

Research Article

Assignments of the $X(4140)$, $X(4500)$, $X(4630)$, and $X(4685)$ Based on the QCD Sum Rules

Zhi-Gang Wang 

Department of Physics, North China Electric Power University, Baoding 071003, China

Correspondence should be addressed to Zhi-Gang Wang; zgwang@aliyun.com

Received 27 October 2021; Accepted 19 November 2021; Published 20 December 2021

Academic Editor: Ruilin Zhu

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In this article, we take into account our previous calculations based on the QCD sum rules, and tentatively assign the $X(4630)$ as the $D_s^* \bar{D}_{s1} - D_{s1} \bar{D}_s^*$ tetraquark molecular state or $[cs]_p [\bar{c}\bar{s}]_A + [cs]_A [\bar{c}\bar{s}]_p$ tetraquark state with the $J^{PC} = 1^{-+}$, and assign the $X(3915)$ and $X(4500)$ as the 1S and 2S $[cs]_A [\bar{c}\bar{s}]_A$ tetraquark states, respectively, with the $J^{PC} = 0^{++}$. Then, we extend our previous works to investigate the LHCb's new tetraquark candidate $X(4685)$ as the first radial excited state of the $X(4140)$ with the QCD sum rules and obtain the mass $M_X = 4.70 \pm 0.12$ GeV, which is in very good agreement with the experimental value $4684 \pm 7_{-16}^{+13}$ MeV. Furthermore, we investigate the two-meson scattering state contributions in details and observe that the two-meson scattering states alone cannot saturate the QCD sum rules, the contributions of the tetraquark states play an unsubstitutable role, and we can saturate the QCD sum rules with or without the two-meson scattering states.

1. Introduction

In 2009, the CDF collaboration observed an evidence for the $X(4140)$ in the $J/\psi\phi$ mass spectrum for the first time with a significance of larger than 3.8σ [1]. Subsequently, the existence of the $X(4140)$ was confirmed by the CDF, CMS, and D0 collaborations [2–5]. In 2016, the LHCb collaboration accomplished the first full amplitude analysis of the $B^+ \rightarrow J/\psi\phi K^+$ decays and acquired a good description of the experimental data in the 6D phase space and confirmed the $X(4140)$ and $X(4274)$ and determined the spin-parity-charge-conjugation $J^{PC} = 1^{++}$ [6, 7]. Furthermore, the LHCb collaboration also observed two new exotic hadrons $X(4500)$ and $X(4700)$ in the $J/\psi\phi$ mass spectrum and determined the quantum numbers $J^{PC} = 0^{++}$ [6, 7]. The Breit-Wigner masses and widths are

$$\begin{aligned} X(4140): M &= 4146.5 \pm 4.5_{-2.8}^{+4.6} \text{ MeV}, \Gamma = 83 \pm 21_{-14}^{+21} \text{ MeV}, \\ X(4274): M &= 4273.3 \pm 8.3_{-3.6}^{+17.2} \text{ MeV}, \Gamma = 56 \pm 11_{-11}^{+8} \text{ MeV}, \\ X(4500): M &= 4506 \pm 11_{-15}^{+12} \text{ MeV}, \Gamma = 92 \pm 21_{-20}^{+21} \text{ MeV}, \\ X(4700): M &= 4704 \pm 10_{-24}^{+14} \text{ MeV}, \Gamma = 120 \pm 31_{-33}^{+42} \text{ MeV}. \end{aligned} \quad (1)$$

Recently, the LHCb collaboration accomplished an improved full amplitude analysis of the $B^+ \rightarrow J/\psi\phi K^+$ decays using 6 times larger signal yields than previously analyzed and observed a hidden-charm and hidden-strange tetraquark candidate $X(4685)$ and ($X(4630)$) in the mass spectrum of the $J/\psi\phi$ with a significance of 15σ (5.5σ), the favored assignment of the spin parity is $J^P = 1^+(1^-)$, and the Breit-Wigner mass and width are $4684 \pm 7_{-16}^{+13}$ MeV ($4626 \pm 16_{-110}^{+18}$ MeV) and $126 \pm 15_{-41}^{+37}$ MeV ($174 \pm 27_{-73}^{+134}$ MeV), respectively [8]. Furthermore, the LHCb collaboration also observed two new tetraquark (molecular) state candidates $Z_{cs}(4000)$ and $Z_{cs}(4220)$ in the mass spectrum of the $J/\psi K^+$ with the preferred spin parity $J^P = 1^+$ and updated the experimental values of the masses and widths of the $X(4500)$ and $X(4700)$ [8]. The $X(4140)$, $X(4274)$, $X(4500)$, $X(4630)$, $X(4685)$, and $X(4700)$ were observed in the mass spectrum of the $J/\psi\phi$, and their quantum numbers are $J^{PC} = 0^{++}$, 1^{++} , and 2^{++} for the S-wave couplings and 0^{-+} , 1^{-+} , 2^{-+} , and 3^{-+} for the P-wave couplings. In the present work, we discuss the possible assignments of the $X(4140)$, $X(4500)$, $X(4630)$, and $X(4685)$ based on the QCD sum rules.

The article is arranged as follows: in Section 2, we discuss the possible assignments of the $X(4630)$ and $X(4500)$ based on the QCD sum rules; in Section 3, we get the QCD sum rules for the masses and pole residues of the tetraquark states $X(4140)/X(4685)$ with the $J^{PC} = 1^{++}$; in Section 4, we obtain numerical results and give discussions, and Section 5 is aimed to get a conclusion.

2. Possible Assignments of the $X(4630)$ and $X(4500)$ Based on the QCD Sum Rules

In this section, we discuss the possible assignments of the $X(4630)$ and $X(4500)$ according to our previous calculations with the QCD sum rules.

In Ref. [9], we construct the color-singlet-color-singlet type four-quark current $J_{\mu\nu}(x)$ to investigate the $D_s^* \bar{D}_{s1} - D_{s1} \bar{D}_s^*$ molecular state:

$$J_{\mu\nu}(x) = \frac{1}{\sqrt{2}} \left[\bar{s}(x) \gamma_\mu c(x) \bar{c}(x) \gamma_\nu \gamma_5 s(x) - \bar{s}(x) \gamma_\nu \gamma_5 c(x) \bar{c}(x) \gamma_\mu s(x) \right]. \quad (2)$$

The current $J_{\mu\nu}(x)$ has definite charge conjugation $C = 1$ but has not definite parity, and the components $J_{0i}(x)$ and $J_{ij}(x)$ have positive parity and negative parity, respectively, where the space indexes $i, j = 1, 2, 3$. The neutral current $J_{\mu\nu}(x)$ couples potentially to the $D_s^* \bar{D}_{s1} - D_{s1} \bar{D}_s^*$ two-meson scattering states or tetraquark molecular states $X_{D_s^* \bar{D}_{s1} - D_{s1} \bar{D}_s^*}$ with the quantum numbers $J^{PC} = 1^{++}$ and 1^{-+} , where we use the symbols D_s^* and D_{s1} to represent the color-neutral clusters with the same quantum numbers as the physical D_s^* and D_{s1} mesons, respectively. In the QCD sum rules, we choose the local currents, and it is better to call the $X_{D_s^* \bar{D}_{s1} - D_{s1} \bar{D}_s^*}$ as the color-singlet-color-singlet type tetraquark state than call it as the tetraquark molecular state. The traditional hidden-flavor mesons, such as the $q\bar{q}$, $c\bar{c}$, and $b\bar{b}$ quarkonia, have the normal quantum numbers $J^{PC} = 0^{-+}, 0^{++}, 1^{--}, 1^{+-}, 1^{++}, 2^{--}, 2^{+-}, 2^{++}, \dots$. The components $J_{0i}(x)$ and $J_{ij}(x)$ couple potentially to the $J^{PC} = 1^{++}$ and 1^{-+} tetraquark molecular states, respectively. We construct projection operators to project out the contribution of the $J^{PC} = 1^{-+}$ component unambiguously and explore the $D_s^* \bar{D}_{s1} - D_{s1} \bar{D}_s^*$ tetraquark molecular state with the exotic quantum numbers $J^{PC} = 1^{-+}$ using the QCD sum rules and acquire the prediction [9],

$$M_X = 4.67 \pm 0.08 \text{ GeV}, \quad (3)$$

which happens to coincide with the mass of the $X(4630)$ from the LHCb collaboration, $M_{X(4630)} = 4626 \pm 16_{-110}^{+18} \text{ MeV}$ [8].

The calculations based on the Bethe-Salpeter equation combined with the heavy meson effective Lagrangian also indicate that there exists such a $D_s^* \bar{D}_{s1} - D_{s1} \bar{D}_s^*$ tetraquark molecular state with the exotic quantum numbers $J^{PC} = 1^{-+}$ [10, 11]. The predictions in Refs. [9–11] were achieved before the discovery of the $X(4630)$. Whether or not the

predictions of the QCD sum rules are reliable, the experimental data can reply. After the discovery of the $X(4630)$ by the LHCb collaboration, Yang et al. study the charmonium-like molecules with hidden-strange via the one-boson exchange mechanism and assign the $X(4630)$ to be the $D_s^* \bar{D}_{s1}$ molecular state with the quantum numbers $J^{PC} = 1^{-+}$ [12].

As long as the diquark-antidiquark type tetraquark states are concerned, we usually take the scalar (S), pseudoscalar (P), vector (V), axialvector (A), and tensor (T) diquark operators without introducing explicit P -waves as the elementary building blocks to construct the interpolating currents. The tensor currents have both vector and axialvector components, and we construct projection operators to project out the spin parity $J^P = 1^-$ and 1^+ components explicitly and denote the corresponding operators as \tilde{V} and \tilde{A} , respectively, to avoid ambiguity. In Ref. [13], we choose the diquark-antidiquark type vector currents $J_\mu^-(x)$ and $J_\mu^+(x)$:

$$\begin{aligned} J_\mu^-(x) &= \frac{\epsilon^{ijk} \epsilon^{imn}}{\sqrt{2}} \left\{ s^j(x) C c^k(x) \bar{s}^m(x) \gamma_\mu C \bar{c}^n(x) - s^j(x) C \gamma_\mu c^k(x) \bar{s}^m(x) C \bar{c}^n(x) \right\}, \\ J_\mu^+(x) &= \frac{\epsilon^{ijk} \epsilon^{imn}}{\sqrt{2}} \left\{ s^j(x) C c^k(x) \bar{s}^m(x) \gamma_\mu C \bar{c}^n(x) + s^j(x) C \gamma_\mu c^k(x) \bar{s}^m(x) C \bar{c}^n(x) \right\}, \end{aligned} \quad (4)$$

to interpolate the $[cs]_P [\bar{c}\bar{s}]_A - [cs]_A [\bar{c}\bar{s}]_P$ -type and $[cs]_P [\bar{c}\bar{s}]_A + [cs]_A [\bar{c}\bar{s}]_P$ -type tetraquark states with the quantum numbers $J^{PC} = 1^{--}$ and 1^{-+} , respectively, and investigate their properties with the QCD sum rules, where the i, j, k, m, n are color indexes. We acquire the predictions:

$$\begin{aligned} M_{X(4630)} &= 4.63_{-0.08}^{+0.11} \text{ GeV for } J^{PC} = 1^{-+}, \\ M_{Y(4660)} &= 4.70_{-0.10}^{+0.14} \text{ GeV for } J^{PC} = 1^{--}, \end{aligned} \quad (5)$$

which happen to coincide with the masses of the $X(4630)$ and $Y(4660)$, respectively [13], and support assigning the $X(4630)$ and $Y(4660)$ to be the tetraquark states with the symbolic quark constituents $c\bar{c}s\bar{s}$ and with the quantum numbers $J^{PC} = 1^{-+}$ and 1^{--} , respectively. The prediction of the mass $4.63_{-0.08}^{+0.11} \text{ GeV}$ was achieved long before the discovery of the $Y(4630)$.

In Refs. [14, 15], we construct the diquark-antidiquark type currents to explore the $[cs]_S [\bar{c}\bar{s}]_S$, $[cs]_P [\bar{c}\bar{s}]_P$, $[cs]_A [\bar{c}\bar{s}]_A$, and $[cs]_V [\bar{c}\bar{s}]_V$ tetraquark states with the quantum numbers $J^{PC} = 0^{++}$ concordantly via the QCD sum rules, and the numerical results support assigning the $X(3915)$ and $X(4500)$ to be the ground state and first radial excited state of the $[cs]_A [\bar{c}\bar{s}]_A$ tetraquark states, respectively, with the quantum numbers $J^{PC} = 0^{++}$ and assigning the $X(4700)$ to be the ground state $[cs]_V [\bar{c}\bar{s}]_V$ tetraquark states with the quantum numbers $J^{PC} = 0^{++}$. Furthermore, we also obtain the potability that assigning the $X(3915)$ to be the ground

TABLE 1: The energy gaps between the ground states and first radial excited states of the hidden-charm tetraquark states with the possible assignments.

J^{PC}	1S	2S	Energy gaps
1^{+-}	$Z_c(3900)$	$Z_c(4430)$	591 MeV
0^{++}	$X(3915)$	$X(4500)$	588 MeV
1^{+-}	$Z_c(4020)$	$Z_c(4600)$	576 MeV
1^{++}	$X(4140)$	$X(4685)$	566 MeV

state $[cs]_S[\bar{c}\bar{s}]_S$ tetraquark state with the $J^{PC} = 0^{++}$ [15]. Our predictions,

$$\begin{aligned} M_{X(4500)} &= 4.50_{-0.09}^{+0.08} \text{ GeV}, \\ M_{X(4700)} &= 4.70_{-0.09}^{+0.08} \text{ GeV}, \end{aligned} \quad (6)$$

are in very good agreement with the LHCb improved measurement $M_{X(4500)} = 4474 \pm 3 \pm 3 \text{ MeV}$ and $M_{X(4700)} = 4694 \pm 4_{-3}^{+16} \text{ MeV}$ [8]. Other assignments of the $X(4500)$ and $X(4700)$, such as the D -wave $cs\bar{c}\bar{s}$ tetraquark states with the $J^P = 0^+$, are also possible [16], and more theoretical and experimental works are still needed to obtain definite conclusion.

In summary, according to the (possible) quantum numbers, decay modes, and energy gaps, we can assign the $X(3915)$ and $X(4500)$ as the ground state and first radial excited state of the hidden-charm tetraquark states with the $J^{PC} = 0^{++}$ [14, 17], assign the $Z_c(3900)$ and $Z_c(4430)$ as the ground state and first radial excited state of the hidden-charm tetraquark states with the $J^{PC} = 1^{+-}$, respectively [18–20], and assign the $Z_c(4020)$ and $Z_c(4600)$ as the ground state and first radial excited state of the hidden-charm tetraquark states with the $J^{PC} = 1^{+-}$, respectively [21, 22]. If we assign the $X(4685)$ to be the first radial excited state of the $X(4140)$ tentatively, we can get the energy gap 566 MeV, and it is reasonable, see Table 1.

Moreover, in Ref. [17], Lebed and Polosa assign the $X(3915)$ and $X(4140)$ to be the $J^{PC} = 0^{++}$ and 1^{++} diquark-antidiquark type hidden-charm tetraquark states $[cs]_S[\bar{c}\bar{s}]_S$ and $[cs]_A[\bar{c}\bar{s}]_S + [cs]_S[\bar{c}\bar{s}]_A$, respectively, based on the effective Hamiltonian with the spin-spin and spin-orbit interactions. In Ref. [23], we construct the $[sc]_{\bar{A}}[\bar{s}\bar{c}]_A + [sc]_A[\bar{s}\bar{c}]_{\bar{A}}$ type and $[sc]_{\bar{V}}[\bar{s}\bar{c}]_V - [sc]_V[\bar{s}\bar{c}]_{\bar{V}}$ type axialvector currents with the quantum numbers $J^{PC} = 1^{++}$ to interpolate the $X(4140)$ and observe that only the $[sc]_{\bar{V}}[\bar{s}\bar{c}]_V - [sc]_V[\bar{s}\bar{c}]_{\bar{V}}$ type current can reproduce the mass and width of the $Y(4140)$ in a consistent way.

3. The $X(4140)/X(4685)$ as the 1S/2S Axialvector Tetraquark States

In this section, we extend our previous work [23] to investigate the $X(4685)$ as the first radial excitation of the $X(4140)$ with the QCD sum rules and discuss the possible assignment

of the $X(4685)$ as the tetraquark state having the quantum numbers $J^{PC} = 1^{++}$.

Firstly, we write down the two-point correlation function $\Pi_{\mu\mu'}(p)$ in the QCD sum rules:

$$\Pi_{\mu\mu'}(p) = i \int d^4x e^{ip \cdot x} \langle 0 | T \{ J_\mu(x) J_{\mu'}^\dagger(0) \} | 0 \rangle, \quad (7)$$

where

$$\begin{aligned} J_\mu(x) &= \frac{\varepsilon^{ijk} \varepsilon^{imn}}{\sqrt{2}} \left[s^{Tj}(x) C \sigma_{\mu\nu} c^k(x) \bar{s}^m(x) \gamma_5 \gamma^\nu C \bar{c}^{Tn}(x) \right. \\ &\quad \left. - s^{Tj}(x) C \gamma^\nu \gamma_5 c^k(x) \bar{s}^m(x) \sigma_{\mu\nu} C \bar{c}^{Tn}(x) \right]. \end{aligned} \quad (8)$$

The current $J_\mu(x)$ couples potentially to the $[sc]_{\bar{V}}[\bar{s}\bar{c}]_V - [sc]_V[\bar{s}\bar{c}]_{\bar{V}}$ tetraquark states with the $J^{PC} = 1^{++}$. The tensor diquark operator $\varepsilon^{ijk} s^{Tj}(x) C \sigma_{\mu\nu} c^k(x)$ has both the spin parity $J^P = 1^+$ and 1^- components, and we project out the 1^- component via multiplying the tensor diquark operator by the vector antidiquark operator $\varepsilon^{imn} \bar{s}^m(x) \gamma_5 \gamma^\nu C \bar{c}^{Tn}(x)$. In Ref. [23], we observe that the current $J_\mu(x)$ can reproduce the mass and width of the $Y(4140)$ satisfactorily.

At the hadron side, we isolate the ground state (X) and first radial excited state (X') contributions, which are supposed to be the pole contributions from the $X(4140)$ and $X(4685)$, respectively,

$$\begin{aligned} \Pi_{\mu\mu'}(p) &= \left(\frac{\lambda_X^2}{M_X^2 - p^2} + \frac{\lambda_{X'}^2}{M_{X'}^2 - p^2} + \dots \right) \left(-g_{\mu\mu'} + \frac{p_\mu p_{\mu'}}{p^2} \right) + \dots \\ &= \Pi(p^2) \left(-g_{\mu\mu'} + \frac{p_\mu p_{\mu'}}{p^2} \right) + \dots, \end{aligned} \quad (9)$$

where the pole residues or decay constants $\lambda_{X^{(\prime)}}$ are defined by $\langle 0 | J_\mu(0) | X^{(\prime)}(p) \rangle = \lambda_{X^{(\prime)}} \varepsilon_\mu$, and the ε_μ is the polarization vectors of the axialvector tetraquark states $X^{(\prime)}$.

A hadron, such as the usually called quark-antiquark type meson, tree-quark type baryon, diquark-antidiquark type tetraquark state, and diquark-diquark-antiquark type pentaquark state, has definite quantum numbers and more than one Fock states. Any current operator with the same quantum numbers and same quark structure as a Fock state in the hadron couples potentially to this hadron; in other words, it has nonvanishing coupling to this hadron. Generally speaking, we can construct several current operators to interpolate a hadron or construct a current operator to interpolate several hadrons. Actually, a hadron has one or two main Fock states, and we call a hadron as a tetraquark state if its main Fock component is of the diquark-antidiquark type.

In the present work, the diquark-antidiquark type local four-quark current operator $J_\mu(x)$ having the quantum numbers $J^{PC} = 1^{++}$ couples potentially to the diquark-antidiquark type tetraquark states with the same quantum numbers $J^{PC} = 1^{++}$. On the other hand, this local current $J_\mu(x)$ can be rearranged into a special superposition of a series of color-singlet-color-singlet type currents through the Fierz transformation both in the Dirac spinor space and color space:

$$\begin{aligned}
2\sqrt{2}J_\mu(x) = & -\bar{c}(x)\sigma_{\mu\nu}\gamma_5 c(x)\bar{s}(x)\gamma^\nu s(x) + \bar{s}(x)\sigma_{\mu\nu}\gamma_5 s(x)\bar{c}(x)\gamma^\nu c(x) \\
& - 3i\bar{c}(x)\gamma_\mu\gamma_5 c(x)\bar{s}(x)s(x) + 3i\bar{s}(x)\gamma_\mu\gamma_5 s(x)\bar{c}(x)c(x) \\
& - \bar{c}(x)\sigma_{\mu\nu}s(x)\bar{s}(x)\gamma^\nu\gamma_5 c(x) + \bar{s}(x)\sigma_{\mu\nu}c(x)\bar{c}(x)\gamma^\nu\gamma_5 s(x) \\
& - 3\bar{c}(x)\gamma_\mu s(x)\bar{s}(x)i\gamma_5 c(x) + 3\bar{s}(x)\gamma_\mu c(x)\bar{c}(x)i\gamma_5 s(x),
\end{aligned} \tag{10}$$

which couple potentially to the tetraquark molecular states or two-meson scattering states having the quantum numbers $J^{PC} = 1^{++}$. The diquark-antidiquark type tetraquark states can be viewed as a special superposition of a series of color-singlet-color-singlet molecular states and embody the net effects, and vice versa.

The diquark-antidiquark type tetraquark state can be plausibly described by two diquarks in a double well potential which are separated by a barrier [24, 25], the spatial distance between the diquark and antidiquark leads to smaller wave-function overlap between the quark and antiquark constituents, and the repulsive barrier or spatial distance frustrates the Fierz rearrangements or recombinations between the quarks and antiquarks and therefore suppresses hadronizing to the meson-meson pairs [24–27].

If the color-singlet-color-singlet type components in Eq. (10), such as $\bar{c}(x)\sigma_{\mu\nu}\gamma_5 c(x)\bar{s}(x)\gamma^\nu s(x)$ and $\bar{s}(x)\sigma_{\mu\nu}\gamma_5 s(x)\bar{c}(x)\gamma^\nu c(x)$, only couple potentially to the two-meson (TM) scattering states, we obtain the correlation function $\Pi_{TM}(p^2)$ at the hadron side:

$$\begin{aligned}
\Pi_{TM}(p^2) = & \frac{1}{768\pi^2} \int_{m_{J/\psi}^2}^{\infty} ds \frac{1}{s-p^2} \frac{\lambda^{1/2}(s, m_{J/\psi}^2, m_\phi^2)}{s} \bar{\rho}_{J/\psi\phi}(s) \\
& + \frac{1}{1536\pi^2} f_\phi^2 m_\phi^2 f_{h_c}^2 \int_{m_{h_c\phi}^2}^{\infty} ds \frac{1}{s-p^2} \frac{\lambda^{1/2}(s, m_{h_c}^2, m_\phi^2)}{s} \bar{\rho}_{h_c\phi}(s) \\
& + \frac{1}{1536\pi^2} f_{J/\psi}^2 m_{J/\psi}^2 f_{h_s}^2 \int_{m_{h_s J/\psi}^2}^{\infty} ds \frac{1}{s-p^2} \frac{\lambda^{1/2}(s, m_{h_s}^2, m_{J/\psi}^2)}{s} \bar{\rho}_{h_s J/\psi}(s) \\
& + \frac{3}{512\pi^2} f_{f_0}^2 m_{f_0}^2 f_{\chi_{c1}}^2 m_{\chi_{c1}}^2 \int_{m_{\chi_{c1} f_0}^2}^{\infty} ds \frac{1}{s-p^2} \frac{\lambda^{1/2}(s, m_{\chi_{c1}}^2, m_{f_0}^2)}{s} \bar{\rho}_{\chi_{c1} f_0}(s) \\
& + \frac{3}{512\pi^2} f_{f_1}^2 m_{f_1}^2 f_{\chi_{c0}}^2 m_{\chi_{c0}}^2 \int_{m_{\chi_{c0} f_1}^2}^{\infty} ds \frac{1}{s-p^2} \frac{\lambda^{1/2}(s, m_{\chi_{c0}}^2, m_{f_1}^2)}{s} \bar{\rho}_{\chi_{c0} f_1}(s) \\
& + \frac{3}{512\pi^2} f_{f_0}^2 m_{f_0}^2 f_{\eta_c}^2 \int_{m_{f_0\eta_c}^2}^{\infty} ds \frac{1}{s-p^2} \frac{\lambda^{3/2}(s, m_{\eta_c}^2, m_{f_0}^2)}{s^2} \\
& + \frac{3}{512\pi^2} f_{\chi_{c0}}^2 m_{\chi_{c0}}^2 f_{\eta}^2 \int_{m_{\chi_{c0}\eta}^2}^{\infty} ds \frac{1}{s-p^2} \frac{\lambda^{3/2}(s, m_{\chi_{c0}}^2, m_{\eta}^2)}{s^2} \\
& + \frac{1}{768\pi^2} f_{D_{s1}}^2 m_{D_{s1}}^2 f_{T,D_s^*}^2 \int_{m_{D_{s1} T,D_s^*}^2}^{\infty} ds \frac{1}{s-p^2} \frac{\lambda^{1/2}(s, m_{D_{s1}}^2, m_{D_s^*}^2)}{s} \bar{\rho}_{D_{s1} D_s^*}(s) \\
& + \frac{1}{192\pi^2} \int_{m_{D_s D_s^*}^2}^{\infty} ds \frac{1}{s-p^2} \frac{\lambda^{1/2}(s, m_{D_s}^2, m_{D_s^*}^2)}{s} \bar{\rho}_{D_s D_s^*}(s) \\
& + \frac{3}{256\pi^2} \frac{f_{D_s}^2 m_{D_s}^4 f_{D_{s0}}^2}{m_c^2} \int_{m_{D_s D_{s0}}^2}^{\infty} ds \frac{1}{s-p^2} \frac{\lambda^{3/2}(s, m_{D_s}^2, m_{D_{s0}}^2)}{s^2} \\
& + (J/\psi\phi \rightarrow \psi'\phi) + (J/\psi\phi \rightarrow \psi''\phi) + (h_c\phi \rightarrow h'_c\phi) + \dots,
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
\bar{\rho}_{J/\psi\phi}(s) = & f_\phi^2 m_\phi^2 f_{T,J/\psi}^2 \left(-s + 8m_{J/\psi}^2 - m_\phi^2 + \frac{(m_{J/\psi}^2 - m_\phi^2)^2}{s} + \frac{(s - m_{J/\psi}^2)^2}{m_\phi^2} \right) \\
& + 2f_{J/\psi} m_{J/\psi} f_\phi m_\phi f_{T,\phi} f_{T,J/\psi} \left(5s - 4m_{J/\psi}^2 - 4m_\phi^2 - \frac{(m_{J/\psi}^2 - m_\phi^2)^2}{s} \right) \\
& + f_{J/\psi}^2 m_{J/\psi}^2 f_{T,\phi}^2 \left(-s + 8m_\phi^2 - m_{J/\psi}^2 + \frac{(m_{J/\psi}^2 - m_\phi^2)^2}{s} + \frac{(s - m_\phi^2)^2}{m_{J/\psi}^2} \right), \\
\bar{\rho}_{h_c\phi}(s) = & -2s - 10m_{h_c}^2 - 2m_\phi^2 + \frac{2m_{h_c}^4 - 3m_{h_c}^2 m_\phi^2}{s} + \frac{2s^2 - 3sm_{h_c}^2}{m_\phi^2} + \frac{m_{h_c}^6}{sm_\phi^2}, \\
\bar{\rho}_{h_s J/\psi}(s) = & -2s - 10m_{h_s}^2 - 2m_{J/\psi}^2 + \frac{2m_{J/\psi}^4 - 3m_{h_s}^2 m_{J/\psi}^2}{s} \\
& + \frac{2s^2 - 3sm_{h_s}^2}{m_{J/\psi}^2} + \frac{m_{h_s}^6}{sm_{J/\psi}^2}, \\
\bar{\rho}_{\chi_{c1} f_0}(s) = & 10 + \frac{m_{\chi_{c1}}^2 - 2m_{f_0}^2}{s} + \frac{s - 2m_{f_0}^2}{m_{\chi_{c1}}^2} + \frac{m_{f_0}^4}{sm_{\chi_{c1}}^2}, \\
\bar{\rho}_{\chi_{c0} f_1}(s) = & 10 + \frac{m_{f_1}^2 - 2m_{\chi_{c0}}^2}{s} + \frac{s - 2m_{\chi_{c0}}^2}{m_{f_1}^2} + \frac{m_{\chi_{c0}}^4}{sm_{f_1}^2}, \\
\bar{\rho}_{D_{s1} D_s^*}(s) = & -2s - 10m_{D_s^*}^2 - 2m_{D_{s1}}^2 + \frac{2m_{D_{s1}}^4 - 3m_{D_s^*}^2 m_{D_{s1}}^2}{s} \\
& + \frac{2s^2 - 3sm_{D_s^*}^2}{m_{D_{s1}}^2} + \frac{m_{D_s^*}^6}{sm_{D_{s1}}^2}, \\
\bar{\rho}_{D_s D_s^*}(s) = & \left(\frac{f_{D_s} f_{T,D_s^*} (s - m_{D_s}^2 - m_{D_s^*}^2)}{4} - \frac{3f_{D_s} m_{D_s}^2 f_{D_s^*} m_{D_s^*}}{2m_c} \right)^2 \\
& \left(10 + \frac{m_{D_s}^2 - 2m_{D_s^*}^2}{s} + \frac{s - 2m_{D_s}^2}{m_{D_s^*}^2} + \frac{m_{D_s}^4}{sm_{D_s^*}^2} \right) \\
& - \left(\frac{f_{D_s}^2 f_{T,D_s^*}^2 (s - m_{D_s}^2 - m_{D_s^*}^2)}{8} - \frac{3f_{D_s}^2 m_{D_s}^2 f_{D_s^*} m_{D_s^*} f_{T,D_s^*}}{4m_c} \right) \\
& \cdot \left(s + m_{D_s}^2 + 2m_{D_s^*}^2 + \frac{3m_{D_s}^2 m_{D_s^*}^2 - m_{D_s}^4 - 3m_{D_s^*}^4}{s} \right) \\
& + \frac{3sm_{D_s}^2 - s^2 - 3m_{D_s}^4}{m_{D_s^*}^2} + \frac{m_{D_s}^6}{sm_{D_s^*}^2} + \frac{f_{D_s}^2 f_{T,D_s^*}^2}{4} \\
& \cdot \left(-s^2 - m_{D_s}^4 + m_{D_s^*}^4 + m_{D_s}^2 m_{D_s^*}^2 + sm_{D_s^*}^2 + \frac{3sm_{D_s^*}^2}{2} \right. \\
& \left. + \frac{m_{D_s}^8}{4sm_{D_s^*}^2} + \frac{6m_{D_s}^2 m_{D_s^*}^4 - 4m_{D_s}^4 m_{D_s^*}^2 + m_{D_s^*}^6 - 4m_{D_s}^6}{4s} \right. \\
& \left. + \frac{s^3 - 4s^2 m_{D_s}^2 + 6sm_{D_s}^4 - 4m_{D_s}^6}{4m_{D_s^*}^2} \right),
\end{aligned} \tag{12}$$

$$\begin{aligned}
\lambda(a, b, c) = & a^2 + b^2 + c^2 - 2ab - 2ac - 2bc, m_{J/\psi\phi} = m_{J/\psi} \\
& + m_\phi, m_{h_c\phi} = m_{h_c} + m_\phi, m_{h_s J/\psi} = m_{h_s} + m_{J/\psi}, m_{f_0\chi_{c1}} = m_{f_0} + \\
& m_{\chi_{c1}}, m_{f_1\chi_{c0}} = m_{f_1} + m_{\chi_{c0}}, m_{f_0\eta_c} = m_{f_0} + m_{\eta_c}, m_{\chi_{c0}\eta} = m_{\chi_{c0}} +
\end{aligned}$$

m_η , $m_{D_{s1}D_s^*} = m_{D_{s1}} + m_{D_s^*}$, $m_{D_sD_s^*} = m_{D_s} + m_{D_s^*}$, $m_{D_sD_{s0}} = m_{D_s} + m_{D_{s0}}$, and we have taken the standard definitions of the decay constants:

$$\begin{aligned}
\langle 0 | \bar{c}(0) \sigma_{\mu\nu} c(0) | J/\psi(p) \rangle &= if_{J/\psi}^T (\varepsilon_\mu p_\nu - \varepsilon_\nu p_\mu), \\
\langle 0 | \bar{s}(0) \sigma_{\mu\nu} s(0) | \phi(p) \rangle &= if_\phi^T (\varepsilon_\mu p_\nu - \varepsilon_\nu p_\mu), \\
\langle 0 | \bar{c}(0) \sigma_{\mu\nu} s(0) | D_s^*(p) \rangle &= if_{D_s^*}^T (\varepsilon_\mu p_\nu - \varepsilon_\nu p_\mu), \\
\langle 0 | \bar{c}(0) \sigma_{\mu\nu} c(0) | h_c(p) \rangle &= if_{h_c} \varepsilon_{\mu\nu\alpha\beta} \varepsilon^\alpha p^\beta, \\
\langle 0 | \bar{s}(0) \sigma_{\mu\nu} s(0) | h_s(p) \rangle &= if_{h_s} \varepsilon_{\mu\nu\alpha\beta} \varepsilon^\alpha p^\beta, \\
\langle 0 | \bar{c}(0) \gamma_\mu c(0) | J/\psi(p) \rangle &= f_{J/\psi} m_{J/\psi} \varepsilon_\mu, \\
\langle 0 | \bar{s}(0) \gamma_\mu s(0) | \phi(p) \rangle &= f_\phi m_\phi \varepsilon_\mu, \\
\langle 0 | \bar{c}(0) \gamma_\mu s(0) | D_s^*(p) \rangle &= f_{D_s^*} m_{D_s^*} \varepsilon_\mu, \\
\langle 0 | \bar{c}(0) \gamma_\mu \gamma_5 c(0) | \chi_{c1}(p) \rangle &= f_{\chi_{c1}} m_{\chi_{c1}} \varepsilon_\mu, \\
\langle 0 | \bar{s}(0) \gamma_\mu \gamma_5 s(0) | f_1(p) \rangle &= f_{f_1} m_{f_1} \varepsilon_\mu, \\
\langle 0 | \bar{c}(0) \gamma_\mu \gamma_5 s(0) | D_{s1}(p) \rangle &= f_{D_{s1}} m_{D_{s1}} \varepsilon_\mu, \\
\langle 0 | \bar{c}(0) \gamma_\mu \gamma_5 c(0) | \eta_c(p) \rangle &= if_{\eta_c} p_\mu, \\
\langle 0 | \bar{s}(0) \gamma_\mu \gamma_5 s(0) | \eta(p) \rangle &= if_\eta p_\mu, \\
\langle 0 | \bar{c}(0) \gamma_\mu \gamma_5 s(0) | D_s(p) \rangle &= if_{D_s} p_\mu, \\
\langle 0 | \bar{c}(0) c(0) | \chi_{c0}(p) \rangle &= f_{\chi_{c0}} m_{\chi_{c0}}, \\
\langle 0 | \bar{s}(0) s(0) | f_0(p) \rangle &= f_{f_0} m_{f_0}, \\
\langle 0 | \bar{c}(0) i\gamma_5 s(0) | D_s(p) \rangle &= \frac{f_{D_s} m_{D_s}^2}{m_c}, \\
\langle 0 | \bar{c}(0) \gamma_\mu s(0) | D_{s0}(p) \rangle &= f_{D_{s0}} p_\mu.
\end{aligned} \tag{13}$$

The ε_μ is the polarization vectors of the vector and axialvector mesons.

We accomplish the operator product expansion for the correlation function $\Pi_{\mu\mu'}(p)$ up to the vacuum condensates of dimension 10 consistently [28–30]. In calculations, we assume dominance of the intermediate vacuum state tacitly, just like in previous works [28–30], and insert the intermediate vacuum state alone in all the channels, and vacuum saturation works well in the large N_c limit [31]. Up to now, almost in all the QCD sum rules for the multiquark states, vacuum saturation is assumed for the higher dimensional vacuum condensates, except in some cases, the parameter $\rho > 1$

$$\langle 0 | : \bar{q}_\alpha^i q_\beta^j \bar{q}_\lambda^m q_\tau^n : | 0 \rangle = \frac{\rho}{16N_c^2} \langle \bar{q}q \rangle^2 (\delta_{ij} \delta_{mn} \delta_{\alpha\beta} \delta_{\lambda\tau} - \delta_{in} \delta_{jm} \delta_{\alpha\tau} \delta_{\beta\lambda}), \tag{14}$$

which parameterizes deviations from the factorization hypothesis, is introduced by hand for the sake of fine-tuning [32], where $q = u, d, s$, the i, j, m , and n are color indexes, and the α, β, λ and τ are Dirac spinor indexes.

In the original works, Shifman, Vainshtein, and Zakharov took the factorization hypothesis based on two reasons [33, 34]. The first one is the rather large value of the quark condensate $\langle \bar{q}q \rangle$, the second one is the duality between the quark and physical states, and they reproduce each other, counting both the quark and physical states (beyond the vacuum states) that maybe lead to a double counting [33, 34].

In the QCD sum rules for the $q\bar{q}$, $q\bar{Q}$, and $Q\bar{Q}$ mesons, the $\langle \bar{q}q \rangle^2$ are always accompanied with the fine-structure constant $\alpha_s = g_s^2/4\pi$ and play a minor important (or tiny) role, and the deviation from $q = 1$, for example, $q = 2 \sim 3$, cannot make much difference in the numerical predictions; though in some cases, the values $q > 1$ can lead to better QCD sum rules [35, 36]. However, in the QCD sum rules for the multiquark states, the $\langle \bar{q}q \rangle^2$ play an important role, large values, for example, if we take the value $q = 2$ in the present case, we can obtain the uncertainties $\delta M_{X'} = +0.08$ GeV and $\delta M_{X'} = +0.09$ GeV, which are of the same order of the total uncertainties from other parameters. Sometimes, large values of the ρ can destroy the platforms in the QCD sum rules for the multiquark states [37].

The true values of the higher dimensional vacuum condensates remain unknown or poorly known; if the true values $q > 1$ or $\gg 1$, the QCD sum rules for the multiquark states have considerably large systematic uncertainties and are less reliable than those of the conventional mesons and baryons [38]. We just make predictions for the multiquark masses with the QCD sum rules based on vacuum saturation and then confront them to the experimental data in the future to examine the theoretical calculations.

After the analytical expression of the QCD spectral density was acquired, we take the quark-hadron duality below the continuum thresholds s_0 and s'_0 by including the contributions of the 1S state and 1S plus 2S states, respectively, and perform Borel transform in regard to the variable $P^2 = -p^2$ to obtain the two QCD sum rules:

$$\begin{aligned}
\lambda_X^2 \exp\left(-\frac{M_X^2}{T^2}\right) &= \int_{4m_c^2}^{s_0} ds \rho(s) \exp\left(-\frac{s}{T^2}\right) = \Pi(\tau), \\
\lambda_X^2 \exp\left(-\frac{M_X^2}{T^2}\right) + \lambda_{X'}^2 \exp\left(-\frac{M_{X'}^2}{T^2}\right) &= \int_{4m_c^2}^{s'_0} ds \rho(s) \exp\left(-\frac{s}{T^2}\right) = \Pi'(\tau),
\end{aligned} \tag{15}$$

TABLE 2: The Borel windows, continuum threshold parameters, and ideal energy scales of the spectral densities, pole contributions, masses and pole residues for the axialvector tetraquark states.

	$T^2(\text{GeV}^2)$	$\sqrt{s_0}(\text{GeV})$	$\mu(\text{GeV})$	Pole	$M(\text{GeV})$	$\lambda(\text{GeV}^5)$
$X(4140)$	2.7 – 3.3	4.7 ± 0.1	2.0	(41 – 69)%	4.14 ± 0.10	$(4.30 \pm 0.85) \times 10^{-2}$
$X(4685)$	2.7 – 3.3	5.1 ± 0.1	3.0	(69 – 90)%	4.70 ± 0.12	$(1.08 \pm 0.17) \times 10^{-1}$

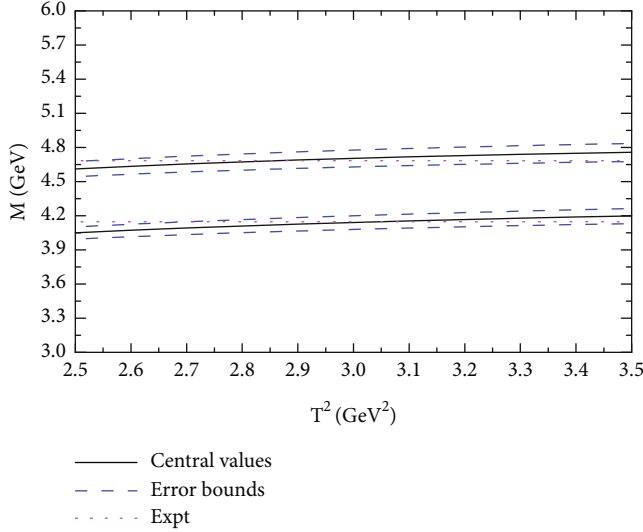


FIGURE 1: The masses of the X and X' with variations of the Borel parameter T^2 , where the expt stands for the experimental values of the masses of the $X(4140)$ and $X(4685)$, respectively.

where $\tau = 1/T^2$. For the explicit expression of the spectral density $\rho(s)$ at the quark and gluon level, one can consult Ref. [23]. We define $D^n = (-d/d\tau)^n$ with $n = 0, 1, 2, \dots$ and then acquire the QCD sum rules for the masses:

$$M_X^2 = \frac{D\Pi(\tau)}{\Pi(\tau)}, \quad (16)$$

$$M_{X'}^2 = \frac{b + \sqrt{b^2 - 4c}}{2}, \quad (17)$$

$$\lambda_{X'}^2 = \frac{(D - M_X^2)\Pi_{\text{QCD}}(\tau)}{M_{X'}^2 - M_X^2} \exp(\tau M_{X'}^2), \quad (18)$$

where

$$\begin{aligned} b &= \frac{D^3 \otimes D^0 - D^2 \otimes D}{D^2 \otimes D^0 - D \otimes D}, \\ c &= \frac{D^3 \otimes D - D^2 \otimes D^2}{D^2 \otimes D^0 - D \otimes D}, \\ D^j \otimes D^k &= D^j \Pi'(\tau) D^k \Pi'(\tau), \end{aligned} \quad (19)$$

the indexes $i = 1, 2$ and $j, k = 0, 1, 2, 3$. For the technical details in obtaining the QCD sum rules in Eqs. (17) and (18), one can consult Refs. [14, 20, 39].

On the other hand, if we saturate the hadron side of the QCD sum rules with the contributions of the two-meson scattering states, we obtain the following two QCD sum rules:

$$\begin{aligned} \Pi_{TM}(T^2) &= \frac{1}{768\pi^2} \int_{m_{J/\psi}^2}^{s_0'} ds \frac{\lambda^{1/2}(s, m_{J/\psi}^2, m_\phi^2)}{s} \bar{\rho}_{J/\psi\phi}(s) \exp\left(-\frac{s}{T^2}\right) \\ &+ \frac{1}{1536\pi^2} f_\phi^2 m_\phi^2 f_{h_c}^2 \int_{m_{h_c\phi}^2}^{s_0'} ds \frac{\lambda^{1/2}(s, m_{h_c}^2, m_\phi^2)}{s} \bar{\rho}_{h_c\phi}(s) \exp\left(-\frac{s}{T^2}\right) \\ &+ \frac{1}{1536\pi^2} f_{J/\psi}^2 m_{J/\psi}^2 f_{h_s}^2 \int_{m_{h_s J/\psi}^2}^{s_0'} ds \frac{\lambda^{1/2}(s, m_{h_s}^2, m_{J/\psi}^2)}{s} \bar{\rho}_{h_s J/\psi}(s) \exp\left(-\frac{s}{T^2}\right) \\ &\cdot \left(-\frac{s}{T^2}\right) + \frac{3}{512\pi^2} f_{\chi_{c1}}^2 m_{\chi_{c1}}^2 f_{\chi_{c1}}^2 \int_{m_{\chi_{c1}}^2}^{s_0'} ds \frac{\lambda^{1/2}(s, m_{\chi_{c1}}^2, m_{f_0}^2)}{s} \bar{\rho}_{\chi_{c1} f_0}(s) \exp\left(-\frac{s}{T^2}\right) \\ &\cdot \left(-\frac{s}{T^2}\right) + \frac{3}{512\pi^2} f_{\chi_{c0}}^2 m_{\chi_{c0}}^2 f_{\chi_{c0}}^2 \int_{m_{\chi_{c0}}^2}^{s_0'} ds \frac{\lambda^{1/2}(s, m_{\chi_{c0}}^2, m_{f_1}^2)}{s} \bar{\rho}_{\chi_{c0} f_1}(s) \exp\left(-\frac{s}{T^2}\right) \\ &\cdot \left(-\frac{s}{T^2}\right) + \frac{3}{512\pi^2} f_{\eta_c}^2 m_{\eta_c}^2 f_{\eta_c}^2 \int_{m_{\eta_c}^2}^{s_0'} ds \frac{\lambda^{3/2}(s, m_{\eta_c}^2, m_{f_0}^2)}{s^2} \exp\left(-\frac{s}{T^2}\right) \\ &+ \frac{3}{512\pi^2} f_{\chi_{c0}}^2 m_{\chi_{c0}}^2 f_{\eta_c}^2 \int_{m_{\chi_{c0}\eta_c}^2}^{s_0'} ds \frac{\lambda^{3/2}(s, m_{\chi_{c0}}^2, m_{\eta_c}^2)}{s^2} \exp\left(-\frac{s}{T^2}\right) \\ &+ \frac{1}{768\pi^2} f_{D_{s1}}^2 m_{D_{s1}}^2 f_{T, D_s'}^2 \int_{m_{D_{s1} D_s'}^2}^{s_0'} ds \frac{\lambda^{1/2}(s, m_{D_{s1}}^2, m_{D_s'}^2)}{s} \bar{\rho}_{D_{s1} D_s'}(s) \exp\left(-\frac{s}{T^2}\right) \\ &\cdot \left(-\frac{s}{T^2}\right) + \frac{1}{192\pi^2} \int_{m_{D_s D_s'}^2}^{s_0'} ds \frac{\lambda^{1/2}(s, m_{D_s}^2, m_{D_s'}^2)}{s} \bar{\rho}_{D_s D_s'}(s) \exp\left(-\frac{s}{T^2}\right) \\ &+ \frac{3}{256\pi^2} \frac{f_{D_s}^2 m_{D_s}^4 f_{D_{s0}}^2}{m_c^2} \int_{m_{D_s D_{s0}}^2}^{s_0'} ds \frac{\lambda^{3/2}(s, m_{D_s}^2, m_{D_{s0}}^2)}{s^2} \exp\left(-\frac{s}{T^2}\right) \\ &+ (J/\psi\phi \rightarrow \psi'\phi) + (J/\psi\phi \rightarrow \psi''\phi) + (h_c\phi \rightarrow h'_c\phi) + \dots = \kappa \Pi'(T^2), \end{aligned} \quad (20)$$

$$\frac{d}{d(1/T^2)} \Pi_{TM}(T^2) = \kappa \frac{d}{d(1/T^2)} \Pi'(T^2). \quad (21)$$

In Eqs. (20) and (21), we introduce the parameter κ to measure the deviations from 1; if $\kappa \approx 1$, we can acquire the conclusion tentatively that the two-meson scattering states can saturate the QCD sum rules.

4. Numerical Results and Discussions

At the QCD side, we choose the conventional values of the vacuum condensates $\langle \bar{q}q \rangle = -(0.24 \pm 0.01 \text{ GeV})^3$, $\langle \bar{s}s \rangle = (0.8 \pm 0.1) \langle \bar{q}q \rangle$, $\langle \bar{s}g_s \sigma Gs \rangle = m_0^2 \langle \bar{s}s \rangle$, $m_0^2 = (0.8 \pm 0.1) \text{ GeV}^2$, and $\langle \alpha_s GG/\pi \rangle = (0.33 \text{ GeV})^4$ at the energy scale $\mu = 1 \text{ GeV}$ [33, 34, 40, 41] and prefer the modified minimal subtracted masses $m_c(m_c) = (1.275 \pm 0.025) \text{ GeV}$ and $m_s(\mu = 2 \text{ GeV}) = (0.095 \pm 0.005) \text{ GeV}$ from the Particle Data Group [42]. In

addition, we consider the energy-scale dependence of the input parameters:

$$\begin{aligned}\langle\bar{s}s\rangle(\mu) &= \langle\bar{s}s\rangle(1\text{GeV}) \left[\frac{\alpha_s(1\text{GeV})}{\alpha_s(\mu)} \right]^{\frac{12}{33-2n_f}}, \\ \langle\bar{s}g_s\sigma Gs\rangle(\mu) &= \langle\bar{s}g_s\sigma Gs\rangle(1\text{GeV}) \left[\frac{\alpha_s(1\text{GeV})}{\alpha_s(\mu)} \right]^{\frac{2}{33-2n_f}}, \\ m_c(\mu) &= m_c(m_c) \left[\frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right]^{\frac{12}{33-2n_f}}, \\ m_s(\mu) &= m_s(2\text{GeV}) \left[\frac{\alpha_s(\mu)}{\alpha_s(2\text{GeV})} \right]^{\frac{12}{33-2n_f}}, \\ \alpha_s(\mu) &= \frac{1}{b_0 t} \left[1 - \frac{b_1}{b_0^2} \frac{\log t}{t} + \frac{b_1^2(\log^2 t - \log t - 1) + b_0 b_2}{b_0^4 t^2} \right],\end{aligned}\quad (22)$$

where $t = \log(\mu^2/\Lambda^2)$, $b_0 = 33 - 2n_f/12\pi$, $b_1 = 153 - 19n_f/24\pi^2$, $b_2 = 2857 - 5033/9n_f + 325/27n_f^2/128\pi^3$, $\Lambda = 210\text{ MeV}$, 292 MeV , and 332 MeV for the flavors $n_f = 5, 4$, and 3 , respectively [42, 43], and evolve all the input parameters to the typical energy scales μ with the flavor number $n_f = 4$, which satisfy the energy scale formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2M_c)^2}$ with the updated value of the effective charmed quark mass $M_c = 1.82\text{ GeV}$ [44, 45], to extract the masses of the hidden-charm tetraquark states. We tentatively assign the $X(4140)$ and $X(4685)$ to be the ground state and first radial excited state of the hidden-charm tetraquark states, respectively, and the corresponding pertinent energy scales of the spectral densities at the quark-gluon level are $\mu = 2.0\text{ GeV}$ and 3.0 GeV , respectively.

In Ref. [23], we obtain the Borel window $T^2 = 2.7 - 3.3\text{ GeV}^2$, continuum threshold parameter $\sqrt{s_0} = 4.7 \pm 0.1\text{ GeV}$, and pole contribution (41–69)% for the $X(4140)$, then acquire the mass and pole residue $M_X = 4.14 \pm 0.10\text{ GeV}$ and $\lambda_X = (4.30 \pm 0.85) \times 10^{-2}\text{ GeV}^5$, which are all shown plainly in Table 2. In the present work, we choose the same Borel parameter $T^2 = 2.7 - 3.3\text{ GeV}^2$ as in Ref. [23] and assume that the energy gap between the first and second radial excited states is about $0.3 \sim 0.5\text{ GeV}$, take the continuum threshold parameter as $\sqrt{s_0'} = 5.1 \pm 0.1\text{ GeV}$, and then obtain the pole contribution (69–90%), and it is large enough to extract the mass of the first radial excited state. Moreover, the convergent behaviors of the operator product expansion are very good, and the contributions from the vacuum condensates of dimension 10 in the QCD sum rules for the $X(4140)$ and $X(4685)$ are $<1\%$ and $\ll 1\%$, respectively.

Finally, we take into account all the uncertainties of the input parameters and get the mass and pole residue of the first radial excited state X' , which are shown plainly in Table 2 and Figure 1. From the table, we observe that the

predicted mass $M_{X'} = 4.70 \pm 0.12\text{ GeV}$ is in very good agreement with the experimental value $M_{X(4685)} = 4684 \pm 7_{-16}^{+13}\text{ MeV}$ from the LHCb collaboration [8].

In Figure 1, we plot the predicted masses M_X and $M_{X'}$ with variations of the Borel parameter T^2 ; from the figure, we can see clearly that there appear rather flat platforms both for the ground state and first radial excited state, and we are confident to obtain reliable predictions. In addition, we present the experimental values of the masses of the $X(4140)$ and $X(4685)$, which happen to lie in the center regions of the predicted values.

If the masses of the ground state $X(4140)$, first radial excited state $X(4685)$, second radial excited state X'' , etc, satisfy the Regge trajectory,

$$M_n^2 = \alpha(n-1) + \alpha_0, \quad (23)$$

where the α and α_0 are some constants to be fitted experimentally, and the n is the radial quantum number. We take the masses of the ground state and first radial excited state, $M_{X(4140)} = 4118\text{ MeV}$ and $M_{X(4685)} = 4684\text{ MeV}$ [8], as input parameters to fit the parameters α and α_0 , and obtain the mass of the second radial excited state, $M_{X''} = 5.19 \pm 0.10\text{ GeV}$, which is consistent with the continuum threshold parameter $\sqrt{s_0'} = 5.1 \pm 0.1\text{ GeV}$, and the contamination from the second radial excited state is avoided; here, we add an uncertainty $\delta = \pm 0.1\text{ GeV}$ to the mass $M_{X'}$, according to Table 2. Now, we reach the conclusion tentatively that the calculations are self-consistent.

The values $M_{X'} = 5.09\text{ GeV}$, 5.19 GeV and 5.29 GeV correspond to the continuum threshold parameters $\sqrt{s_0'} = 5.0\text{ GeV}$, 5.1 GeV , and 5.2 GeV , respectively, and have the relation $M_{X'} > \sqrt{s_0'}$ and the contamination from the second radial excited state can be neglected. At the beginning, we assume that the energy gap between the first and second radial excited states is about $0.3 \sim 0.5\text{ GeV}$ and tentatively take the continuum threshold parameter as $\sqrt{s_0'} = 5.1 \pm 0.1\text{ GeV}$ to obtain the mass of the X' , then resort to the Regge trajectory to check whether or not such a choice is self-consistent. Fortunately, such a choice happens to be satisfactory. On the other hand, if it is not self-consistent, we can choose another value of the $\sqrt{s_0'}$, then repeat the same routine to obtain self-consistent $M_{X'}$, $M_{X''}$, and $\sqrt{s_0'}$ via trial and error. In all the calculations, we should obtain flat Borel platforms to suppress the dependence on the Borel parameters.

In Refs. [9, 13, 14, 23], we investigate the hidden-charm tetraquark (molecular) states via the QCD sum rules using the energy scale formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2M_c)^2}$ to choose the pertinent energy scales of the spectral densities at the quark and gluon level, which can enhance the pole contributions remarkably and improve the convergent behaviors of

TABLE 3: The possible assignments of the LHCb's X states based on the predictions from the QCD sum rules, where the superscript * denotes the mass of the $X(4140)$ taken from Ref. [23].

	J^{PC}	1S [mass (GeV)]	2S [mass (GeV)]	References
$D_s^* \bar{D}_{s1} - D_{s1} \bar{D}_s^*$	1^{-+}	$X(4630) [4.67 \pm 0.08]$		[9]
$[cs]_p [\bar{c} \bar{s}]_A + [cs]_A [\bar{c} \bar{s}]_p$	1^{-+}	$X(4630) [4.63^{+0.11}_{-0.08}]$		[13]
$[cs]_A [\bar{c} \bar{s}]_A$	0^{++}	$X(3915) [3.92^{+0.19}_{-0.18}]$	$X(4500) [4.50^{+0.08}_{-0.09}]$	[14]
$[sc]_{\bar{V}} [\bar{s} \bar{c}]_{\bar{V}} - [sc]_{\bar{V}} [\bar{s} \bar{c}]_{\bar{V}}$	1^{++}	$X(4140) [4.14 \pm 0.10]$	$X(4685) [4.70 \pm 0.12]$	[23]*

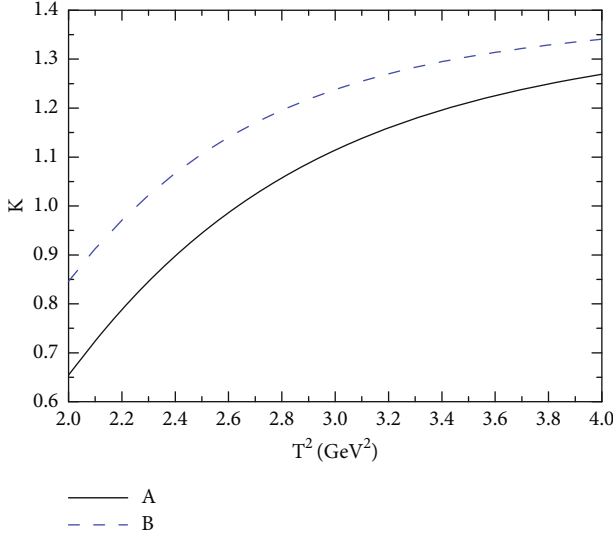


FIGURE 2: The values of the κ with variations of the Borel parameter T^2 , where the A and B come from the QCD sum rules in Eqs. (20) and (21), respectively.

the operator product expansion remarkably. It is a unique feature of our works. The predictions $M_{D_s^* \bar{D}_{s1} - D_{s1} \bar{D}_s^*} = 4.67 \pm 0.08$ GeV [9], $M_{[cs]_p [\bar{c} \bar{s}]_A + [cs]_A [\bar{c} \bar{s}]_p} = 4.63^{+0.11}_{-0.08}$ GeV [13], $M_{[cs]_A [\bar{c} \bar{s}]_A, 1S} = 3.92^{+0.19}_{-0.18}$ GeV, $M_{[cs]_A [\bar{c} \bar{s}]_A, 2S} = 4.50^{+0.08}_{-0.09}$ GeV [14], $M_{[sc]_{\bar{V}} [\bar{s} \bar{c}]_{\bar{V}} - [sc]_{\bar{V}} [\bar{s} \bar{c}]_{\bar{V}}, 1S} = 4.14 \pm 0.10$ GeV [23], and $M_{[sc]_{\bar{V}} [\bar{s} \bar{c}]_{\bar{V}} - [sc]_{\bar{V}} [\bar{s} \bar{c}]_{\bar{V}}, 2S} = 4.70 \pm 0.12$ GeV based on the QCD sum rules support assigning the $X(4630)$ as the $D_s^* \bar{D}_{s1} - D_{s1} \bar{D}_s^*$ tetraquark molecular state [9] or $[cs]_p [\bar{c} \bar{s}]_A + [cs]_A [\bar{c} \bar{s}]_p$ tetraquark state with the quantum numbers $J^{PC} = 1^{-+}$ [13], assigning the $X(3915)$ and $X(4500)$ as the 1S and 2S $[cs]_A [\bar{c} \bar{s}]_A$ tetraquark states, respectively, with the quantum numbers $J^{PC} = 0^{++}$, and assigning the $X(4140)$ and $X(4685)$ as the 1S and 2S $[sc]_{\bar{V}} [\bar{s} \bar{c}]_{\bar{V}} - [sc]_{\bar{V}} [\bar{s} \bar{c}]_{\bar{V}}$ tetraquark states, respectively, with the quantum numbers $J^{PC} = 1^{++}$ [23]. The predictions with the possible assignments are given plainly in Table 3. We should bear in mind that other assignments of the $X(4500)$ and $X(4700)$, such as the D -wave $c\bar{s}\bar{c}$ tetraquark states with the $J^P = 0^+$, are also possible [16], and more theoretical and experimental works are still needed to obtain definite conclusion.

Now, we explore the outcome in the case of saturating the hadron side of the QCD sum rules with the two-meson scattering states. At the hadron side of the QCD sum rules in Eqs. (20) and (21), we choose the parameters $m_{J/\psi} =$

3.0969 GeV, $m_{\eta_c} = 2.9839$ GeV, $m_{h_c} = 3.52538$ GeV, $m_{\chi_{c0}} = 3.41471$ GeV, $m_{\chi_{c1}} = 3.51067$ GeV, $m_{\phi} = 1.019461$ GeV, $m_{h_1} = 1.416$ GeV, $m_{f_1} = 1.4263$ GeV, $m_{f_0} = 1.506$ GeV, $m_{\eta} = 0.547862$ GeV, $m_{D_s} = 1.969$ GeV, $m_{D_s^*} = 2.1122$ GeV, $m_{D_{s0}} = 2.318$ GeV, $m_{D_{s1}} = 2.4596$ GeV, $m_{\psi'} = 3.6861$ GeV, and $m_{\psi''} = 4.0396$ GeV from the Particle Data Group [42]; $m_{h'_c} = 3.9560$ GeV from the Godfrey-Isgur model [46]; $f_{J/\psi} = 0.418$ GeV, $f_{\eta_c} = 0.387$ GeV, $f_{J/\psi}^T = 0.410$ GeV, and $f_{h_c} = 0.235$ GeV from the lattice QCD [47]; $f_{\chi_{c1}} = 0.338$ GeV, $f_{\chi_{c0}} = 0.359$ GeV [48], $f_{\phi} = 0.231$ GeV, $f_{\phi}^T = 0.200$ GeV [49, 50], $f_{\eta} = 1.34 f_{\pi}$ [51], $f_{h_1} = 0.183$ GeV, $f_{f_1} = 0.211$ GeV [52], $f_{f_0} = 0.490$ GeV [53], $f_{D_s} = 0.240$ GeV, $f_{D_s^*} = 0.308$ GeV, $f_{D_{s0}} = 0.333$ GeV ($m_c/m_{D_{s0}}$), and $f_{D_{s1}} = 0.345$ GeV [54] from the QCD sum rules; $f_{\pi} = 0.130$ GeV, $f_{\psi'} = 0.295$ GeV, and $f_{\psi''} = 0.187$ GeV extracted from the experimental data [42]; and $f_{D_s^*}^T = f_{D_s^*}$, $f_{\psi'}^T = f_{\psi'}$, $f_{\psi''}^T = f_{\psi''}$, and $f_{h'_c} = f_{h'_c} (f_{\psi'}/f_{J/\psi}) = 0.166$ GeV estimated in the present work (also in Ref. [27]).

In Figure 2, we plot the values of the κ with variations of the Borel parameter T^2 for the central values of the input parameters. From Figure 2, we can see that the values of the κ increase monotonically and quickly with the increase of the Borel parameter T^2 , no platform appears, which indicates that the QCD sum rules in Eqs. (20) and (21) are not satisfactory, and the two-meson scattering states alone cannot saturate the QCD sum rules at the hadron side.

In Ref. [27], we investigate the $Z_c(3900)$ with the QCD sum rules in details by including all the two-meson scattering state contributions and nonlocal effects between the diquark and antidiquark constituents. We observe that the two-meson scattering states alone cannot saturate the QCD sum rules at the hadron side, and just like in the present case, the contribution of the $Z_c(3900)$ (or pole term) plays an unsubstitutable role, and we can saturate the QCD sum rules with or without the two-meson scattering state contributions. We expect the conclusion is also applicable in the present case.

Now, we explore the two-meson scattering state contributions; besides, the tetraquark states X_c and X'_c and take account of all the contributions:

$$\Pi_{\mu\nu}(p) = -\frac{\hat{\lambda}_X^2}{p^2 - \hat{M}_X^2 + \Sigma_{J/\psi\phi}(p^2) + \dots} g_{\mu\nu} - \frac{\hat{\lambda}_{X'}^2}{p^2 - \hat{M}_{X'}^2 + \Sigma_{J/\psi\phi}(p^2) + \dots} g_{\mu\nu} + \dots \quad (24)$$

We choose the bare masses and pole residues $\widehat{M}_X, \widehat{M}_{X'}, \widehat{\lambda}_X,$ and $\widehat{\lambda}_{X'}$ to absorb the divergent terms in the self-energies $\Sigma_{J/\psi\phi}(p^2), \dots$. The renormalized self-energies satisfy the relations $p^2 - M_{X,R}^2 + \bar{\Sigma}_{J/\psi\phi}(p^2) + \dots = 0$ and $p^2 - M_{X',R}^2 + \bar{\Sigma}_{J/\psi\phi}(p^2) + \dots = 0$, where the subscripts R represent the \overline{MS} masses, and the overlines above the self-energies represent that the divergent terms have been subtracted. The tetraquark states X_c and X'_c have finite widths and are unstable particles, and the relations should be modified, $p^2 - M_{X,R}^2 + \text{Re}\bar{\Sigma}_{J/\psi\phi}(p^2) + \dots = 0$, $p^2 - M_{X',R}^2 + \text{Re}\bar{\Sigma}_{J/\psi\phi}(p^2) + \dots = 0$, $\text{Im}\bar{\Sigma}_{J/\psi\phi}(p^2) + \dots = \sqrt{p^2}\Gamma_X(p^2)$, and $\text{Im}\bar{\Sigma}_{J/\psi\phi}(p^2) + \dots = \sqrt{p^2}\Gamma_{X'}(p^2)$. The renormalized self-energies contribute a finite imaginary part to modify the dispersion relation:

$$\begin{aligned} \Pi_{\mu\nu}(p) = & -\frac{\lambda_X^2}{p^2 - M_X^2 + i\sqrt{p^2}\Gamma_X(p^2)} g_{\mu\nu} \\ & -\frac{\lambda_{X'}^2}{p^2 - M_{X'}^2 + i\sqrt{p^2}\Gamma_{X'}(p^2)} g_{\mu\nu} + \dots, \end{aligned} \quad (25)$$

where $M_{X^{(\prime)}}^2 = M_{X^{(\prime),R}^2} + \bar{\Sigma}_{J/\psi\phi}(M_{X^{(\prime)}}^2)$.

We can take account of the finite width effects by the simple replacements of the hadronic spectral densities:

$$\lambda_{X^{(\prime)}}^2 \delta(s - M_{X^{(\prime)}}^2) \longrightarrow \lambda_{X^{(\prime)}}^2 \frac{1}{\pi} \frac{M_{X^{(\prime)}} \Gamma_{X^{(\prime)}}(s)}{(s - M_{X^{(\prime)}}^2)^2 + M_{X^{(\prime)}}^2 \Gamma_{X^{(\prime)}}^2(s)}, \quad (26)$$

where

$$\Gamma_{X^{(\prime)}}(s) = \Gamma_{X^{(\prime)}} \frac{M_{X^{(\prime)}}}{\sqrt{s}} \sqrt{\frac{s - (m_{J/\psi} + m_\phi)^2}{M_{X^{(\prime)}}^2 - (m_{J/\psi} + m_\phi)^2}}. \quad (27)$$

The Γ_X and $\Gamma_{X'}$ are the physical decay widths. Then, the hadron sides of the QCD sum rules undergo the following changes:

$$\begin{aligned} \lambda_{X^{(\prime)}}^2 \exp\left(-\frac{M_{X^{(\prime)}}^2}{T^2}\right) & \longrightarrow \lambda_{X^{(\prime)}}^2 \int_{(m_{J/\psi} + m_\phi)^2}^{s_0^{\dagger}} ds \frac{1}{\pi} \frac{M_{X^{(\prime)}} \Gamma_{X^{(\prime)}}(s)}{(s - M_{X^{(\prime)}}^2)^2 + M_{X^{(\prime)}}^2 \Gamma_{X^{(\prime)}}^2(s)} \exp\left(-\frac{s}{T^2}\right), \\ & = (0.81 \sim 0.82 \lambda_X)^2 ((0.97 \sim 0.98 \lambda_{X'})^2 \exp\left(-\frac{M_{X^{(\prime)}}^2}{T^2}\right)), \\ \lambda_{X^{(\prime)}}^2 M_{X^{(\prime)}}^2 \exp\left(-\frac{M_{X^{(\prime)}}^2}{T^2}\right) & \longrightarrow \lambda_{X^{(\prime)}}^2 \int_{(m_{J/\psi} + m_\phi)^2}^{s_0^{\dagger}} ds s \frac{1}{\pi} \frac{M_{X^{(\prime)}} \Gamma_{X^{(\prime)}}(s)}{(s - M_{X^{(\prime)}}^2)^2 + M_{X^{(\prime)}}^2 \Gamma_{X^{(\prime)}}^2(s)} \exp\left(-\frac{s}{T^2}\right), \\ & = (0.82 \sim 0.83 \lambda_X)^2 ((0.96 \sim 0.97 \lambda_{X'})^2 M_{X^{(\prime)}}^2 \exp\left(-\frac{M_{X^{(\prime)}}^2}{T^2}\right)). \end{aligned} \quad (28)$$

We can absorb the numerical factors $0.81 \sim 0.82$, $0.82 \sim 0.83$, $0.97 \sim 0.98$, and $0.96 \sim 0.97$ into the pole residues safely, and the two-meson scattering states cannot affect the masses M_X and $M_{X'}$ significantly [55]. Again, we obtain the conclusion, the pole terms or tetraquark states play an unsubstitutable role, we can saturate the QCD sum rules with or without the two-particle scattering state contributions, and the two-particle scattering states can only modify the pole residues [27].

In the present work, we choose the local four-quark current $J_\mu(x)$, while the traditional mesons are spatial extended objects and have average spatial sizes $\sqrt{\langle r^2 \rangle} \neq 0$, for example, $\sqrt{\langle r^2 \rangle} = 0.41$ fm (0.42 fm) for the J/ψ ([56, 57]) and

$\sqrt{\langle r^2 \rangle} = 0.63$ fm for the $\phi(1020)$ [58]. On the other hand, the diquark-antidiquark type tetraquark states have the average spatial sizes $\langle r \rangle = 0.5 \sim 0.7$ fm [59]. The $J/\psi, \phi(1020), X(4140)$, and $X(4685)$ have average spatial sizes of the same order, and the couplings to the continuum states $J/\psi\phi$, etc. can be neglected, as the overlappings of the wave-functions are small enough.

5. Conclusion

At the first step, we take into account our previous calculations based on the QCD sum rules and make possible assignments of the LHCb's new particles $X(4630)$ and $X(4500)$.

We tentatively assign the $X(4630)$ to be the $D_s^* \bar{D}_{s1} - D_{s1} \bar{D}_s^*$ tetraquark molecular state or $[cs]_P [\bar{c}\bar{s}]_A + [cs]_A [\bar{c}\bar{s}]_P$ tetraquark state with the quantum numbers $J^{PC} = 1^{-+}$ and assign the $X(3915)$ and $X(4500)$ to be the 1S and 2S $[cs]_A [\bar{c}\bar{s}]_A$ tetraquark states, respectively, with the quantum numbers $J^{PC} = 0^{++}$ according to the predicted masses.

Then, we extend our previous works to explore the $X(4685)$ as the first radial excited state of the $X(4140)$ with the QCD sum rules and obtain the value of the mass $M_{X(4685)} = 4.70 \pm 0.12 \text{ GeV}$, which is in very good agreement with the experimental value $M_{X(4685)} = 4684 \pm 7^{+13}_{-16} \text{ MeV}$ from the LHCb collaboration, and support assigning the $X(4140)$ and $X(4685)$ as the 1S and 2S $[sc]_{\bar{V}} [\bar{s}\bar{c}]_{\bar{V}} - [sc]_{\bar{V}} [\bar{s}\bar{c}]_{\bar{V}}$ tetraquark states, respectively, with the quantum numbers $J^{PC} = 1^{++}$. Furthermore, we investigate the two-meson scattering state contributions in details and observe that the two-meson scattering state contributions alone cannot saturate the QCD sum rules at the hadron side, the contributions of the tetraquark states (or pole terms) play an unsubstitutable role, we can saturate the QCD sum rules with or without the two-meson scattering state contributions, the two-meson scattering state contributions can only modify the pole residues, and the predictions of the tetraquark masses are robust.

Data Availability

All data included in this manuscript are available upon request by contacting with the corresponding author.

Conflicts of Interest

The author declares that he/she has no conflicts of interest.

Acknowledgments

This work is supported by the National Natural Science Foundation, Grant Number 12175068.

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