag{2.32}$$

To the limit of c^{-2}, (2.32) reduces to

$$\Gamma^{0}{}_{02} = \Gamma^{0}{}_{20} = 0. \tag{2.33}$$

$$\Gamma^{1}{}_{00} = \frac{1}{c^2}\left[g_{00}\right]\frac{\partial f(t,r,\theta)}{\partial r}. \tag{2.34}$$

$$\Gamma^{1}{}_{00} = \frac{1}{c^2}\left[1-\frac{2k}{c^2 r}-\frac{2kt}{c^2 r}-\frac{kt^2}{c^2 r}\right]\left[\frac{k}{r^2}\exp\left(t-\frac{r\theta}{c}\right)+\frac{k\theta}{cr}\exp\left(t-\frac{r\theta}{c}\right)\right]. \tag{2.35}$$

To the limit of c^{-2}, (2.35) reduces to

$$\Gamma^{1}{}_{00} = \frac{k}{c^2 r^2}+\frac{kt}{c^2 r^2}+\frac{kt^2}{2c^2 r^2}. \tag{2.36}$$

$$\Gamma^{1}{}_{10} = \Gamma^{1}{}_{01} = \frac{1}{c^2}\left[g_{11}\right]\frac{\partial f(t,r,\theta)}{\partial t}. \tag{2.37}$$

$$\Gamma^{1}{}_{01} = \Gamma^{1}{}_{10} = \frac{1}{c^2}\left[1+\frac{2k}{c^2 r}+\frac{2kt}{c^2 r}+\frac{kt^2}{c^2 r}\right]\left[\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]. \tag{2.38}$$

To the limit of c^{-2}, (2.38) reduces to

$$\Gamma^{1}{}_{10} = \Gamma^{1}{}_{01} = \frac{k}{c^2 r}+\frac{kt}{c^2 r}+\frac{kt^2}{2c^2 r}. \tag{2.39}$$

$$\Gamma^{1}{}_{11} = \frac{1}{c^2}\left[g_{11}\right]\frac{\partial f(t,r,\theta)}{\partial r}. \quad (2.40)$$

$$\Gamma^{1}{}_{11} = -\frac{1}{c^2}\left[1-\frac{2k}{c^2 r}-\frac{2kt}{c^2 r}-\frac{kt^2}{c^2 r}\right]\left[\frac{k}{r^2}\exp\left(t-\frac{r\theta}{c}\right)+\frac{k\theta}{cr}\exp\left(t-\frac{r\theta}{c}\right)\right]. \quad (2.41)$$

To the limit of c^{-2}, (2.41) reduces to

$$\Gamma^{1}{}_{11} = -\frac{k}{c^2 r^2}-\frac{kt}{c^2 r^2}-\frac{kt^2}{2c^2 r^2}. \quad (2.42)$$

$$\Gamma^{1}{}_{12} = \Gamma^{1}{}_{21} = \frac{1}{c^2}\left[g_{11}\right]\frac{\partial f(t,r,\theta)}{\partial \theta}. \quad (2.43)$$

$$\Gamma^{1}{}_{12} = \Gamma^{1}{}_{21} = -\frac{1}{c^2}\left[1-\frac{2k}{c^2 r}-\frac{2kt}{c^2 r}-\frac{kt^2}{c^2 r}\right]\left[\frac{k}{c}\exp\left(t-\frac{r\theta}{c}\right)\right]. \quad (2.44)$$

To the limit of c^{-2}, (2.44) reduces to

$$\Gamma^{1}{}_{12} = \Gamma^{1}{}_{21} = 0. \quad (2.45)$$

$$\Gamma^{1}{}_{22} = -r\left[1-\frac{2k}{c^2 r}\exp\left(t-\frac{r\theta}{c}\right)\right]. \quad (2.46)$$

To the limit of c^{-2}, (2.46) reduces to

$$\Gamma^{1}{}_{22} = -r+\frac{2k}{c^2}+\frac{2kt}{c^2}+\frac{2kt^2}{c^2}. \quad (2.47)$$

$$\Gamma^{1}{}_{33} = -r\sin^2\theta\left[1-\frac{2k}{c^2 r}\exp\left(t-\frac{r\theta}{c}\right)\right]. \quad (2.48)$$

To the limit of c^{-2}, (2.47) reduces to

$$\Gamma^{1}{}_{33} = -\sin^2\theta\left[r-\frac{2k}{c^2}-\frac{2kt}{c^2}-\frac{kt^2}{c^2}\right]. \quad (2.49)$$

$$\Gamma^{2}{}_{00} = \frac{1}{c^2 r^2}\frac{\partial f}{\partial \theta}\left(-\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right). \quad (2.50)$$

To the limit of c^{-2}, (2.50) reduces to

$$\Gamma^{2}{}_{00} = \frac{1}{c^3 r^2}\left(-\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right) = 0. \quad (2.51)$$

$$\Gamma^2{}_{11} = \frac{1}{c^2 r^2}\left[1 - \frac{2k}{c^2 r}\exp\left(t - \frac{r\theta}{c}\right)\right]^{-2} \frac{\partial}{\partial\theta}\left(-\frac{k}{r}\exp\left(t - \frac{r\theta}{c}\right)\right). \tag{2.52}$$

To the limit of c^{-2}, (2.52) reduces to

$$\Gamma^2{}_{11} = \frac{1}{c^2 r^2}\left[1 + \frac{4k}{c^2 r}\exp\left(t - \frac{r\theta}{c}\right)\right]\left[\frac{k}{c}\exp\left(t - \frac{r\theta}{c}\right)\right] = 0. \tag{2.53}$$

$$\Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{r}. \tag{2.54}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta. \tag{2.55}$$

$$\Gamma^3{}_{13} = \Gamma^3{}_{31} = \frac{1}{r}. \tag{2.56}$$

$$\Gamma^3{}_{23} = \Gamma^3{}_{32} = \cot\theta. \tag{2.57}$$

$$\Gamma^\mu{}_{\alpha\beta} = 0. \tag{2.58}$$

3. MOTION OF PARTICLES OF NON-ZERO REST MASS WITHIN THIS FIELD

The general relativistic equation of motion for particles of non-zero rest masses in a gravitational field [4,7,11-12] is given by

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0. \tag{2.59}$$

The time equation of motion in this field [4] is given as

$$\frac{d}{d\tau}\left(\ln \dot{t}\right) + \frac{d}{d\tau}\left[\ln\left(1 + \frac{2}{c^2} f(t, r, \theta)\right)\right] = 0. \tag{2.60}$$

Substituting (2.11) into (2.60) gives

$$\frac{d}{d\tau}\left(\ln \dot{t}\right) + \frac{d}{d\tau}\left[\ln\left(1 - \frac{2k}{c^2 r}\exp(t - \frac{r\theta}{c})\right)\right] = 0. \tag{2.61}$$

Simplifying (2.61) using binomial theorem and limiting the result to the order of c^{-2} gives

$$\frac{d}{d\tau}\left(\ln \dot{t}\right) + \frac{d}{d\tau}\left[\ln\left(1 - \frac{2k}{c^2 r} - \frac{2kt}{c^2 r} - \frac{kt^2}{c^2 r}\right)\right] = 0. \tag{2.62}$$

Simplifying (2.62) gives

$$\dot{t} = A\left(1 - \frac{2k}{c^2 r} - \frac{2kt}{c^2 r} - \frac{kt^2}{c^2 r}\right)^{-1}. \tag{2.63}$$

As $t \to \tau,\ f(t, r, \theta) \to 0$ and the constant $A \equiv 1$. Hence,

$$\dot{t} = \left(1 - \frac{2k}{c^2 r} - \frac{2kt}{c^2 r} - \frac{kt^2}{c^2 r}\right)^{-1}. \quad (2.64)$$

The radial, polar and azimuthal angle equations of motion [4] are given as

$$\ddot{r} + \frac{1}{c^2}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]\frac{\partial f(t,r,\theta)}{\partial r}\dot{t} - \frac{2}{c}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial t}\dot{t}\,\dot{r} -$$

$$\frac{1}{c^2}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial r}\dot{r}^2 - \frac{2}{c^2}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial t}\dot{r}\,\dot{\theta} -$$

$$r\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]\dot{\theta}^2 - r\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]\sin^2\theta\,\dot{\phi} = 0. \quad (2.65)$$

$$\ddot{\theta} + \frac{1}{r^2}\frac{\partial f(t,r,\theta)}{\partial \theta}\dot{t}^2 - \frac{1}{c^2 r^2}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-2}\frac{\partial f(t,r,\theta)}{\partial \theta}\dot{r}^2 - \sin\theta\cos\theta\,\dot{\phi}^2 + \frac{2}{r}\dot{r}\,\dot{\theta} = 0. \quad (2.66)$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\,\dot{\theta} = 0. \quad (2.67)$$

Substituting (2.11) into (2.65)-(2.67) and simplifying to the limit of c^{-2} gives,

$$\ddot{r} + \left[\frac{k}{c^2 r} + \frac{kt}{c^2 r} + \frac{kt^2}{2c^2 r^2}\right]\dot{t}^2 + \left[-\frac{k}{c^2 r^2} - \frac{kt}{c^2 r^2} - \frac{kt^2}{2c^2 r^2}\right]\dot{r}^2 + \left[-r + \frac{2k}{c^2} + \frac{2kt}{c^2} + \frac{kt^2}{c^2}\right]\dot{\theta}^2 +$$

$$\left[\frac{k}{c^2 r} + \frac{kt}{c^2 r} + \frac{kt^2}{2c^2 r^2}\right]\dot{t}\,\dot{r} + \left[-r + \frac{2k}{c^2} + \frac{2kt}{c^2} + \frac{kt^2}{c^2}\right]\sin^2\theta\,\dot{\phi}^2 = 0. \quad (2.68)$$

$$\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 + \frac{2}{r}\dot{r}\,\dot{\theta} = 0. \quad (2.69)$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\,\dot{\theta} = 0. \quad (2.70)$$

For pure radial motion, $\dot{\theta} = \dot{\phi} \equiv 0$, hence (2.68) becomes

$$\ddot{r} + \left[\frac{k}{c^2 r} + \frac{kt}{c^2 r} + \frac{kt^2}{2c^2 r^2}\right]\dot{t}^2 + \left[-\frac{k}{c^2 r^2} - \frac{kt}{c^2 r^2} - \frac{kt^2}{2c^2 r^2}\right]\dot{r}^2 + \left[\frac{k}{c^2 r} + \frac{kt}{c^2 r} + \frac{kt^2}{2c^2 r^2}\right]\dot{t}\,\dot{r} = 0. \quad (2.71)$$

Integrating (2.70) that is the polar motion of the test particles that has radial dependence gives

$$\dot{\phi} = \frac{A}{r^2}, \quad (2.72)$$

where A is the constant of integration.

This motion has an inverse square dependence on the radial distance.

4. MOTION OF PARTICLES IN THE EQUATORIAL PLANE WITHIN THIS FIELD

The Lagrangian in the space time exterior to any astrophysical body is defined by [1,13],

$$L = \frac{1}{c}\left(-g_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau} \right)^{\frac{1}{2}}. \quad (2.73)$$

Thus in this field, the Lagrangian becomes

$$L = \frac{1}{c}\left(-g_{00}\left(\frac{dt}{d\tau}\right)^2 - g_{11}\left(\frac{dr}{d\tau}\right)^2 - g_{22}\left(\frac{d\theta}{d\tau}\right)^2 - g_{33}\left(\frac{d\phi}{d\tau}\right)^2 \right)^{\frac{1}{2}}. \quad (2.74)$$

Considering orbit in the equatorial plane of a homogeneous spherical mass,

$$\theta \equiv \frac{\pi}{2}$$

Then, the Lagrangian equation reduces to (2.75) by substituting equation (2.14) and (2.16) into (2.74):

$$L = \frac{1}{c}\left(-\left(1 + \frac{2f(t,r,\theta)}{c^2}\right)\dot{t}^2 + \left(1 + \frac{2f(t,r,\theta)}{c^2}\right)^{-1} \dot{r}^2 \right)^{\frac{1}{2}}. \quad (2.75)$$

It is an established fact that $L = \in$,with $\in = 1$ for time like orbits and $\in = 0$,for null orbits [1]. Setting $L = \in$,in equation (2.75) and squaring both sides yields;

$$c^2 \in^2 = -\left(1 + \frac{2f(t,r,\theta)}{c^2}\right)\dot{t}^2 + \left(1 + \frac{2f(t,r,\theta)}{c^2}\right)^{-1} \dot{r}^2. \quad (2.76)$$

Orbital shape (which is a function of azimuthal angle) is paramount in most applications of general relativity. Therefore it is very important to transform equation (2.76) in terms of the azimuthal angle ϕ.

Using the following transformation, with $r = r(\phi)$ and $u(\phi) = \dfrac{1}{r(\phi)}$ then,

$$\dot{r} = \dot{\phi}\frac{dr}{d\phi} \text{ or } \dot{r} = \frac{l}{1+r^2}\frac{dr}{d\phi}. \quad (2.77)$$

But

$$\frac{dr}{d\phi} = \frac{dr}{du}\frac{du}{d\phi} \text{ or } \frac{dr}{d\phi} = -u^2\frac{du}{d\phi} \quad (2.78)$$

and thus,

$$\dot{r}=\frac{l}{1+r^2}\frac{dr}{d\phi}, \tag{2.79}$$

Substituting (2.77)-(2.79) into (2.76), gives

$$\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left(1+\frac{2f(t,r,\theta)}{c^2}\right)-\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-1}\dot{t}^2-c^2\in^2=0. \tag{2.80}$$

Substituting (2.11) into (2.80), and for $\in=1$, we have

$$\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{2k}{c^2r}\exp\left(t-\frac{r\theta}{c}\right)\right]-\left[1-\frac{2k}{c^2r}\exp\left(t-\frac{r\theta}{c}\right)\right]^{-1}\dot{t}^2-c^2=0. \tag{2.81}$$

Multiplying (2.81) by $\left(1-\frac{2k}{c^2r}\exp\left(t-\frac{r\theta}{c}\right)\right)^{-1}$ gives,

$$\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2-\left(1-\frac{2k}{c^2r}\exp\left(t-\frac{r\theta}{c}\right)\right)^{-2}\dot{t}^2-\left(1-\frac{2k}{c^2r}\exp\left(t-\frac{r\theta}{c}\right)\right)^{-1}c^2=0. \tag{2.82}$$

Simplifying the exponential terms and ignoring higher terms gives,

$$\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2-\left[1-\frac{2k}{c^2r}\left(1+t+\frac{t^2}{2}\right)\right]^{-2}\dot{t}^2-\left[1-\frac{2k}{c^2r}\left(1+t+\frac{t^2}{2}\right)\right]^{-1}c^2=0. \tag{2.83}$$

For $\in=0$ which correspond to the equation of motion of light on null geodesic and substituting (2.11) into (2.80) gives,

$$\dot{t}^2\left[1+\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]=\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]. \tag{2.84}$$

Multiplying (2.84) by $\left[1+\frac{2}{c^2}\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right)\right]^{-1}$ and simplifying gives,

$$\dot{t}^2=\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2\left[1-\frac{4k}{c^2r}\exp\left(t-\frac{r\theta}{c}\right)+\frac{4k}{c^4r^2}\exp 2\left(t-\frac{r\theta}{c}\right)\right]. \tag{2.85}$$

In the order of c^0 : equation (2.85) reduces to

$$\dot{t}^{2}=\frac{1}{\left(1+u^{2}\right)^{2}}\left(\frac{du}{d\phi}\right)^{2}. \tag{2.86}$$

which on further simplification reduces to (2.87)

$$\dot{t}=\frac{1}{\left(1+u^{2}\right)}\left(\frac{du}{d\phi}\right). \tag{2.87}$$

In order of c^{-2} : equation (2.85) reduces to

$$\dot{t}=\frac{1}{1+u^{2}}\frac{du}{d\phi}\left[1-\frac{2k}{c^{2}r}\exp\left(t-\frac{r\theta}{c}\right)\right]. \tag{2.88}$$

Expanding $\left[1-\frac{2k}{c^{2}r}\exp\left(t-\frac{r\theta}{c}\right)\right]$ exponential term and ignoring higher terms gives,

$$\dot{t}=\frac{1}{1+u^{2}}\frac{du}{d\phi}\left[1-\frac{2k}{c^{2}r}\left(1+t+\frac{t^{2}}{2}\right)\right]. \tag{2.89}$$

This is the photon equation of motion in this region.

5. CONCLUSION

The completely defined metric tensors for this field are given in equations (2.3)-(2.5), (2.8)-(2.10), (2.14), (2.16), (2.18) and (2.20) while the completely defined coefficients of affine connections are given in equations (2.24), (2.27), (2.30), (2.33), (2.36), (2.39), (2.42), (2.45), (2.47), (2.49), (2.51), (2.54)- (2.58).

Equation (2.64) is the expression for the variation of the time on a clock moving in this gravitational field. It is of same form as that in Schwarzschild's gravitational field, though our result contained additional terms not found in the result obtained by Schwarzschild. The additional corrections terms in our results can be use to investigate the existence of gravitational waves even though the effect is very weak. Additionally, our results can be use in satellite communications likewise it can also be use in the study of black holes.

Interestingly, our expression differs from [1]. In this article he obtained $\dot{t}$ not as an exponential function dependent on his unknown function $f(\eta,\xi)$. Thus, our expression in its merit stands out uniquely, as an extension of the results in Schwarzschild's field.

The radial equation of motion (2.71) can be use to obtain the complete instantaneous speed of a particle of nonzero rest mass in this field.

Equation (2.72) is the polar equation of motion which is an inverse square equation that depends on the radial distance as obtained by [4,11].

Equation (2.83) is the planetary equation of motion for test particles in the region of rotating homogenous spherical mass which when solved will reveal the perihelion precision of planetary orbits within this field.

Instructively, equation (2.89) is the first constructed equation of motion for photon in this field using variable gravitational scalar potential. This equation contains additional terms that are not found in the well-known Schwartzchild's equation, which shows the effect of gravity, radial distance and polar angle to the time equation of motion of photons in the equatorial plane of a rotating homogeneous spherical body.

The door is therefore open for the study of other astrophysical phenomena within this field such as the gravitational red-shift by the sun, time dilation, length contraction, Riemann-Christoffel tensors e.t.c using the metric tensors and coefficient of affine connections for this field.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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