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Research Article

The Ground-State Calculations for Some Nuclei by Mesonic Potential of Nucleon-Nucleon Interaction

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The interaction of nucleon-nucleon (NN) has certain physical characteristics, indicated by nucleon, and meson degrees of freedom. The main purpose of this work is calculating the ground-state energies of ${}^{2}_{1}H$ and ${}^{4}_{2}He$ through the two-body system with the exchange of mesons (π, σ, and ω) that mediated between two nucleons. This paper investigates the NN interaction based on the quasirelativistic decoupled Dirac equation and self-consistent Hartree-Fock formulation. We construct a one-boson exchange potential (OBEP) model, where each nucleon is treated as a Dirac particle and acts as a source of pseudoscalar, scalar, and vector fields. The potential in the present work is analytically derived with two static functions of meson, the single-particle energy-dependent (SPED) and generalized Yukawa (GY) functions; the parameters used in meson functions are just published ones (mass, coupling constant, and cutoff parameters). The theoretical results are compared to other theoretical models and their corresponding experimental data; one can see that the SPED function gives more satisfied agreement than the GY function in the case of the considered nuclei.

1. Introduction

One of the aims of nuclear structure theory is to derive the ground-state properties. Such properties are related to the constituents of matter, which are represented in the physics of elementary particles with their characteristics (electric charge, mass, spin, etc.) and how each particle interacts with others [1]. Yukawa (1934) introduced an assumption of some sort of field to be the reason of attraction between proton and neutron. This field is quantized, characterized by the force of the short range, and its mass equals 300 times of electron called Yukawa particle (meson). Meson is a Greek word, which means intermediate, and this is the right description for meson which transmits the nuclear force between hadrons; meson can participate in weak, strong, and electromagnetic interactions with a net electric charge. All mesons are unstable and their lifetimes reach to hundredths of microseconds. Each meson is characterized by quantum numbers, principal number ($n = 0, 1, \cdots$), orbital angular momentum $l = n - 1$ (indicating the orbiting of quarks around each other), magnetic number ($m = -l, \cdots, l$), and spin $s = 0$ for singlet state (1 for triplet state). The description of quantum numbers can be illustrated by using the nuclear shell model [2].

The interaction between each nucleon with all other nucleons generates an average potential field where each nucleon moves. The rules of Pauli exclusion principle govern the occupation of orbital quantum states in the shell model and postulate that under the meson exchange between two nucleons, the wave function is a totally antisymmetrical product wave function. The interacted nucleons have a potential (the nuclear mean field) characterized by its dependence on the position coordinates. There is an ability to calculate the mean-field potential for electrons or in nuclei; the calculation methods are very similar, but the interactions are different. The NN interaction is a fundamental problem in nuclear physics. It had a variant success in describing the nuclear properties; this determination ranged from the empirical picture to fit the experimental data and to derive it microscopically from the bare NN potential. Thus, there is no unique NN potential to be the start point [3–5].

A microscopic description was provided in nuclear models to include the elementary interaction between nucleons. The original attempts to find the fundamental theory of nuclear forces [6–10] were not so successful. The reason for their failure was the pion dynamics which has been restricted by the chiral symmetry [11, 12]. The quantum description and field methods were included in making the potential structure such as the Partovi-lomon model [13], Stony Brook-group [14], Paris-group [15], Nijmegen-group [16], and Bonn-group potentials [12]. The successful theoretical models were based on OBEP, including one-meson exchange and multimeson exchange, plus short-range phenomenology. Inside a nucleus, there is a fact that nucleons move quasi-independently from one to another which achieves the concept of nuclear mean field (NMF); this fact relies on Hartree-Fock interaction.

The microscopic description of degrees of freedom related to nucleon and meson has to depend on a relativistic quantum field to include the full structure of the medium (spin structure) which is associated with the fermion field of Dirac equation [17] and the bound-state energies. This can be found by solving the Dirac equation which leads to investigating the ground-state energies of nuclei which is ensured by the calculation of the NMF potential with Dirac-Hartree-Fock [18].

It is known that the NN interaction can be distinguished into three parts: the first part is the long range at $r \geq 2\,\text{fm}$ originated from pseudoscalar mesons; the second part is the medium range at $1\,\text{fm} \leq r \leq 2\,\text{fm}$, which mainly comes from the exchange of scalar meson (σ which is a fictitious scalar meson responsible for attraction); and the third part is the short range at $r \leq 1\,\text{fm}$, from the vector meson $(\rho, \omega, \cdots)$ exchange. In order to have a potential of NN interaction, there were many models that serve this point [19–25]. After little development in nuclear properties, the dominant part of the interaction is central, having a strong repulsion at a short range ($r \leq 0.7\,\text{fm}$) and attraction force at an intermediate range ($>1\,\text{fm}$). There is a cancelation of major static effects between vector and scalar mesons to maintain the stability of nucleus [26].

Now, a variant number of pseudoscalar, scalar, and vector mesons are found; the vast advance of OBEP models related to NN interaction not only for free parameter reduction but also for the accuracy and fitting them with experimental data [12, 27]. The development of quantum field theory and boson field Lagrangian by Heisenberg, Pauli, Dirac, and Rosenfeld in 1930 allows the meson field coordinates to depend on themselves by Yukawa in 1935. Firstly, Yukawa suggested the conjunction of a scalar field coordinate of mesons and then extended to include vector fields by Proca (1936), and Kemmer (1938) embraced the pseudoscalar, axial vector, and antisymmetric tensor. Till now, there are a large number of modifications in the vector-scalar combinations as well as pseudoscalar and pseudovector mesons.

The present work represents a motivated model of determining the ground state for deuteron (^{2}H) and helium (^{4}He) based on the NN interaction, the potential related to OBEP with the exchange of pseudoscalar meson (π), scalar meson (σ), and vector meson (ω). This potential is derived analytically with two static functions of mesons; it relies on the Dirac-Hartree-Fock equation. Then, we compare the obtained theoretical results with others and their corresponding experimental data.

This paper is arranged as follows. Section 2 is devoted to explain the theoretical analysis in details, with three subsections, 2.1, 2.2, and 2.3. Subsection 2.1 refers to how this model uses the Hartree-Fock equation with the Dirac Hamiltonian and how it deals with the wave functions. Subsection 2.2 is related to the mathematical treatment of each term in the Hamiltonian equation. Subsection 2.3 represents the potential of our model. Section 3 represents the results of the potential and ground-state energy for the selected nuclei. Finally, Section 4 is the conclusion.

2. Theoretical Analysis

There are several models which determine the structure and properties of the nucleus through the strong force between nucleons in the nucleus [28, 29]. The states of the nucleus are bounded due to this strong force. The nature of nucleon interactions can be described by studying it as a two-body problem. The general wave equation used in such models has the form

$$\widehat{H}|\Psi\rangle = E|\Psi\rangle, \tag{1}$$

where $\widehat{H}$ represents the general Hamiltonian operator and E is the eigen energy. We studied the interaction through two bodies via OBEP between two fermions (nucleons), so the convenient representation of the energy is the relativistic form of the Dirac equation.

Thus, the accurate interaction of the nuclear system can be described by the Dirac Hamiltonian which includes all fermion interactions and is given by [30–32]

$$\widehat{H} = \sum_{i}^{A} c\vec{\alpha}_i \cdot \vec{p}_i + (\beta_i - I) m_i c^2 - \widehat{T} + \frac{1}{2}\sum_{i\neq j}^{A} V_{ij}. \tag{2}$$

Since I is the unit matrix, $\vec{\alpha}$ and β are (4×4) Dirac matrices, m_i is the nucleon mass, $\vec{p}_i$ is the momentum of the system, $\widehat{T}$ is kinetic energy operator, and V_{ij} is the potential energy between fermions' pairs, we ignore three and many body interactions in the present work. The total kinetic energy of the nucleon equals the total energy subtracted from the rest of the mass energy [33].

$$\widehat{T} = E - Mc^2, \tag{3}$$

where $M = Am_i$, in which A is the number of nucleons and E is the total relativistic energy which has the form

$$E = Mc^2\sqrt{1 + \frac{p^2}{M^2c^2}}. \tag{4}$$

The kinetic energy can be decomposed into two contributions: the first one is the relative space contribution $\widehat{T}_{\mathrm{r}}$, and the other is the center of mass contribution $\widehat{T}_{\mathrm{cm}}$ [34, 35].

$$\widehat{T} = \sum_i \widehat{T}_{\mathrm{r}} - \widehat{T}_{\mathrm{cm}} = \frac{\left(\sum_i^A p_i\right)^2}{2mA} - \frac{\sum_i^A p_i^2}{2mA}. \tag{5}$$

The second part of Equation (5) can be neglected. This neglects the center of mass term $(\sum_i^A (p_i^2/2mA))$ according to [36]. Applying the binomial theorem for E and substituting into Equation (3), the relativistic kinetic energy $\widehat{T}$ takes the form

$$\widehat{T} = \frac{\left(\sum_i^A p_i\right)^2}{2mA} = \frac{1}{2m}\sum_{i=1}^{A} p_i^2 - \frac{2}{mA}\sum_{i<j}^{A} p_{ij}^2, \tag{6}$$

where $p_{ij} = 1/2(p_i - p_j)$ is the relative momentum of the two nucleon systems. By substituting Equation (6) into Equation (2), this leads to the effective nuclear Hamiltonian operator.

$$\widehat{H} = \sum_i^A c\vec{\alpha}_i.\vec{p}_i + (\beta_i - I)m_ic^2 - \frac{1}{2m}p^2 + \sum_{i<j}^{A} V_{ij} + \frac{2}{mA}\sum_{i<j}^{A} p_{ij}^2. \tag{7}$$

In Hartree-Fock theory, we seek the best single state given by the lowest energy expectation value of this Hamiltonian.

2.1. Variational and Modified Hartree-Fock Wave Function. One is able to ensure the antisymmetry of the fermions' wave functions with the aid of a Slater determinant and Hartree product to have the convenient form in calculating the ground-state energy as the following wave function which is suitable for fermions [33]. So, the wave function of nucleus $\Psi(r)$ becomes

$$\Psi(r) = \frac{1}{\sqrt{A!}}\det \psi_i\left(\vec{r}_i\right), \tag{8}$$

where ψ_i is the nucleon wave function which can be expanded as

$$\psi_i\left(\vec{r}_i\right) = \sum_\alpha C_{i\alpha} F_\alpha\left(\vec{r}_i\right) \tag{9}$$

where $C_{i\alpha}$ is the oscillator constant and $F_\alpha(\vec{r}_i)$ is the oscillator wave function which has two components, radial component Φ_α and spin component χ_α.

$$|F_a\rangle = \left|\begin{matrix}\Phi_a \\ X_a\end{matrix}\right\rangle. \tag{10}$$

The two components have the following relation between them [17, 37], as

$$\chi = \left(1 - \frac{\varepsilon - \nu}{2Mc^2}\right)\frac{\vec{\sigma}\cdot\vec{p}}{2mc}\phi. \tag{11}$$

The principle of antisymmetry of the wave function was not clarified by the Hartree method only, but the accurate picture of the ground-state energy calculations should have the Hartree-Fock approximation besides the Slater determinant as in Equation (8) for the wave function. To facilitate the calculation of the wave function, we will use Equation (11) which enables replacing between the two parts of the oscillator wave function with the quantities of 1, 2, and 3. Here, we are dealing with the ground state, so $(\varepsilon - \nu)/c^2$ makes the value of the second term very small and can be neglected.

$$\chi \cong \frac{\vec{\sigma}\cdot\vec{p}}{2mc}\phi \tag{12}$$

The wave functions for two nucleons i and j have the formula for bra part $\langle\Phi_a(r_i)\phi_\gamma(r_j)|$ and ket part $|\phi_\beta(r_i)\phi_\delta(r_j)\rangle$ as the bracket needs two wave functions in each side of the bracket:

$$\begin{aligned}\left\langle\phi_\alpha(r_i)\phi_\gamma(r_j)\right| = &\sum_{m_{l_\alpha}m_{s_\alpha}}\sum_{m_{l_\gamma}m_{s_\gamma}}\left(l_\alpha s_\alpha m_{l_\alpha} m_{s_\alpha}|j_\alpha M_\alpha\right) \\ &\cdot\left(l_\gamma s_\gamma m_{l_\gamma} m_{s_\gamma}\middle|j_\gamma M_\gamma\right)\left\langle\phi_{n_\alpha l_\alpha m_{l_\alpha}}(r_i)\phi_{n_\gamma l_\gamma m_{l_\gamma}}(r_j)\right| \\ &\cdot\left\langle\chi_{m_{s_\alpha}}^{1/2}\chi_{m_{s_\gamma}}^{1/2}\right|\left\langle\widehat{P}_{T_\alpha}\widehat{P}_{T_\gamma}\right|,\end{aligned} \tag{13}$$

where $(l_\alpha s_\alpha m_{l_\alpha} m_{s_\alpha}|j_\alpha M_\alpha)$ is the Clebsch-Gordon coefficient, $\chi_{m_{s_\alpha}}^{1/2}$ is the spin function, and $\widehat{P}_{T_\alpha}$ is the function of isotopic spin. The two wave functions depend on r_i and r_j which can be merged to one wave by changing the special coordinates for it that converts to the relative and center of mass coordinates (see Appendix A for more details). Then, we have the formula

$$\begin{aligned}\left\langle\phi_\alpha(r_i)\phi_\gamma(r_j)\right| = &\sum_{m_{l_\alpha}m_{s_\alpha}}\sum_{m_{l_\gamma}m_{s_\gamma}}\sum_{lS}\sum_{\lambda\mu}\sum_{nlNL}\sum_{mM}\sum_{sm_s}\sum_{T} \\ &\cdot\left(l_\alpha s_\alpha m_{l_\alpha} m_{s_\alpha}|j_\alpha M_\alpha\right)\left(l_\gamma s_\gamma m_{l_\gamma} m_{s_\gamma}\middle|j_\gamma M_\gamma\right) \\ &\cdot\left(l_\alpha l_\gamma m_{l_\alpha} m_{l_\gamma}|\lambda\mu\right)\langle n_\alpha l_\alpha n_\gamma l_\gamma|NLnl\rangle \\ &\cdot(lSm_l m_S|JM)(LlMm|\lambda\mu) \\ &\cdot\left(s_\alpha s_\gamma m_{s_\alpha} m_{s_\gamma}|SM_s\right)\left(s_\alpha s_\gamma T_\alpha T_\gamma|TM_T\right) \\ &\cdot\langle\phi_{nlm}(r)\phi_{NLM}(R)|\left\langle\chi_{m_s}^{S}(i,j)\right|\langle\widehat{P}_T(i,j)|,\end{aligned} \tag{14}$$

where $\langle n_\alpha l_\alpha n_\gamma l_\gamma \mid NLnl\rangle$ is the Talmi-Moshinsky bracket and $\phi_{nlm}(r) = R_{nlm} Y_{nlm}$, with radial function R_{nlm} and Y_{nlm} the spherical harmonics; the same treatment happens to the ket part. The bracket of the spin function is $\langle \chi^S_{m_s}(i,j) \mid \chi^S_{m_s}(i,j)\rangle = 1$, and the isotopic function is $\langle \widehat{P}_T(i,j) \mid \widehat{P}_T(i,j)\rangle = 1$. This formula is convenient for two-body interaction as in Deuteron, and the number of nucleons of the Helium nucleus should be emerged in an equation through adding $\Sigma^4_{i<j=1}$. The bracket for spherical functions equals one as ϑ and φ are not affected here, but the distance r does. We have the solution of radial wave function as an oscillator with the Laguerre function (Leigh, Ritz, and Galerkin) method [38] where the wave function can be expanded in terms of a complete set with basis set:

$$R_{nlm} = \left[\frac{2n!}{\Gamma(n+l+3/2)}\right]^{1/2} \left(\frac{1}{b}\right)^{3/2} \left(\frac{r}{b}\right)^{l} \exp\left(\frac{-1}{2}\left(\frac{r}{b}\right)^2\right) L_n^{l+1/2}\left(\frac{r}{b}\right)^2. \tag{15}$$

l represents the angular momentum, $L_n^{l+(1/2)}$ is the associated Laguerre polynomial [39], and the length parameter $b = \sqrt{(\hbar c)^2/(mc^2\,\hbar\omega)}$, where m is the mass of the considered particles (nucleons) and ω is the oscillator frequency. The simplest shell model should have the overall size of the nucleus through the scale of this parameter, and it is related to the number density of nucleons or equivalent to $\hbar\omega = 45\, A^{-(1/3)} - 25A^{-(2/3)}$ according to the equilibrium density A of the even-even nucleus [40].

2.2. The Handling of the Kinetic Energy Term. Using Equations (9), (10), and (7), we obtain the relativistic modified Hartree-Fock equations, and we apply the Lagrange multiplier method for seeking the minimum point of the expression:

$$\begin{aligned}\sum_{i\alpha\beta} h_i C^*_{i\alpha} C_{i\beta}\left\langle F_\alpha \mid \widetilde{F_\beta}\right\rangle &= \sum_{i\alpha\beta} C^*_{i\alpha} C_{i\beta}\left\langle F_\alpha(r) \mid c\vec{\alpha}_i.\vec{p}_i + (\beta_i - I)m_i c^2 \right.\\ &\left. - \frac{1}{2m}p_i^2 \mid F_\beta\right\rangle + \sum_{i<j}\sum_{\alpha\gamma\beta\delta} C^*_{i\alpha}C_{i\beta}C^*_{j\gamma}C_{j\delta}\\ &\cdot\left\langle F_\alpha F_\gamma \mid \frac{2}{Am}p_{ij}^2 + V_{ij} \mid \widetilde{F_\beta F_\delta}\right\rangle.\end{aligned} \tag{16}$$

Differentiating Equation (16) with respect to $C^*_{i\alpha}$ which is the conjugate of the oscillator constant, one has

$$\begin{aligned}&\sum_{i\alpha\beta} C_{i\beta}\left\langle F_\alpha \mid c\alpha_i.p_i + (\beta_i - I)m_i c^2 - \frac{1}{2m}p_i^2 \mid F_\beta\right\rangle\\ &+ \sum_{i<j}\sum_{\alpha\gamma\beta\delta} C_{i\beta}C^*_{j\gamma}C_{j\delta}\left\langle F_\alpha F_\gamma \mid \frac{2}{Am}p_{ij}^2\right.\\ &\left. + V_{ij} \mid \widetilde{F_\beta F_\delta}\right\rangle - \sum_{i\alpha\beta} h_i C_{i\beta}\left\langle F_\alpha \mid F_\beta\right\rangle = 0.\end{aligned} \tag{17}$$

Treating the first bracket as H_1,

$$\begin{aligned}\sum_{i\alpha\beta} C_{i\beta}\left\langle F_\alpha \mid \widehat{H}_1 \mid F_\beta\right\rangle &= \sum_{i\alpha\beta} C_{i\beta}\left\langle F_\alpha \mid c\vec{\alpha}_i.\vec{p}_i \right.\\ &\left. + (\beta_i - I)m_i c^2 - \frac{1}{2m}p_i^2 \mid F_\beta\right\rangle.\end{aligned} \tag{18}$$

Taking into account Dirac matrices [26, 41], $\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0\end{pmatrix}$, $\beta = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}$, the unit matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ and $p_i = p$:

$$\begin{aligned}\left\langle F_\alpha \mid H_1 \mid F_\beta\right\rangle &= \left\langle \phi_\alpha \mid \frac{(\vec{\sigma}.\vec{p})(\vec{\sigma}.\vec{p})}{2m} \mid \phi_\beta\right\rangle\\ &+ \left\langle \phi_\alpha \mid \frac{(\vec{\sigma}.\vec{p})(\vec{\sigma}.\vec{p})}{2m} \mid \phi_\beta\right\rangle\\ &- \left\langle \phi_\alpha \mid \frac{(\vec{\sigma}.\vec{p})(\vec{\sigma}.\vec{p})}{2m} \mid \phi_\beta\right\rangle\\ &- \left\langle \phi_\alpha \mid \frac{p^2}{2m} \mid \phi_\beta\right\rangle - \left\langle \phi_\alpha \mid \frac{(\vec{\sigma}.\vec{p})(\vec{\sigma}.\vec{p})p^2}{(2m)(4m^2c^2)} \mid \phi_\beta\right\rangle = 0.\end{aligned} \tag{19}$$

We ignore the last term for both simplicity and avoiding the fourth power of momentum and speed of light $(p^4/8m^3c^4)$; hence, we have the kinetic term being vanished.

2.3. The Construction of the Potential through One-Boson Exchange. After the treatment of the kinetic energy, the two-body Hamiltonian becomes

$$H = \sum_{i<j}\left(\frac{2}{Am}P_{ij}^2 + V_{ij}\right). \tag{20}$$

The first term in the right hand side is the remainder part results from the treatment of kinetic term in the Dirac equation as Equation (6), and $V_{ij}(r)$ is the potential according to the two-body interaction. The relativistic form of one-meson exchange potential between two nucleons (i, j) based on the degrees of freedom was associated with three, pseudoscalar, scalar, and vector mesons:

$$V_{ij}(r) = V_\pi(r) + V_\sigma(r) + V_\omega(r), \tag{21}$$

$$\begin{aligned}V_{\text{ps}}(r) &= \gamma_i^o\gamma_i^5\gamma_j^o\gamma_5^j J_{\text{ps}},\\ V_\sigma &= -\gamma_i^o\gamma_j^o J_\sigma,\\ V_\omega(r) &= \gamma_i^o\gamma_j^o \overleftarrow{\gamma}_i^{\mu}\overrightarrow{\gamma}_j^{\mu} J_\omega,\end{aligned} \tag{22}$$

where $\overrightarrow{\gamma}_i^{\mu}\overrightarrow{\gamma}_j^{\mu} = [\gamma_i^o\gamma_j^o - \overrightarrow{\gamma}_i\overrightarrow{\gamma}_j]$.

$$\gamma_i^o = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \gamma^5 = i\gamma^o\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \vec{\gamma}_i = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \tag{23}$$

The Dirac representation for mesons' functions will be used to have Dirac matrices corresponding to Pauli spin matrices [30, 42]. Substitute Equations (23) and (22) into Eq. (21) to get V_π, V_σ, and V_ω. We add π-meson as a pseudoscalar one to the previous two mesons because it ties the mesons with the nucleus as it is the fare one. We seek for the stability of the nucleus, and the exchange of pion meson increases the stability of the nucleus. The attractive behavior is represented in scalar (σ) meson, and the repulsive behavior is represented in vector (ω) meson. So, the physics of nucleon potential is maintained. The Fock exchange between the two wave functions spatially is introduced after interaction in the potential, briefly in the ($\tilde{}$) symbol on the right part (ket part).

$$\begin{aligned}
\left\langle F_\alpha F_\gamma | V_{ij} | \widetilde{F_\beta F_\delta} \right\rangle &= \left\langle F_\alpha F_\gamma | V_\pi(r) | \widetilde{F_\beta F_\delta} \right\rangle + \left\langle F_\alpha F_\gamma | V_\sigma(r) | \widetilde{F_\beta F_\delta} \right\rangle \\
&+ \left\langle F_\alpha F_\gamma \middle| V_{\omega(r)} \middle| \widetilde{F_\beta F_\delta} \right\rangle = \langle \phi_\alpha | \widetilde{\left\langle \phi_\gamma \middle| J_\pi \middle| \chi_\beta \right\rangle} | \chi_\delta \rangle \\
&+ \langle \phi_\alpha | \widetilde{\left\langle \chi_\gamma \middle| J_\pi \middle| -\phi_\beta \right\rangle} | \chi_\delta \rangle + \langle \chi_\alpha | \widetilde{\left\langle \phi_\gamma \middle| J_\pi \middle| \chi_\beta \right\rangle} | -\phi_\delta \rangle \\
&+ \langle \chi_\alpha | \widetilde{\left\langle \chi_\gamma \middle| J_\pi \middle| -\phi_\beta \right\rangle} | -\phi_\delta \rangle - \left\langle \left(\phi_\alpha \phi_\gamma \right) | J_\sigma | \widetilde{\left(\phi_\beta \phi_\delta \right)} \right\rangle \\
&+ \left\langle \left(\phi_\alpha \chi_\gamma \right) | J_\sigma | \widetilde{\left(\chi_\beta \phi_\delta \right)} \right\rangle + \left\langle \left(\chi_\alpha \phi_\gamma \right) | J_\sigma | \widetilde{\left(\phi_\beta \chi_\delta \right)} \right\rangle \\
&- \left\langle \left(\chi_\alpha \chi_\gamma \right) | J_\sigma | \widetilde{\left(\chi_\beta \chi_\delta \right)} \right\rangle + \left\langle \left(\phi_\alpha \phi_\gamma \right) | J_\omega | \widetilde{\left(\phi_\beta \phi_\delta \right)} \right\rangle \\
&+ \left\langle \left(\phi_\alpha \chi_\gamma \right) | J_\omega | \widetilde{\left(\chi_\beta \phi_\delta \right)} \right\rangle + \left\langle \left(\chi_\alpha \phi_\gamma \right) | J_\omega | \widetilde{\left(\phi_\beta \chi_\delta \right)} \right\rangle \\
&+ \left\langle \left(\chi_\alpha \chi_\gamma \right) | J_\omega | \left(\chi_\beta \chi_\delta \right) \right\rangle \\
&- \left\langle \left(\phi_\alpha \phi_\gamma \right) \middle| J_\omega \left(\vec{\sigma}_i \cdot \vec{\sigma}_j \right) \middle| \widetilde{\left(\chi_\beta \chi_\delta \right)} \right\rangle \\
&- \left\langle \left(\phi_\alpha \chi_\gamma \right) \middle| J_\omega \left(\vec{\sigma}_i \cdot \vec{\sigma}_j \right) \middle| \widetilde{\left(\phi_\beta \chi_\delta \right)} \right\rangle \\
&- \left\langle \left(\chi_\alpha \phi_\gamma \right) \middle| J_\omega \left(\vec{\sigma}_i \cdot \vec{\sigma}_j \right) \middle| \widetilde{\left(\chi_\beta \phi_\delta \right)} \right\rangle \\
&- \left\langle \left(\chi_\alpha \chi_\gamma \right) \middle| J_\omega \left(\vec{\sigma}_i \cdot \vec{\sigma}_j \right) \middle| \widetilde{\left(\phi_\beta \phi_\delta \right)} \right\rangle .
\end{aligned} \tag{24}$$

According to the relation between ϕ and χ in Equation (12) and defining the momentum for each nucleon (i, j) [33, 43], $p_i = p_r + (1/2)p_R$, $p_j = -p_r + (1/2)p_R$, and $p_i = p'_i$, $p_j = p'_j$, and $p_r = p$.

Substituting those relations into Equation (B.1), We will apply some important relations [44] on this equation. We have used two static functions for the meson degree of freedom in the NN interaction, GY and SPED, with $(k = \pi, \sigma, \omega)$. These forms were used to carry out our calculations for a Hartree-Fock problem (HF). The first function [30] is represented by

$$(J_k)_{\mathrm{GY}} = g_k \hbar c \left(\frac{\exp(-\mu_k r)}{r} - \frac{\exp(-\lambda_k r)}{r} \left(1 + \frac{\lambda_k^2 - \mu_k^2}{2\lambda_k} r \right) \right). \tag{25}$$

We have the following: g_k^2 is the meson-nucleon coupling constant, λ_k is a parameter related to the structure function of the form factor, and $\mu_k = mc/h$ is the range of i-meson associated with the meson mass. The second function has the form [33]

$$(J_k)_{\mathrm{SPED}} = g_k \hbar c \left(\frac{\lambda_k^2}{\lambda_k^2 - \mu_k^2} \right) \left(\frac{\exp(-\mu_k r)}{r} - \frac{\exp(-\lambda_k r)}{r} \right). \tag{26}$$

The details to obtain the following equation is explained in Appendices B and C:

$$\begin{aligned}
\left\langle F_\alpha F_\gamma \mid V_{ij} \mid \widetilde{F_\beta F_\delta} \right\rangle &= \left\langle \phi_\alpha \phi_\gamma \right| - J_\sigma + J_\omega + \frac{1}{4m^2c^2} \\
&\times \left[2J_\sigma p^2 - 2\hbar^2 \left\{ \frac{dJ_\sigma}{dr} \frac{d}{dr} \right\} + \frac{2}{r} \frac{dJ_\sigma}{dr} \left[\vec{S}.\vec{L} \right] \right. \\
&+ 2J_\omega p^2 - 2\hbar^2 \left\{ \frac{dJ_\omega}{dr} \frac{d}{dr} \right\} \\
&+ \frac{2}{r} \frac{dJ_\omega}{dr} [S \cdot L] - 6J_\omega p^2 + 6\hbar^2 \left\{ \frac{dJ_\omega}{dr} \frac{d}{dr} \right\} \\
&- \frac{6}{r} \frac{dJ_\omega}{dr} \left[\vec{S} \cdot \vec{L} \right] + J_\omega \left(\vec{\sigma}_i \cdot \vec{\sigma}_j \right) \frac{2}{\hbar^2} (S \cdot p)^2 \\
&- J_\omega \left(\vec{\sigma}_i \cdot \vec{\sigma}_j \right) p^2 + \frac{2}{\hbar^2} (S \cdot p)^2 J_\omega \left(\vec{\sigma}_i \cdot \vec{\sigma}_j \right) \\
&\left. - p^2 J_\omega \left(\vec{\sigma}_i \cdot \vec{\sigma}_j \right) \right] + \frac{1}{4m^2c^2} \\
&\times \left[-J_\omega \left(\vec{\sigma}_i \cdot \vec{\sigma}_j \right) \frac{2}{\hbar^2} (S \cdot P_R)^2 + J_\omega \left(\vec{\sigma}_i \cdot \vec{\sigma}_j \right) p_R^2 \right. \\
&\left. + (1/2) p_R^2 J_\sigma + (1/2) p_R^2 J_\omega \right] + \frac{-\hbar^2}{m^2c^2} (2S(S+1) - 3) \\
&\times \left(\frac{dJ_\pi}{dr} \frac{d}{dr} \right) - \frac{1}{mc^2} \left(\frac{2}{\hbar^2} (S \cdot \hat{n})^2 - 1 \right) J_\pi \left(\hbar\omega \left(2n + l + \frac{3}{2} \right) \right. \\
&\left. - \frac{1}{2} m\omega^2 r^2 \right) \left| \widetilde{\phi_\beta \phi_\delta} \right\rangle .
\end{aligned} \tag{27}$$

With total spin operator S and the meson function $J(r)$, to simplify the solution and get the result, we suppose the nucleons of equal masses, so the relative mass $\mu = (m_1 m_2 / (m_1 + m_2))$ and center mass $M = m_1 + m_2$.

$$\begin{aligned}\langle F_\alpha F_\gamma \mid V_{ij} \mid F_\beta F_\delta \rangle = &\Big\langle \phi_\alpha\phi_\gamma \Big| - J_\sigma + J_\omega + \frac{1}{8\mu^2 c^2}\left[-\hbar^2\left\{\frac{dJ_\sigma}{dr}\frac{d}{dr}\right\}\right.\\ &+ \frac{1}{r}\frac{dJ_\sigma}{dr}\left[\frac{\hbar^2}{2}\left[J(J+1) - L(L+1) - S(S+1)\right]\right]\Bigg]\\ &+ 2\hbar^2\left\{\frac{dJ_\omega}{dr}\frac{d}{dr}\right\} - \frac{2}{r}\frac{dJ_\omega}{dr}\left[\frac{\hbar^2}{2}\left[J(J+1) - L(L+1) - S(S+1)\right]\right]\\ &+ \frac{1}{4\mu c^2}\left[J_\sigma(r)\left((\hbar\omega(2n+l+3/2)) - 1/2\mu\omega^2 r^2\right)\right.\\ &- 2J_\omega(r)\left((\hbar\omega(2n+l+3/2)) - 1/2\mu\omega^2 r^2\right)\\ &+ 4(2S(S+1)-3)\left(\frac{2}{\hbar^2}(S.\hat{n})^2 - 1\right)J_\omega((\hbar\omega(2n+l+3/2))\\ &\left.- 1/2\mu\omega^2 r^2)\right] + \frac{1}{Mc^2}\left[-2(2S(S+1)-3)\left(\frac{2}{\hbar^2}(S.\hat{n})^2 - 1\right)\right.\\ &\cdot J_\omega\left((\hbar\omega(2N+L+3/2)) - (1/2)M\omega^2 R^2\right)\\ &\left.+ (J_\sigma + J_\omega)\left((\hbar\omega(2N+L+3/2)) - (1/2)M\omega^2 R^2\right)J_\omega\right]\\ &+ \frac{-\hbar^2}{m^2c^2}(2S(S+1)-3)\left(\frac{dJ_\pi}{dr}\frac{d}{dr}\right) - \frac{1}{mc^2}\left(\frac{2}{\hbar^2}(S.\hat{n})^2 - 1\right)\\ &\cdot J_\pi\left(\hbar\omega\left(2n+l+\frac{3}{2}\right) - \frac{1}{2}m\omega^2 r^2\right)\Big|\widetilde{\phi_\beta\phi_\delta}\Big\rangle.\end{aligned} \tag{28}$$

We substitute $(S\cdot\hat{n})$ from [45]. The determination of the energy eigen values requires the diagonalization of the Hamiltonian matrix whose elements are calculated with the functions of Equation (7). We have Equation (28) to show that our model can determine a satisfied result for S state with the meson functions J_σ, J_ω, and J_{omega}; the value of mesons' wave functions depends on distance r which is determined as follows:

(i) In the case of repulsive meson exchange ω, the lower value of (r) is taken as the hadron radius $\simeq 0.5\,\text{fm}$ and the upper value is calculated using the following equation, where R is the range of meson and μ is the mass of meson:

$$R = \frac{1}{\mu} \tag{29}$$

(ii) In the case of the attractive meson exchange π and σ, the upper limit of the previous case is taken as the lower limit in this case and the upper limit is determined using Equation (29)

In the present work, we have applied our model to calculate the ground states of ^{2}H and ^{4}He nuclei ($A = 2$, $A = 4$), respectively. We have determined for two nucleons and four nucleons in $1S_{1/2}$-state according to ^nX_j where $n = 1, j = l + s$, and X represents the state. Table 1 shows the group of parameters used for π, σ, and ω mesons. The set of parameters are I and II that include mass μ, coupling constant (g), and the cutoff λ parameters.

Table 1: The meson parameters for OBEP for different sets.

	Ref	Meson	Mass (MeV)	Coupling constant g_i	Cutoff parameter λ (MeV)
Set I	[46]	π	138.03	14.9	2000
		σ	700	16.07	2000
		ω	782.6	28	1300
Set II	[46]	π	138.03	14.40	1700
		σ	710	18.37	2000
		ω	782.6	24.50	1850

We have determined the ratio R, to ensure the accuracy between the calculated results and the experimental data [47].

$$R = \frac{E_{\text{theor.}}}{E_{\text{exp.}}}, \tag{30}$$

where E_{theor} is the calculated ground state and E_{exp} the experimental one. We can also determine the binding energy per nucleon E/A for the studied nuclei as [48]

$$\frac{E}{A} = -\frac{E_{\text{g.s.}}}{A}, \tag{31}$$

with the mass number A and the total ground state energy $E_{\text{g.s.}}$.

3. Results and Discussion

Table 1 represents the group of parameters used for π, σ, and ω mesons. The set of parameters are I and II that include the mass of meson, the coupling constant (g), and the cutoff parameter (λ). The potential is elaborated to calculate the ground-state energies for the H^2, He^4 nuclei. The results are listed in Tables 2 and 3 in comparison with the experimental data. The ratio between the present work and experimental one is estimated for both cases, in other words, by using the potential extracted from GY and SPED functions.

We have examined potential Equation (28) to calculate the ground-state energy of H^2 and He^4 nuclei using two static meson functions (GY and SPED) with two sets of parameters listed in Table 1 which shows the different sets of the used parameters and for different exchange mesons, σ and ω mesons and π, σ, and ω mesons. The potential for different cases is plotted in Figures 1–8.

Figures 1–8 illustrate the potential $V(r)$ in MeV versus r in fm for H^2 and He^4 nuclei for both cases (GY and SPED) and for different sets of parameters (I and II) (Table 1).

Potential energy Equation (28) is illustrated in Figures 1–8 by two sets of parameters. We have checked them with different meson exchange function GY and SPED. So, we categorize our results into two groups: for the set I parameter with GY and SPED calculated within σ and ω and for the set II parameter the same above for both nuclei (H^2 and He^4).

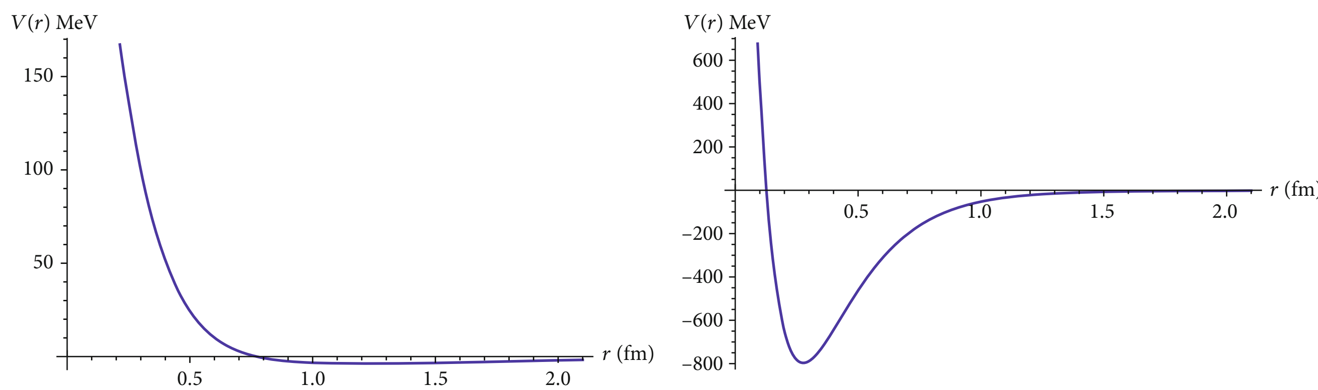
All cases are calculated again within σ, ω, and π, in other words, by adding the third exchange meson "π" which works

Table 2: The ground-state energy of ^{2}H with the aid of Table 1.

Parameter sets [46]	Meson exchange	Present work (GY)	Present work (SPED)	Others	exp. [49–51]	Ratio GY	Ratio SPED	E/A GY	E/A SPED	E/A exp. [52]
I	σ, ω	−2.916	−2.041	-2.215[53]		1.311	0.918	1.458	1.0205	
II		−3.486	−1.973		−2.224	1.567	0.887	1.743	0.9865	1.112
I	π, σ, ω	−2.199	−2.248	-2.220		0.989	1.011	1.0995	1.124	
II		−2.168	−2.204	±0.179[54]		0.975	0.991	1.084	1.102	

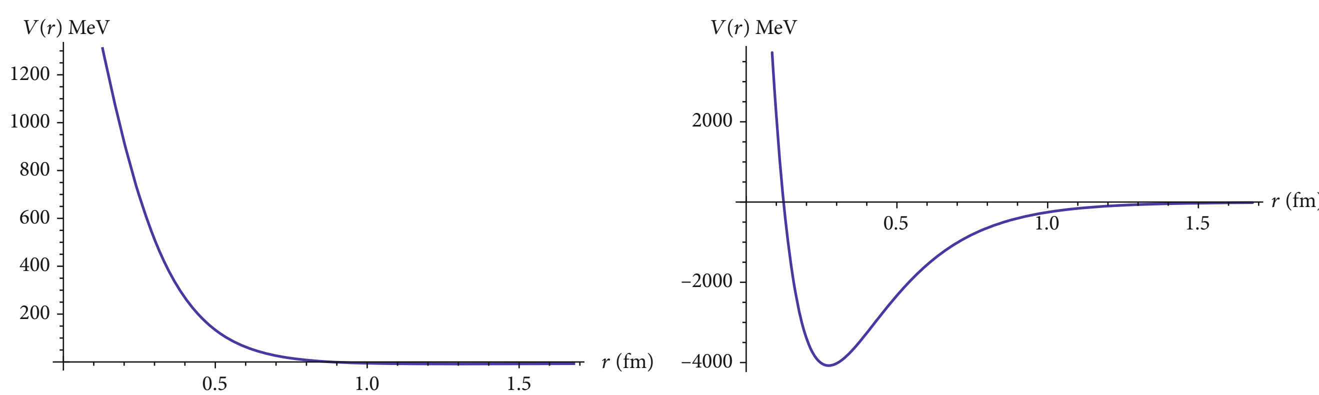
Table 3: The ground-state energy of ^{4}He with the aid of Table 1.

Parameter sets [46]	Meson exchange	Present work (GY)	Present work (SPED)	[55]	exp. [47, 56]	Ratio GY	Ratio SPED	E/A GY	E/A SPED	E/A exp.
I	σ, ω	−22.372	−20.238			1.0966	0.992	5.593	5.0595	
II		−22.751	−21.556	-21.385	-20.4±0.3	1.115	1.057	5.6877	5.389	5.1
I	π, σ, ω	−22.637	−20.375			1.109	0.999	5.659	5.0937	
II		−21.871	−20.337			1.072	0.997	5.4677	5.08425	



(a) (b)

Figure 1: The potential with GY and SPED functions, respectively, of ^{2}H nuclei as OBEP through the exchange of σ and ω mesons, with parameter I.



(a) (b)

Figure 2: The potential with GY and SPED functions, respectively, of ^{4}He nuclei as OBEP through the exchange of σ and ω mesons, with parameter I.

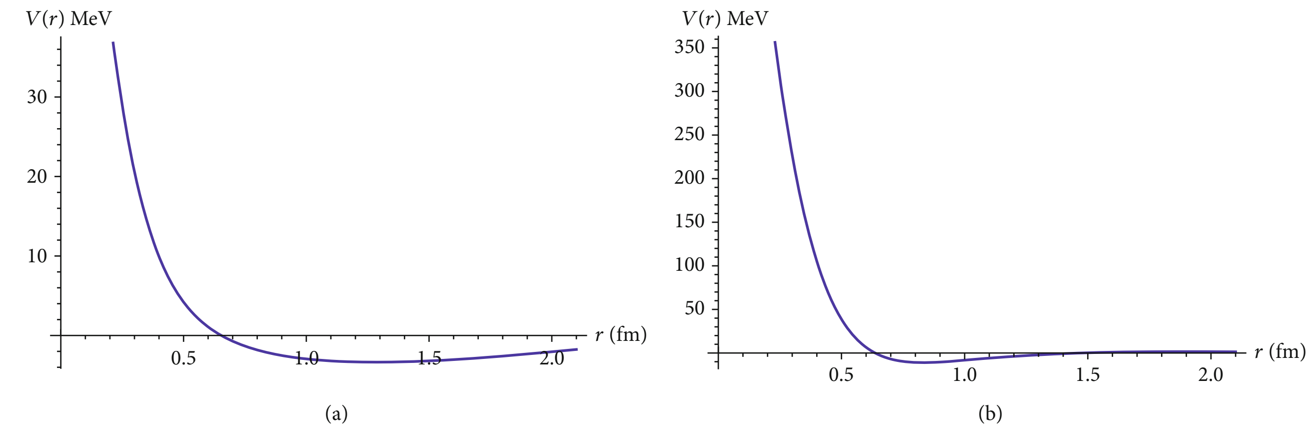


Figure 3: The potential with GY and SPED functions, respectively, of ^{2}H nuclei as OBEP through the exchange of σ and ω mesons, with parameter II.

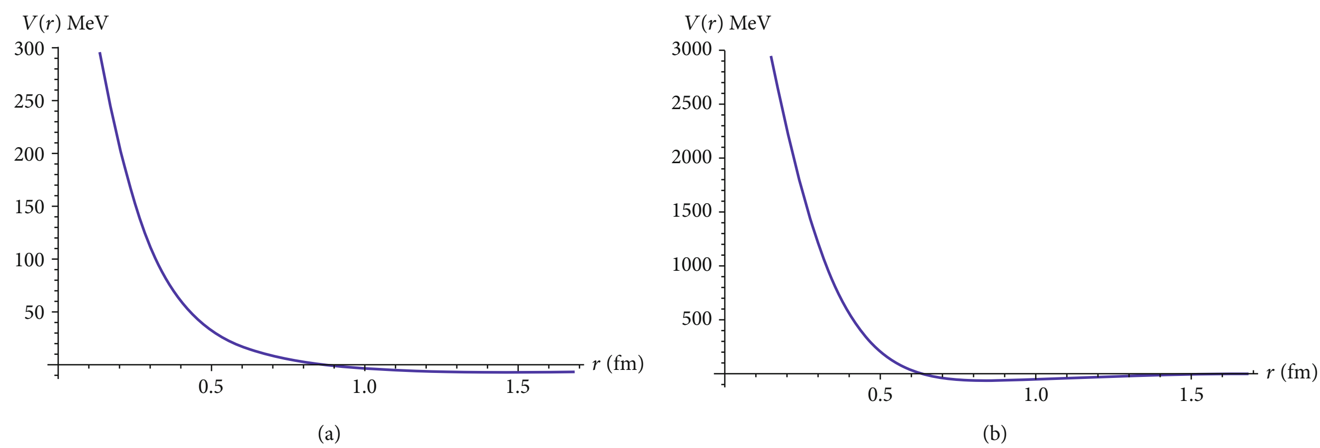


Figure 4: The potential with GY and SPED functions, respectively, of ^{4}He nuclei as OBEP through the exchange of σ and ω mesons, with parameter II.

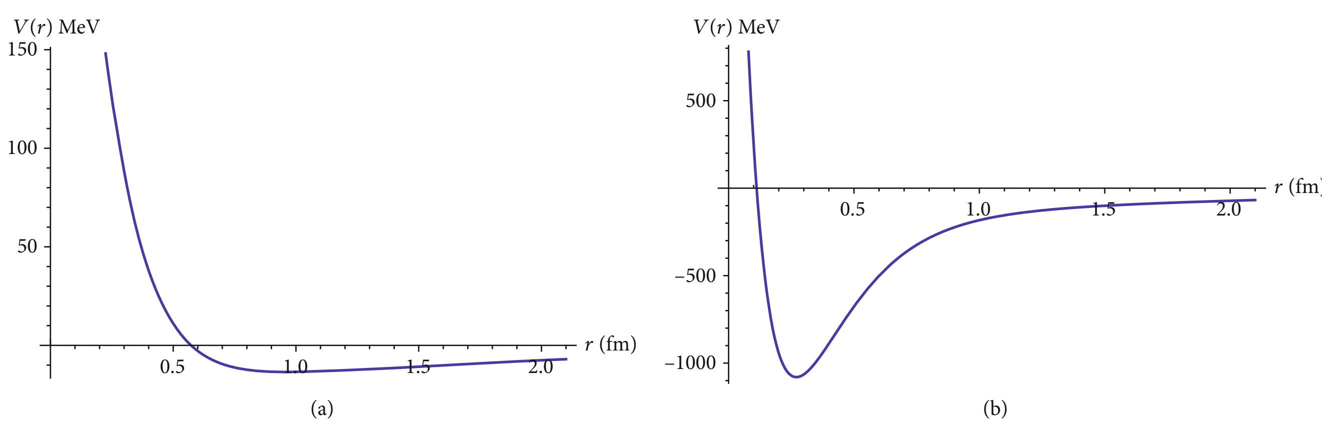


Figure 5: The potential with GY and SPED functions, respectively, of ^{2}H nuclei as OBEP through the exchange of π, σ, and ω mesons, with parameter I.

attractively at large r. Figures 1–4 represent category "I, II" for two-meson exchange, respectively, and Figures 5–8 represent category "I, II" for three-meson exchange. Figure 1(a) shows the potential by using the GY meson function in which the effect of repulsive potential due to ω meson appears at quite large distance, while the attractive part does not appear clearly as the depth of the potential is very small, the attractive part began with $r\sim 2.0$ fm near to the diameter of H^2 nuclei and finished at $r\sim 0.7$ fm.

Figure 1(b) is calculated by using the SPED meson function; in this case, a significant attractive potential began with $r\sim 1.1$ fm and ended at $r\sim 0.25$ fm. (Figure 2) represents the

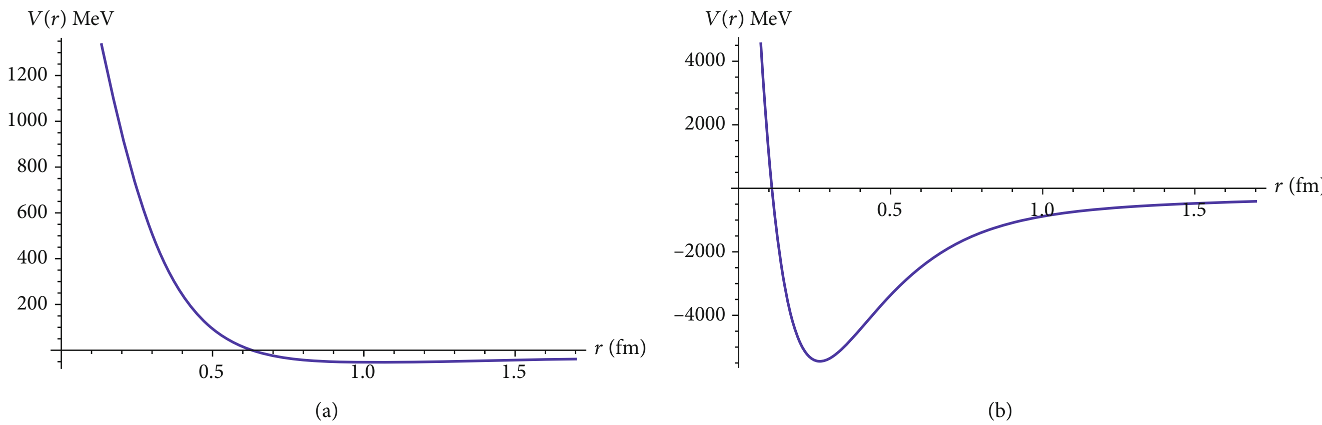


FIGURE 6: The potential with GY and SPED functions, respectively, of ^{4}He nuclei as OBEP through the exchange of π, σ, and ω mesons, with parameter I.

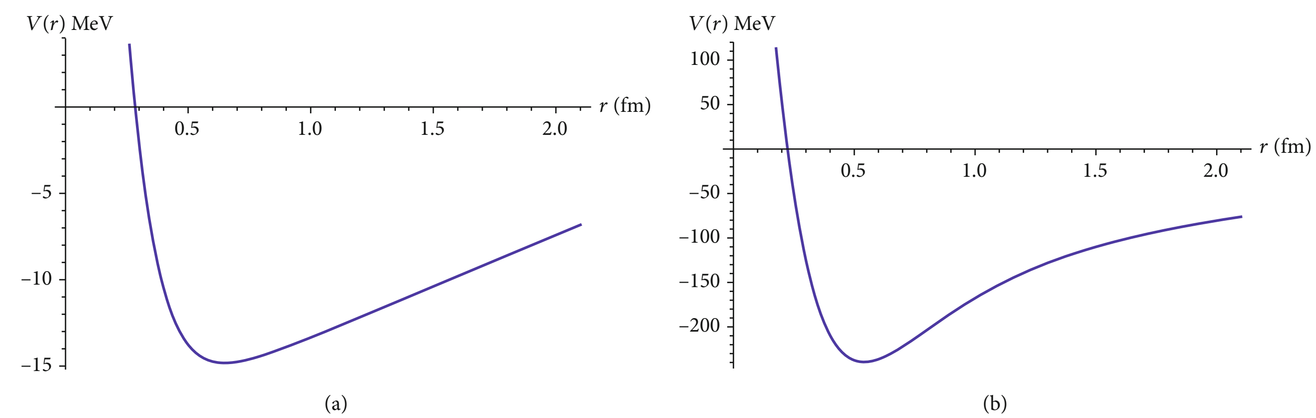


FIGURE 7: The potential with GY and SPED functions, respectively, of ^{2}H nuclei as OBEP through the exchange of π, σ, and ω mesons, with parameter II.

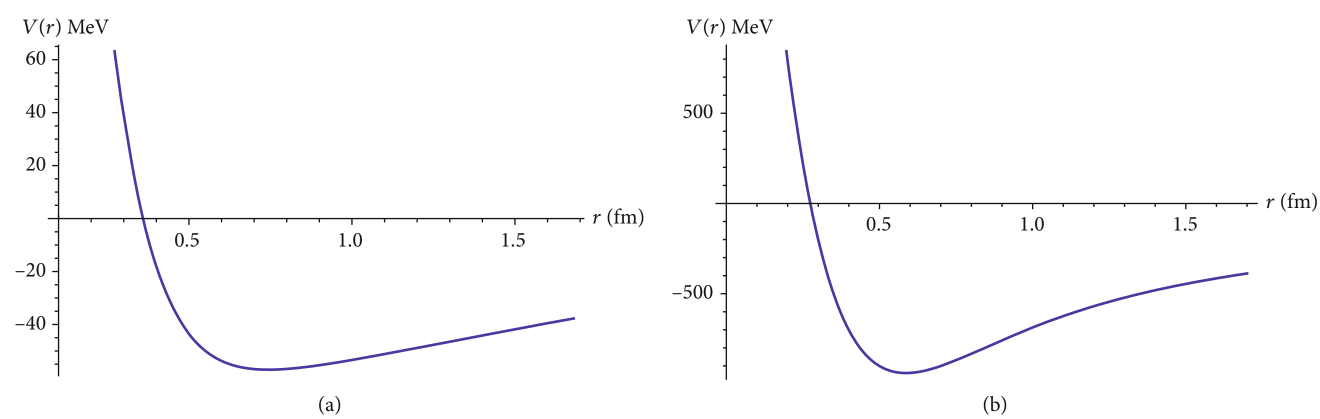


FIGURE 8: The potential with GY and SPED functions, respectively, of ^{4}He nucleus as OBEP through the exchange of π, σ, and ω mesons, with parameter II.

same manner of H^4 nuclei with different values of the potential; the transferred point between the attractive and repulsive parts is similar to the one in (Figure 1), and the beginning point of the attractive part is controlled by the diameter of nuclei.

It can be noticed that the depth of the attractive potential extracted by the SPED function is greater than the one from the GY function for both nuclei (Figures 1 and 2). But it seems that, using set II, the behavior of two functions is close to each other from the transferred point and the depth; the difference between two nuclei is still the values of the attractive potential, and ω meson is the master here in which the repulsive potential has more higher values and the effect of σ has been damped (see Figures 3 and 4). The effect of the third meson exchange π meson is added as shown in Figures 5 and 6 for set I and Figures 7 and 8 for set II. Firstly, again for set I, Figures 5 and 6 represent the potential that behaves as the same before for two-meson exchange, in which the attractive part increased very slowly. The depth of the attractive potential increased significantly, π meson flies at more than $r \sim 1.5$ fm, and the transferred point has different values from the one in the two-meson exchange by using the SPED function. On the other hand, for set II as shown in Figures 7 and 8, GY and SPED have an improvement in their transferred point and depth than in the two-meson exchange; the SPED function is still better than the GY function.

We observe from Tables 2 and 3 two sets of parameters and how many mesons to be exchanged between two nucleons and meson exchanges with two functions, and we listed our results for each one for the selected nuclei. Meanwhile, if the ratio tends to unite, the ground energies would be close to the experimental data. The preferable theoretical value of the ^{2}H nucleus in case of using two mesons is the SPED function for parameter I, and by using three mesons, we have the value of SPED in parameter II to be more accurate than others. It is obvious from Tables 2 and 3 that the ground energy is close to the data in the case of SPED function for set I and set II in comparison with the experimental data. The ^{4}He nucleus has a little different manner; the theoretical value of the SPED function for two mesons in parameter I is better than the value of the GY function. In case of handling three mesons, SPED function in parameter II is the best as shown in Table 3. Using the ratio relation is useful to ensure that our results for the SPED function is better than the GY function, and our attempt to include more two mesons in OBEP analytically is successful in result improvement. The ratio is getting a better result for going on more massive nuclei and encouraged for our potential. We concluded that the used model is well-defined and compatible with the data and even with other models (see [57, 58]). The deuteron ground-state energy is quiet a little different from the numerical data in [54, 59] as a good sign for our constructing potential analytically. The calculation of binding energy per nucleon serves our idea of being the OBEP with three and four mesons in the case of SPED function and gives satisfied values for Deuteron and Helium nuclei comparing with the experimental one.

4. Conclusion

In the framework of quasirelativistic formulation, the meson exchange potential helps in obtaining a potential with few numbers of parameters to calculate the ground state for the light nuclei Deuteron and Helium using two- (σ and ω) and three- (π, σ, and ω) meson exchange. In addition, it was shown that a self-consistent treatment of the semirelativistic nucleon wave function in nuclear state has a great importance in calculations. The difference in masses of σ and ω mesons would not seriously change the main aspect of the concept of relativistic or semirelativistic interaction, providing an average potential of cancelation of the repulsive meson (ω) and the attractive meson (σ) in conjunction with a weak long-range effect (π). This work with OBEP in the Dirac-Hartree-Fock equation gives a close relationship to other recent approaches, based upon different formalisms which tended to support this direction. The ground-state energies for ^{2}H and ^{4}He nuclei are successfully determined through this work and give us a hope to continue with more massive nuclei. The nuclear properties are being clear in our trail to include two more mesons to describe the NN interaction through our potential. The SPED function has a good ability to give us the better shapes of our potential and also better values for energies. We hope that our potential represents a base for the NN interaction with different ranges of energies in the following search.

Appendix

A. Wave Function with the Clebsch-Gordon Coefficient

The wave functions for two nucleons i and j have a form with Clebsch-Gordon coefficients.

$$\left\langle \phi_\alpha(r_i)\phi_\gamma(r_j)\right| = \sum_{m_{l_\alpha} m_{s_\alpha}} \sum_{m_{l_\gamma} m_{s_\gamma}} \left(l_\alpha s_\alpha m_{l_\alpha} m_{s_\alpha} | j_\alpha M_\alpha\right) \cdot \left(l_\gamma s_\gamma m_{l_\gamma} m_{s_\gamma} \middle| j_\gamma M_\gamma\right) \left\langle \phi_{n_\alpha l_\alpha m_{l_\alpha}}(r_i)\phi_{n_\gamma l_\gamma m_{l_\gamma}} \cdot (r_j)\right| \left\langle \chi^{1/2}_{m_{s_\alpha}} \chi^{1/2}_{m_{s_\gamma}} \right| \left\langle \widehat{P}_{T_\alpha} \widehat{P}_{T_\gamma} \right|, \tag{A.1}$$

where l is the orbital angular momentum; s_γ is the spin; the total angular momentum $j_\alpha = l_\alpha + s_\alpha$; $j_\gamma = l_\gamma + s_\gamma$; $M_\alpha = m_{l_\alpha} + m_{s_\alpha}$ in which m_{l_α} is the projection of orbital quantum number; m_{s_α} is the projection of spin quantum number; $M_\gamma = m_{l_\gamma} + m_{s_\gamma}$; and $\widehat{P}_{T_\alpha}$ is the function of isotopic spin. The two wave functions are not connected and depend on r_i and r_j, so the two wave functions need to be connected

$$\left\langle \phi_\alpha(r_i)\phi_\gamma(r_j)\right| = \sum_{m_{l_\alpha} m_{s_\alpha}} \sum_{m_{l_\gamma} m_{s_\gamma}} \sum_{\lambda\mu} \left(l_\alpha s_\alpha m_{l_\alpha} m_{s_\alpha} | j_\alpha M_\alpha\right) \cdot \left(l_\gamma s_\gamma m_{l_\gamma} m_{s_\gamma} \middle| j_\gamma M_\gamma\right) \left(l_\alpha l_\gamma m_{l_\alpha} m_{l_\gamma} | \lambda\mu\right) \cdot \left\langle \phi_{n_\alpha l_\alpha m_{l_\alpha}}(r_i)\phi_{n_\gamma l_\gamma m_{l_\gamma}}(r_j)\right| \cdot \left\langle \chi^{1/2}_{m_{s_\alpha}} \chi^{1/2}_{m_{s_\gamma}} \right| \left\langle \widehat{P}_{T_\alpha} \widehat{P}_{T_\gamma} \right|. \tag{A.2}$$

With $\lambda = l_\alpha + l_\gamma$ and $\mu = m_{l_\alpha} + m_{l_\gamma}$, we can change the special coordinates for each wave function to become one wave that depends on the relative mass and center of mass.

$$\left\langle \phi_\alpha(r_i)\phi_\gamma(r_j)\right| = \sum_{m_{l_\alpha} m_{s_\alpha}} \sum_{m_{l_\gamma} m_{s_\gamma}} \sum_{\lambda\mu} \sum_{nlNL} \left(l_\alpha s_\alpha m_{l_\alpha} m_{s_\alpha} \mid j_\alpha M_\alpha\right) \cdot \left(l_\gamma s_\gamma m_{l_\gamma} m_{s_\gamma} \mid j_\gamma M_\gamma\right)\left(l_\alpha l_\gamma m_{l_\alpha} m_{l_\gamma} \mid \lambda\mu\right) \cdot \langle n_\alpha l_\alpha n_\gamma l_\gamma \mid NLnl\rangle \langle \phi_{NLnl}(r,R)| \left\langle \chi^{1/2}_{m_{s_\alpha}} \chi^{1/2}_{m_{s_\gamma}}\right| \cdot \left\langle \widehat{P}_{T_\alpha} \widehat{P}_{T_\gamma}\right|, \tag{A.3}$$

where $\langle n_\alpha l_\alpha n_\gamma l_\gamma \mid NLnl\rangle$ is the Talmi-Moshinsky bracket, NL is the total center of mass, and nl is the total relative. The wave function $\phi_{NLnl}(r,R)$ can be spitted into the form

$$\left\langle \phi_\alpha(r_i)\phi_\gamma(r_j)\right| = \sum_{m_{l_\alpha} m_{s_\alpha}} \sum_{m_{l_\gamma} m_{s_\gamma}} \sum_{JM} \sum_{\lambda\mu} \sum_{nlNL} \sum_{mM} \left(l_\alpha s_\alpha m_{l_\alpha} m_{s_\alpha} | j_\alpha M_\alpha\right) \cdot \left(l_\gamma s_\gamma m_{l_\gamma} m_{s_\gamma} \middle| j_\gamma M_\gamma\right)\left(l_\alpha l_\gamma m_{l_\alpha} m_{l_\gamma} | \lambda\mu\right) \cdot \langle n_\alpha l_\alpha n_\gamma l_\gamma | NLnl\rangle (lSm_l m_S | JM)(LlMm|\lambda\mu) \cdot \langle \phi_{NLM}(R)\phi_{nlm}(r)| \left\langle \chi^{1/2}_{m_{s_\alpha}} \chi^{1/2}_{m_{s_\gamma}}\right| \left\langle \widehat{P}_{T_\alpha} \widehat{P}_{T_\gamma}\right| \tag{A.4}$$

As L gives the total orbital quantum number in the center of mass, l gives the total orbital quantum number in relative coordinates and $S = s_i + s_j$ is the total spin. Relative to the spin functions and isospin functions to be connected, we have to use them as follows:

$$\left\langle \phi_\alpha(r_i)\phi_\gamma(r_j)\right| = \sum_{m_{l_\alpha} m_{s_\alpha}} \sum_{m_{l_\gamma} m_{s_\gamma}} \sum_{JM} \sum_{\lambda\mu} \sum_{nlNL} \sum_{mM} \sum_{sm_s} \sum_{sm_s} \left(l_\alpha s_\alpha m_{l_\alpha} m_{s_\alpha} \mid j_\alpha M_\alpha\right) \cdot \left(l_\gamma s_\gamma m_{l_\gamma} m_{s_\gamma} \mid j_\gamma M_\gamma\right)\left(l_\alpha l_\gamma m_{l_\alpha} m_{l_\gamma} \mid \lambda\mu\right) \cdot \langle n_\alpha l_\alpha n_\gamma l_\gamma \mid NLnl\rangle (lSm_l m_S \mid JM)(LlMm \mid \lambda\mu) \cdot \left(s_\alpha s_\gamma m_{s_\alpha} m_{s_\gamma} \mid SM_s\right)\left(s_\alpha s_\gamma T_\alpha T_\gamma \mid TM_T\right) \cdot \langle \phi_{NLM}(R)\phi_{nlm}(r)| \left\langle \chi^S_{m_s}(i,j)\right| \langle \widehat{P}_T(i,j)|, \tag{A.5}$$

with $T_{\text{proton}} = -1/2$ and $T_{\text{neutron}} = 1/2$.

B. The Derivation of OBEP through the Exchange of Two Mesons

According to the relation between ϕ and χ in Equation (12), one obtains

$$\begin{aligned}\left\langle F_\alpha F_\gamma \mid V_{ij}(r) \mid \widetilde{F_\beta F_\delta}\right\rangle = &\left\langle \phi_\alpha\phi_\gamma\right| -J_\sigma + J_\omega + \frac{1}{4m^2c^2}\Big[\left(\vec{\sigma}_i\cdot\vec{P}_i\right)J_\sigma\left(\vec{\sigma}_i\cdot\vec{p'}_i\right) \\ &+ \left(\vec{\sigma}_j\cdot\vec{P}_j\right)J_\sigma\left(\vec{\sigma}_j\cdot\vec{p'}_j\right) + \left(\vec{\sigma}_i\cdot\vec{p}_i\right)J_\omega\left(\vec{\sigma}_i\cdot\vec{p'}_i\right) \\ &+ \left(\vec{\sigma}_j\cdot\vec{P}_j\right)J_\omega\left(\vec{\sigma}_j\cdot\vec{p'}_j\right) - J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\left(\vec{\sigma}_j\cdot\vec{p'}_j\right)\left(\vec{\sigma}_i\cdot\vec{p'}_i\right) \\ &- \left(\vec{\sigma}_j\cdot\vec{P}_j\right)J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\left(\vec{\sigma}_i\cdot\vec{p'}_i\right) - \left(\vec{\sigma}_i\cdot\vec{p}_i\right)J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\left(\vec{\sigma}_j\cdot\vec{p'}_j\right) \\ &- \left(\vec{\sigma}_i\cdot\vec{P}_i\right)\left(\vec{\sigma}_j\cdot\vec{P}_j\right)J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\Big] \mid \widetilde{\phi_\beta\phi_\delta}\rangle.\end{aligned} \tag{B.1}$$

Defining the momentum for each nucleon (i and j), $p_i = p_R + (1/2)p_r$, $p_j = -p_r + (1/2)p_r$, and $p_i = p'_i$ and $p_j = p'_j$ [34, 44]. Substituting those relations into Equation (B.1), the dependence of $J(r)$ on the relative distance (r) not on R makes its movement with the center of mass operators more easy, where $p_r = p$ and $(\sigma_i \cdot p_r)(\sigma_i \cdot p_r) = p_r^2$; we obtain

$$\begin{aligned}\left\langle F_\alpha F_\gamma \mid V_{ij}(r) \mid \widetilde{F_\beta F_\delta}\right\rangle = &\left\langle \phi_\alpha\phi_\gamma\right| - J_\sigma + J_\omega \\ &+ \frac{1}{4m^2c^2}\Big[\left(\vec{\sigma}_i\cdot\vec{p}\right)J_\sigma\left(\vec{\sigma}_i\cdot\vec{p}\right) + \left(\vec{\sigma}_j\cdot\vec{p}\right)J_\sigma\left(\vec{\sigma}_j\cdot\vec{p}\right) \\ &+ \left(\vec{\sigma}_i\cdot\vec{p}\right)J_\omega\left(\vec{\sigma}_i\cdot\vec{p}\right) + \left(\vec{\sigma}_j\cdot\vec{p}\right)J_\omega\left(\vec{\sigma}_j\cdot\vec{p}\right) \\ &+ J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\left(\vec{\sigma}_j\cdot\vec{p}\right)\left(\vec{\sigma}_i\cdot\vec{p}\right) + \left(\vec{\sigma}_i\cdot\vec{p}\right)J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right) \\ &\cdot\left(\vec{\sigma}_j\cdot\vec{p}\right) + \left(\vec{\sigma}_j\cdot\vec{p}\right)J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\left(\vec{\sigma}_i\cdot\vec{p}\right) \\ &+ \left(\vec{\sigma}_i\cdot\vec{p}\right)\left(\vec{\sigma}_j\cdot\vec{p}\right)J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right) + J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\left(\vec{\sigma}_j\cdot\vec{p}\right) \\ &\cdot\left(\vec{\sigma}_i\cdot\vec{p}_R\right) - J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\left(\vec{\sigma}_j\cdot\vec{p}_r\right)\left(\vec{\sigma}_i\cdot\vec{p}\right) \\ &+ \left(\vec{\sigma}_j\cdot\vec{p}\right)\left(\vec{\sigma}_i\cdot\vec{p}_r\right)J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right) - \left(\vec{\sigma}_i\cdot\vec{p}\right)\left(\vec{\sigma}_j\cdot\vec{p}_r\right) \\ &\cdot J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\Big] + \frac{1}{8m^2c^2}\Big[\left(\vec{\sigma}_i\cdot\vec{p}\right)\left(\vec{\sigma}_i\cdot\vec{p}_r\right)J_\sigma \\ &+ J_\sigma\left(\vec{\sigma}_i\cdot\vec{p}_r\right)\left(\vec{\sigma}_i\cdot\vec{p}\right) - \left(\vec{\sigma}_j\cdot\vec{p}\right)\left(\vec{\sigma}_j\cdot\vec{p}_r\right)J_\sigma \\ &- J_\sigma\left(\vec{\sigma}_j\cdot\vec{p}_r\right)\left(\vec{\sigma}_j\cdot\vec{p}\right) + \left(\vec{\sigma}_i\cdot\vec{p}\right)\left(\vec{\sigma}_i\cdot\vec{p}_r\right)J_\omega \\ &+ J_\omega\left(\vec{\sigma}_i\cdot\vec{p}_r\right)\left(\vec{\sigma}_i\cdot\vec{p}\right) - \left(\vec{\sigma}_j\cdot\vec{p}\right)\left(\vec{\sigma}_j\cdot\vec{p}_r\right)J_\omega \\ &- J_\omega\left(\vec{\sigma}_j\cdot\vec{p}_r\right)\left(\vec{\sigma}_j\cdot\vec{p}\right) + p_r^2 J_\sigma + p_r^2 J_\omega \\ &- 2J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\left(\vec{\sigma}_j\cdot\vec{p}_r\right)\left(\vec{\sigma}_i\cdot\vec{p}_r\right)\Big]\left|\widetilde{\phi_\beta\phi_\delta}\right\rangle.\end{aligned} \tag{B.2}$$

We will apply some important relations [60]:

(1) $(\vec{\sigma}_1 \cdot \vec{A})(\vec{\sigma}_1 \cdot \vec{B}) = A \cdot B + i\vec{\sigma}_1(A \times B)$

(2) $(\vec{\sigma}_1 \cdot \vec{A})^2 = A^2$

(3) $(\vec{\sigma}_1 \cdot \vec{A})(\vec{\sigma}_2 \cdot \vec{A}) = (2/\hbar^2)(S \cdot A)^2 - A^2$

(4) $(\vec{\sigma} \cdot \vec{A})F(r)(\vec{\sigma} \cdot \vec{A}) = F(r)A^2 - i\hbar\{\nabla F(r) \cdot A + i\sigma[(\nabla F(r)) \times A]\}$

(5) $\vec{\sigma}_i\vec{\sigma}_j = 2\delta_{ij} - \sigma_{ji}$

Include these relations in potential equation.

$$\begin{aligned}\langle F_\alpha F_\gamma \mid V_{ij}(r) \mid \tilde{F}_\beta F_\delta\rangle = &\left\langle \phi_\alpha\phi_\gamma\right| -J_\sigma + J_\omega \\ &+ \frac{1}{4m^2c^2}\Big[\left(\vec{\sigma}_i\cdot\vec{p}\right)J_\sigma\left(\vec{\sigma}_i\cdot\vec{p}\right) + \left(\vec{\sigma}_j\cdot\vec{p}\right)J_\sigma\left(\vec{\sigma}_j\cdot\vec{p}\right) \\ &+ \left(\vec{\sigma}_i\cdot\vec{p}\right)J_\omega\left(\vec{\sigma}_i\cdot\vec{p}\right) + \left(\vec{\sigma}_j\cdot\vec{p}\right)J_\omega\left(\vec{\sigma}_j\cdot\vec{p}\right) \\ &+ J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\left(\vec{\sigma}_j\cdot\vec{p}\right)\left(\vec{\sigma}_i\cdot\vec{p}\right) + \left(\vec{\sigma}_i\cdot\vec{p}\right) \\ &\cdot J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\left(\vec{\sigma}_j\cdot\vec{p}\right) + \left(\vec{\sigma}_j\cdot\vec{p}\right)J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\left(\vec{\sigma}_i\cdot\vec{p}\right) \\ &+ \left(\vec{\sigma}_i\cdot\vec{p}\right)\left(\vec{\sigma}_j\cdot\vec{p}\right)J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\Big] \\ &+ \frac{1}{4m^2c^2}\Big[J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\left(\vec{\sigma}_j\cdot\vec{p}_r\right)\left(\vec{\sigma}_i\cdot\vec{p}_r\right) + \left(\frac{1}{2}\right)p_r^2 J_\sigma \\ &+ \left(\frac{1}{2}\right)p_r^2 J_\omega\cdot\Big] \mid \tilde{\phi}_\beta\phi_\delta\rangle\end{aligned} \tag{B.3}$$

To get the solution of Equation (B.3), we substitute every term as follows, using the relation of angular momentum $L=\vec{r}\times p$ and $\sigma=2S/\hbar$ where S is the spin operator, $P=-i\hbar\nabla$, and $\nabla J_\sigma=(1/r)(dJ_\sigma/dr)r$. According to the previous relations and where $\sigma_j^2=\sigma_x^2+\sigma_y^2+\sigma_z^2=1$ as triplet case for two nucleons,

$$\begin{aligned}\left(\vec{\sigma}_j\cdot\vec{p}\right)J_\omega(r)\left(\vec{\sigma}_j\cdot\vec{p}\right)&=\left(\vec{\sigma}_j\cdot\vec{p}\right)J_\omega(r)\sigma_j^2(\sigma_i\cdot p)\\&=-3J_\omega(r)p^2+3\hbar^2\left\{\frac{dJ_\omega}{dr}\frac{d}{dr}\right\}\\&\quad-\frac{6}{r}\frac{dJ_\omega}{dr}\left[\vec{S}_j\cdot\vec{L}\right],\\\left(\vec{\sigma}_i\cdot\vec{p}\right)J_\omega(r)\left(\vec{\sigma}_i\vec{\sigma}_j\right)\left(\vec{\sigma}_j\cdot\vec{p}\right)&=-3J_\omega(r)p^2+3\hbar^2\left\{\frac{dJ_\omega}{dr}\frac{d}{dr}\right\}\\&\quad-\frac{6}{r}\frac{dJ_\omega}{dr}\left[\vec{S}_i\cdot\vec{L}\right].\end{aligned}\tag{B.4}$$

We substitute those terms in Equation (B.3)

$$\begin{aligned}\left\langle F_\alpha F_\gamma\mid V_{ij}(r)\mid\tilde{F}_\beta F_\delta\right\rangle&=\Big\langle\phi_\alpha\phi_\gamma\Big|-J_\sigma+J_\omega\\&\quad+\frac{1}{4m^2c^2}\left[2J_\sigma(r)p^2-2\hbar^2\left\{\frac{dJ_\sigma}{dr}\frac{d}{dr}\right\}+\frac{2}{r}\frac{dJ_\sigma}{dr}\left[\vec{S}.\vec{L}\right]\right.\\&\quad-4J_\omega(r)p^2+4\hbar^2\left\{\frac{dJ_\omega}{dr}\frac{d}{dr}\right\}-\frac{4}{r}\frac{dJ_\omega}{dr}[S\cdot L]\\&\quad+J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\frac{2}{\hbar^2}(S\cdot p)^2-J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)p^2\\&\quad\left.+\frac{2}{\hbar^2}(S\cdot p)^2J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)-p^2J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\right]\\&\quad+\frac{1}{4m^2c^2}\left[-J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\frac{2}{\hbar^2}(S\cdot p_r)^2+J_\omega\left(\vec{\sigma}_i\cdot\vec{\sigma}_j\right)p_r^2\right.\\&\quad\left.+(1/2)p_r^2J_\sigma+(1/2)p_r^2J_\omega\right]\mid\tilde{\phi}_\beta\phi_\delta\Big\rangle,\end{aligned}\tag{B.5}$$

with total spin operator S and the meson function $J(r)$, using $(S\cdot P)^2=(S\cdot\hat{n})^2P^2$, $(\sigma_i\cdot\sigma_j)=(2/\hbar^2)S^2-3$ and $S\cdot L=(\hbar^2/2)[J(J+1)-L(L+1)-S(S+1)]$ [46]. Quantum mechanics have a magnificent tool; this tool is the harmonic oscillator which is capable of being solved in closed form; it has generally useful approximations and exact solutions of different problems [40]. It solves the differential equations in quantum mechanics. We have the energy of a harmonic oscillator ($\hbar\omega(2n+l+3/2)$) which equals the kinetic energy ($P^2/2m$) added to the potential energy ($(1/2)m\omega^2x^2$) to simplify the solution and get the result. It is slitted in relative harmonic oscillator energy $\hbar\omega(2n+l+3/2)=(P^2/2\mu)+(1/2)\mu\omega^2r^2$ [36, 61], with ω that is the angular frequency, and the center of mass contribution in harmonic oscillator energy $\hbar\omega(2N+L+3/2)=(P^2/2m)+(1/2)M\omega^2R^2$.

We suppose the nucleons of equal masses, so the relative mass $\mu=(m_1m_2)/(m_1+m_2)=m/2$ and the center mass $M=m_1+m_2=2m$.

$$\begin{aligned}&\left\langle F_\alpha F_\gamma\mid V_{ij}(r)\mid\tilde{F}_\beta F_\delta\right\rangle\\&=\Big\langle\phi_\alpha\phi_\gamma\Big|-J_\sigma+J_\omega+\frac{1}{8\mu^2c^2}\left[-\hbar^2\left\{\frac{dJ_\sigma}{dr}\frac{d}{dr}\right\}\right.\\&\quad+\frac{1}{r}\frac{dJ_\sigma}{dr}\left[\frac{\hbar^2}{2}[J(J+1)-L(L+1)-S(S+1)]\right]\\&\quad\left.+2\hbar^2\left\{\frac{dJ_\omega}{dr}\frac{d}{dr}\right\}-\frac{2}{r}\frac{dJ_\omega}{dr}\left[\frac{\hbar^2}{2}[J(J+1)-L(L+1)-S(S+1)]\right]\right]\\&\quad+\frac{1}{4\mu c^2}\left[J_\sigma(r)\left(\frac{p^2}{2\mu}\right)-2J_\omega(r)\left(\frac{p^2}{2\mu}\right)+2J_\omega(2S(S+1)-3)\right.\\&\quad\times\left(\frac{2}{\hbar^2}(S\cdot\hat{n})^2-1\right)\left(\frac{p^2}{2\mu}\right)+2\left(\frac{2}{\hbar^2}(S\cdot\hat{n})^2-1\right)\left(\frac{p^2}{2\mu}\right)\\&\quad\left.\times J_\omega(2S(S+1)-3)\right]+\frac{1}{Mc^2}\left[-2(2S(S+1)-3)J_\omega(r)\right.\\&\quad\left.\times\left(\frac{2}{\hbar^2}(S\cdot\hat{n})^2-1\right)\left(\frac{p_r^2}{2M}\right)+\left(\frac{p_r^2}{2M}\right)J_\sigma+\left(\frac{p_r^2}{2M}\right)J_\omega\right]\Big|\tilde{\phi}_\beta\phi_\delta\Big\rangle.\end{aligned}\tag{B.6}$$

The total formula of two-nucleons interaction through the exchange of two mesons where $p_{ij}=p$ and $A1$ is the mass number of the required nuclei.

$$\begin{aligned}&\left\langle\phi_\alpha(r_i)\phi_\gamma(r_j)\mid H\mid\phi_\beta(r_i)\phi_\delta(r_j)\right\rangle\\&=\sum_{m_{l_\alpha}m_{s_\alpha}}\sum_{m_{l_\gamma}m_{s_\gamma}}\sum_{JM}\sum_{\lambda\mu}\sum_{nlNL}\sum_{mM}\sum_{sm_s}\sum_{T}\left(l_\alpha s_\alpha m_{l_\alpha}m_{s_\alpha}\mid j_\alpha M_\alpha\right)\\&\quad\times\left(l_\gamma s_\gamma m_{l_\gamma}m_{s_\gamma}\mid j_\gamma M_\gamma\right)\left(l_\alpha l_\gamma m_{l_\alpha}m_{l_\gamma}\mid\lambda\mu\right)\left\langle n_\alpha l_\alpha n_\gamma l_\gamma\mid NLnl\right\rangle\\&\quad\times(lSm_lm_S\mid JM)(LlMm\mid\lambda\mu)\left(s_\alpha s_\gamma m_{s_\alpha}m_{s_\gamma}\mid SM_s\right)\\&\quad\times\left(\chi_\alpha\chi_\gamma T_\alpha T_\gamma\mid M_TT\right)\langle R_{NLM}(R)Y_{NLM}R_{nlm}(r)Y_{nlm}|\\&\quad\times\langle\hat{P}_T(i,j)|\frac{4}{A1}\left((\hbar\omega(2n+l+3/2))-1/2\mu\omega^2r^2\right)-J_\sigma+J_\omega\\&\quad+\frac{1}{8\mu^2c^2}\left[-\hbar^2\left\{\frac{dJ_\sigma}{dr}\frac{d}{dr}\right\}+\frac{1}{r}\frac{dJ_\sigma}{dr}\left[\frac{\hbar^2}{2}[J(J+1)-L(L+1)\right.\right.\\&\quad\left.-S(S+1)]\right]+2\hbar^2\left\{\frac{dJ_\omega}{dr}\frac{d}{dr}\right\}-\frac{2}{r}\frac{dJ_\omega}{dr}\left[\frac{\hbar^2}{2}[J(J+1)-L(L+1)\right.\\&\quad\left.\left.-S(S+1)]\right]\right]+\frac{1}{4\mu c^2}\left[J_\sigma(r)\left((\hbar\omega(2n+l+3/2))-1/2\mu\omega^2r^2\right)\right.\\&\quad-2J_\omega(r)\left((\hbar\omega(2n+l+3/2))-1/2\mu\omega^2r^2\right)+2J_\omega(2S(S+1)-3)\\&\quad\times\left(\frac{2}{\hbar^2}(S.\hat{n})^2-1\right)\left((\hbar\omega(2n+l+3/2))-1/2\mu\omega^2r^2\right)\\&\quad+2\left(\frac{2}{\hbar^2}(S.\hat{n})^2-1\right)\left((\hbar\omega(2n+l+3/2))-1/2\mu\omega^2r^2\right)J_\omega\\&\quad\left.\times(2S(S+1)-3)\right]+\frac{1}{Mc^2}\left[-2(2S(S+1)-3)J_\omega(r)\left(\frac{2}{\hbar^2}(S.\hat{n})^2-1\right)\right.\\&\quad\times\left((\hbar\omega(2N+L+3/2))-(1/2)M\omega^2R^2\right)+\left((\hbar\omega(2N+L+3/2))\right.\\&\quad\left.-(1/2)M\omega^2R^2\right)J_\sigma+\left((\hbar\omega(2N+L+3/2))\right.\\&\quad\left.\left.-(1/2)M\omega^2R^2\right)J_\omega\right]\Big|\sum_{m_{l_\beta}m_{s_\beta}}\sum_{m_{l_\delta}m_{s_\delta}}\sum_{JM}\sum_{\lambda\mu}\sum_{nlNL}\sum_{mM}\sum_{sm_s}\sum_{T}\\&\quad\times\left(l_\beta s_\beta m_{l_\beta}m_{s_\beta}\mid j_\beta M_\beta\right)\left(l_\delta s_\delta m_{l_\delta}m_{s_\delta}\mid j_\delta M_\delta\right)\\&\quad\times\left(l_\alpha l_\gamma m_{l_\alpha}m_{l_\gamma}\mid\lambda\mu\right)\left\langle n_\beta l_\beta n_\delta l_\delta\mid NLnl\right\rangle(lSm_lm_S\mid JM)\\&\quad\times(LlMm\mid\lambda\mu)\left(s_\beta s_\delta m_{s_\beta}m_{s_\delta}\mid SM_s\right)\\&\quad\times\left(\chi_\beta\chi_\delta T_\beta T_\delta\mid M_TT\right)|R_{NLM}(R)Y_{NLM}R_{nlm}(r)Y_{nlm}\rangle|\hat{P}_T(i,j)\rangle.\end{aligned}\tag{B.7}$$

C. The Treatment of Pseudoscalar Meson within Our Potential

We choose pion π as a pseudoscalar meson to be added to the previous two mesons, because it is the one which ties the mesons with the nucleus as it is the fare one. We do not choose another pseudoscalar meson as it demands a reaction between two nucleons and we want to calculate the ground state, so we seek for stability of the nucleus and the exchange of pion meson increases the stability of the nucleus. We have the pseudoscalar potential as

$$\begin{aligned}
&\langle F_\alpha F_\gamma \mid V_{\mathrm{ps}}(r) \mid \tilde{F}_\beta F_\delta \rangle \\
&= \left\langle (\phi_\alpha \quad \chi_\alpha) \mid \left\langle (\phi_\gamma \, \chi_\gamma) \mid \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_j \right.\right. \\
&\quad \left.\left. \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_j J_\pi \mid \begin{pmatrix} \phi_\beta \\ \chi_\beta \end{pmatrix} \rangle \mid \begin{pmatrix} \phi_\delta \\ \chi_\delta \end{pmatrix} \right\rangle \\
&= \left\langle \phi_\alpha \mid \left\langle \phi_\gamma \mid J_\pi \mid \tilde{\chi}_\beta \right\rangle \mid \chi_\delta \right\rangle + \left\langle \phi_\alpha \mid \left\langle \chi_\gamma \mid J_\pi \mid \tilde{-}\phi_\beta \right\rangle \mid \chi_\delta \right\rangle \\
&\quad + \left\langle \chi_\alpha \mid \left\langle \phi_\gamma \mid J_\pi \mid \tilde{\chi}_\beta \right\rangle \mid -\phi_\delta \right\rangle + \left\langle \chi_\alpha \mid \left\langle \chi_\gamma \mid J_\pi \mid \tilde{-}\phi_\beta \right\rangle \mid -\phi_\delta \right\rangle .
\end{aligned} \tag{C.1}$$

Substituting from the previous relations in the treatment of two mesons, we obtain

$$\begin{aligned}
&\langle F_\alpha F_\gamma \mid V_{\mathrm{ps}}(r) \mid \tilde{F}_\beta F_\delta \rangle \\
&= \left\langle (\phi_\alpha \quad \phi_\gamma) \mid \frac{1}{4m^2c^2} \left[J_\pi \left(\vec{\sigma}_j \cdot \vec{p}_j \right) - \left(\vec{\sigma}_j \cdot \vec{p}_j \right) J_\pi \left(\vec{\sigma}_i \cdot \vec{p}_i \right) \right.\right. \\
&\quad \left.\left. - \left(\vec{\sigma}_i \cdot \vec{p}_i \right) J_\pi \left(\vec{\sigma}_j \cdot \vec{p}_j \right) + \left(\vec{\sigma}_i \cdot \vec{p}_i \right) \left(\vec{\sigma}_j \cdot \vec{p}_j \right) J_\pi \right] \mid (\phi_\beta \quad \phi_\delta) \right\rangle \\
&= \left\langle (\phi_i \quad \phi_j) \mid \frac{1}{4m^2c^2} \left[-J_\pi \left(\vec{\sigma}_j \cdot \vec{p} \right) \left(\vec{\sigma}_i \cdot \vec{p} \right) \right.\right. \\
&\quad + \left(\vec{\sigma}_j \cdot \vec{p} \right) J_\pi \left(\vec{\sigma}_i \cdot \vec{p} \right) + \left(\vec{\sigma}_i \cdot \vec{p} \right) J_\pi \left(\vec{\sigma}_j \cdot \vec{p} \right) \\
&\quad \left.\left. - \left(\vec{\sigma}_i \cdot \vec{p} \right) \left(\vec{\sigma}_j \cdot \vec{p} \right) J_\pi \right] \mid (\phi_j \quad \phi_i) \right\rangle .
\end{aligned} \tag{C.2}$$

Using the relation $(\vec{\sigma}_i . \vec{p})(\vec{\sigma}_j . \vec{p}) = -\hbar^2 (\vec{\sigma}_i . \vec{\sigma}_j)(dJ_\pi / dr)(d/dr)$ we obtain,

$$\begin{aligned}
&\langle F_\alpha F_\gamma \mid V_{ps}(r) \mid \tilde{F}_\beta F_\delta \rangle \\
&= \left\langle \phi_\alpha \phi_\gamma \mid \frac{1}{4m^2c^2} \left[-J_\pi \left(2(S.\hat{n})^2 P^2 \right) + J_\pi P^2 \right.\right. \\
&\quad \left.\left. - 2\hbar^2 (2S(S+1) - 3) \frac{dJ_\pi}{dr} \frac{d}{dr} - 2(S.\hat{n})^2 P^2 J_\pi + p^2 J_\pi \right] \mid \tilde{\phi}_\beta \phi_\delta \right\rangle \\
&= \left\langle \phi_\alpha \phi_\gamma \mid \frac{-2\hbar^2 c^2}{4m^2 c^4} (2S(S+1) - 3) \left(\frac{dJ_\pi}{dr} \frac{d}{dr} \right) \right. \\
&\quad - \frac{\hbar\omega}{2mc^2} \left(2n + l + \frac{3}{2} \right) \left(2(S.\hat{n})^2 - 1 \right) J_\pi \\
&\quad \left. - \frac{\hbar^2 \omega^2}{16\hbar^2 c^2} \left(J_\pi r^2 \right) \mid \tilde{\phi}_\beta \phi_\delta \right\rangle .
\end{aligned} \tag{C.3}$$

Data Availability

No data were used to support this study. We studied theoretically, and our results are compared to the published experimental one.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Supplementary Materials

Deuteron potential for the GY function and (set I) parameter Deuteron potential for the GY function and (set II) parameter Deuteron potential for the SPED function and (set I) parameter. *(Supplementary Materials)*

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