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Rotational Inertial Physics in Non-Classical Thermoviscous Fluent Continua Incorporating Internal Rotation Rates

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Abstract

In this paper, we derive non-classical continuum theory for physics of compressible and incompressible thermoviscous non-classical fluent continua using the conservation and balance laws (CBL) by incorporating additional physics of internal rotation rates arising from the velocity gradient tensor as well as their time varying rates and the rotational inertial effects. In this non-classical continuum theory time dependent deformation of fluent continua results in time varying rotation rates *i.e.*, angular velocities and angular accelerations at material points. Resistance offered to these by deforming fluent continua results in additional moments, angular momenta and inertial effects due to rotation rates *i.e.*, angular velocities and angular accelerations at the material points. Currently, this physics due to internal rotation rates and inertial effects is neither considered in classical continuum mechanics (CCM) nor in non-classical continuum mechanics (NCCM). In this paper, we present a derivation of conservation and balance laws in Eulerian description: conservation of mass (CM), balance of linear momenta (BLM), balance of angular momenta (BAM), balance of moment of moments (BMM), first and second laws of thermodynamics (FLT, SLT) that include: (i) Physics of internal rotation rates resulting from the velocity gradient tensor; (ii) New physics resulting due to angular velocities and angular accelerations due to spatially varying and time dependent rotation rates. The balance laws derived here are compared with those that only consider the rotational rates but neglect rotational inertial effects and angular accelerations to demonstrate the influence of the new physics. Constitutive variables and their argument tensors are established using conjugate pairs in the entropy inequality, additional desired physics and principle of equipresence when appropriate. Constitutive theories are derived using Helmholtz free energy density as well as representation theorem and integrity (complete basis). It is shown that the mathematical model consisting of the conservation and balance laws and constitutive theories presented in this paper has closure. Influence of new physics in the conservation and balance laws on compressible and incompressible thermoviscous fluent continua is demonstrated due to presence of angular velocities and angular accelerations arising from time varying rotation rates when the deforming fluent continua offer rotational inertial resistance. The fluent continua are considered homogeneous and isotropic. Model problem studies are considered in a follow-up paper.

Keywords

Non-Classical Continuum Mechanics, Internal Rotation Rates, Angular Velocities, Angular Accelerations, Rotational Inertia, Balance of Moment of Moments, Thermoviscous Fluent Continua

1. Introduction, Literature Review and Scope of Work

In the spatial or Eulerian mathematical description of deforming continua such as fluent continua [1] [2] [3] [4] the velocities $\bar{\boldsymbol{v}}$ and the velocity gradient tensor $\bar{\boldsymbol{L}}$ are fundamental measures of deformation physics. In general $\bar{\boldsymbol{v}}$ and $\bar{\boldsymbol{L}}$ vary between material points. Polar decomposition of $\bar{\boldsymbol{L}}$ at material points into stretch rate tensor (left or right) and rotation rate tensor shows that if $\bar{\boldsymbol{L}}$ varies between material points so do the stretch rate and rotation rate tensors. Alternatively, we can additively decompose $\bar{\boldsymbol{L}}$ at a material point into symmetric ($\bar{\boldsymbol{D}}$) and skew-symmetric ($\bar{\boldsymbol{W}}$) tensors in which the symmetric tensor is the first convective time derivative of the Green strain tensor as well as the first convected time derivative of the Almansi strain tensor which are shown to be basis independent [4]. The skew-symmetric tensor is a measure of pure rotation rate, referred to as internal rotation rate tensor or angular velocity tensor. In classical continuum mechanics when considering thermoviscous fluent continua, Cauchy stress tensor $\bar{\boldsymbol{\sigma}}$ is a rate of work conjugate to $\bar{\boldsymbol{D}}$ and $\bar{\boldsymbol{\sigma}}$ is basis independent when only conjugate to $\bar{\boldsymbol{D}}$. In the constitutive theory for Cauchy stress tensor we can also consider higher order convected time derivatives of the strain tensors in which case Cauchy stress tensor is basis dependent *i.e.*, contravariant Cauchy stress tensor $\bar{\boldsymbol{\sigma}}^{(0)}$ or covariant Cauchy stress tensor $\bar{\boldsymbol{\sigma}}_{(0)}$. In CCM, influence of time varying rotation rates at each material point due to $\bar{\boldsymbol{L}}$ is not considered. Surana *et al.* [2] [3] [5] have presented conservation and balance laws for non-classical continuum theory in which additional physics due to time varying rotation rates is incorporated into the conservation and balance laws. Thus, this non-classical continuum theory incorporates $\bar{\boldsymbol{L}}$ in its entirety in the conservation and balance laws. In subsequent papers, yang *et al.* [6] and Surana *et al.* [1] [7] showed that the presence of new physics due to time varying rotation rates requires additional balance law "balance of moment of moments" (BMM). This balance law was originally proposed by Yang *et al.* [6] based on static equilibrium considerations for solid continua. Surana *et al.* [1] [7] showed that the derivation of a balance law must be based on rates and presented derivation of BMM balance law for solid continua [7] as well as fluent continua [1]. Ordered rate constitutive theory for thermoviscous fluent continua incorporating internal rotation rates has been presented by Surana *et al.* [5]. Prior to the work in references [1] [2] [3] [5] [6] [7] a large number of publications have appeared (primarily related to solid continua) under couple stress theories, microtheories (micropolar, microstretch, micromorphic) and their application to beams, plates and shells. In the following, we present a brief review of some of the published works that are pertinent to the work presented in this paper. Surana *et al.* have clearly distinguished these published works from the works in references [1] [2] [3] [5] [7] [8].

In a recent paper, Surana *et al.* [9] presented a comprehensive literature review of published works on non-classical Continuum theories and their applications for solid and fluent continua. In the literature review presented in this paper, we only reference the works related to the non-classical theories and their applications to fluent continua. The works related to micropolar theories, non-local theories, couple stress theories of fluent continua and their applications can be found in references [10]-[29]. The micropolar theories consider micro-deformation of micro-constituents in the continuum and associated homogenization so that the matter at macro scale is isotropic and homogeneous. The theories related to the non-local effects are believed to be originated by Eringen [30] in which a definition of non-local stress tensor is introduced through an integral relationship using the product of macroscopic stress tensor and a distance kernel representing non-local effects. The works by Eringen [18] [19] [20] [21] [22] establish conservation and balance laws, constitutive theories, micromechanics considerations and their use in non-classical theories for fluent continua. Some stability and boundary considerations for non-classical theories are discussed in references [23] [24]. In reference [25] authors present a discussion on a collection of papers related to the macro-micro mechanics' aspects of deformation physics. In reference [26] a micropolar theory is presented for binary media with applications to phase transition of fiber suspensions to show flow during the filling state of injection molding of short fiber reinforced thermoplastics. A similarity solution for boundary problem flow of a polar fluid is given in reference [27]. In references [28] [29] phenomenological theory of ferrofluids and statistical mechanical theory of polar fluids are presented.

The motivation for this work is to present a complete NCCT based on internal rotation rates for fluent continua in the presence of microconstituents. The rotational inertial properties of the fluent continua due to microconstituents necessitate presence of angular acceleration term in the BAM. This is accomplished by extending non-classical theory of references [1] [2] [3] [5] [7] that is based on internal rotation rates at a material point due to $\bar{\boldsymbol{L}}$ to include additional physics that may arise due to time varying internal rotation rates (angular velocities) and the associated angular accelerations. In general the internal rotation rates vary between a material point and its neighbors and are time dependent. Thus, at each material point, in addition to the rotation rates (angular velocities) and associated angular accelerations, the resistance offered by the deforming fluent continua to these angular velocities and angular accelerations results in moments, angular momenta and rotational inertial effects. In this paper, we present derivation of conservation and balance laws and associated constitutive theories for deforming fluent continua in which the internal rotation rates and their material derivatives are considered. It is shown that while the conservation of mass and the balance of linear momenta in the present derivation remain the same as in references [1] [2] [3] [5] [6], the other balance laws require substantial modification and/or inclusion of new details due to additional new physics. It is shown that the new balance law "balance of moment of moments" introduced in reference [1] for non-classical fluent continua considering internal rotation rates only is also needed in the present work. This balance law results in additional three equations in the present work, whereas in the non-classical theories of reference [1] [2] [3] [5] [6] the balance law only establishes symmetry of the Cauchy moment tensor. References [31] [32] also contain some details that can be helpful in context of the work presented here. The complete mathematical model consisting of conservation and balance laws and the constitutive theories in Eulerian description has closure. The theories are presented for compressible as well as incompressible fluent continua. Influence of the new physics on incompressible as well as compressible thermoviscous fluent continua is investigated.

2. Notations, Choice of Basis, Various Measures of Stress, Moment Tensors and Strain Rates Tensors, Internal Rotation Rates and Their Gradients

2.1. Notations

The notations used in this paper conform to reference [4] but are different than conventional notations used in continuum mechanics writings to provide more clarity and transparency. $\boldsymbol{x}$, $\boldsymbol{A}$, V, $\partial\bar{\boldsymbol{A}}$, ∂V refer to material point coordinates (in a fixed Cartesian frame), area, volume, boundary of $\boldsymbol{A}$ and the surface bounding V, all in the reference or undeformed configuration, whereas $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{A}}$, $\bar{V}$, $\partial\bar{\boldsymbol{A}}$, $\partial\bar{V}$ are their counterparts in the current configuration. $\boldsymbol{Q}=\boldsymbol{Q}(\boldsymbol{x},t)$ and $\bar{\boldsymbol{Q}}=\bar{\boldsymbol{Q}}(\bar{\boldsymbol{x}},t)$ are Lagrangian and Eulerian descriptions of a quantity $\boldsymbol{Q}$ at a material point $\boldsymbol{x}$ in the reference configuration with its corresponding location $\bar{\boldsymbol{x}}$ in the current configuration.

A tetrahedron in the undeformed configuration (volume V) with its oblique plane constituting a part of surface ∂V bounding V deforms and rotates in the current configuration. Equilibrium considerations associated with conservation and balance laws require measures of stress, strain rates, etc. associated with the deformed tetrahedron. Two obvious choices are covariant and contravariant bases. If the edges of a tetrahedron in the undeformed configuration represent material lines, then upon finite deformation the material lines will become curved. The tangent vectors to these deformed lines at a material point (a point from which the material lines emanate) forming the edges of the deformed tetrahedron are called contravariant base vectors ($\tilde{\boldsymbol{g}}_i$). The vectors orthogonal to the faces of the deformed tetrahedron (formed by the covariant base vectors) are called contravariant base vectors ($\tilde{\boldsymbol{g}}^i$). Hence, ($\tilde{\boldsymbol{g}}_i$) and ($\tilde{\boldsymbol{g}}^i$) form non-orthogonal covariant and contravariant bases that are reciprocal to each other. Since the covariant basis is tangent to the deformed material lines, the convected time derivatives of the covariant strain tensor ia a physical measure of the strain rate tensor. Likewise, the contravariant directions normal to the faces of the tetrahedron is a natural way to define stress tensor. Thus, we define $\bar{\boldsymbol{\sigma}}^{(0)}$ as contravariant Cauchy stress tensor, $\gamma_{(1)}$ as the first convected time derivative of the Green's strain tensor. These measures are physical as these are related to the faces and edges of the true deformed tetrahedron. Since ($\tilde{\boldsymbol{g}}_i$) and ($\tilde{\boldsymbol{g}}^i$) form reciprocal bases, we could use covariant directions for stress measure and contravariant directions for strain rate measures, *i.e.*, $\bar{\boldsymbol{\sigma}}_{(0)}$ and $\gamma^{(1)}$, covariant Cauchy stress tensor and contravariant strain rate tensor. Mathematically this is justified, however in terms of physics, this description requires ($\tilde{\boldsymbol{g}}_i$) to be normal to the tetrahedron faces and ($\tilde{\boldsymbol{g}}^i$) to be the material lines tangent vectors. In other words, this description requires a new configuration of the actual deformed tetrahedron that is non-physical. When strain rates are small, the two measures are the same as the deformed and undeformed tetrahedron are virtually the same.

2.2. Internal Rotation Rates and Their Gradients

The velocities $\bar{\boldsymbol{v}}$ and the velocity gradients ($\bar{L}_{ij} = \dfrac{\partial \bar{v}_i}{\partial x_j}$) are fundamental measures of deformation physics in fluent continua in Eulerian description, hence these in their entirety must form a basis for a complete thermodynamic framework. Decomposition of $\bar{\boldsymbol{L}}$ into symmetric tensor $\bar{\boldsymbol{D}}$ and skew-symmetric tensor $\bar{\boldsymbol{W}}$. The physics of $\bar{\boldsymbol{D}}$ and $\bar{\boldsymbol{W}}$ exists in all deforming fluent continua. The currently used thermodynamic framework (classical continuum mechanics, CCM) only considers $\bar{\boldsymbol{D}}$. Hence, $\bar{\boldsymbol{W}}$ containing internal rotation rates is not considered at all. Incorporating entirety of $\bar{\boldsymbol{L}}$ in the conservation and balance laws implies that we incorporate the additional physics due to internal rotation rates in the existing thermodynamic framework for fluent continua as the physics due to $\bar{\boldsymbol{D}}$ is already present in CCM. The internal rotation rates can be visualized as the rotation rates about the axes of a triad located at a material point (a location) whose axes are parallel to the axes of the fixed Cartesian x-frame. The velocity gradient tensor $\left[\bar{L}\right]$ can be decomposed into pure rotation rate tensor $\left[{}^t\bar{R}\right]$ and the right and left stretch rates $\left[{}^t\bar{S}_r\right]$ and $\left[{}^t\bar{S}_l\right]$. Then, $\left[{}^t\bar{R}\right]$ is orthogonal and $\left[{}^t\bar{S}_r\right]$ and $\left[{}^t\bar{S}_l\right]$ are symmetric and positive-definite.

$$\left[\bar{L}\right]=\left[{}^{t}\bar{R}\right]\left[{}^{t}\bar{S}_r\right]=\left[{}^{t}\bar{S}_l\right]\left[{}^{t}\bar{R}\right] \tag{1}$$

Let $\left({}^{t}\lambda_i,\{\phi\}_i\right);\ i=1,2,3$ be the eigenpairs of $\left[\bar{L}\right]^{\mathrm{T}}\left[\bar{L}\right]$ in which $\{\phi\}_i^{\mathrm{T}}\{\phi\}_j=\delta_{ij}$, then

$$\left[\bar{L}\right]^{\mathrm{T}}\left[\bar{L}\right]=\left[\bar{\Phi}\right]\left[{}^{t}\bar{\lambda}\right]\left[\bar{\Phi}\right]^{\mathrm{T}}=\left[{}^{t}\bar{S}_r\right]^2 \tag{2}$$

The columns of $\left[\bar{\Phi}\right]$ are eigenvectors of $\{\phi\}_i$ and $\left[{}^{t}\bar{\lambda}\right]$ is a diagonal matrix of the eigenvalues ${}^{t}\lambda_i;\ i=1,2,3$. If we choose

$$\left[{}^{t}\bar{S}_r\right]=\left[\bar{\Phi}\right]\left[\sqrt{{}^{t}\bar{\lambda}}\right]\left[\bar{\Phi}\right]^{\mathrm{T}} \tag{3}$$

then (2) holds, hence definition of $\left[{}^{t}\bar{S}_r\right]$ in (3) is valid. $\left[{}^{t}\bar{R}\right]$ can now be defined using (1).

$$\left[{}^{t}\bar{R}\right]=\left[\bar{L}\right]\left[{}^{t}\bar{S}_r\right]^{-1} \tag{4}$$

Furthermore, using

$$\left[\bar{L}\right]\left[\bar{L}\right]^{\mathrm{T}}=\left[{}^{t}\bar{S}_l\right]^2 \tag{5}$$

and following a similar procedure we can establish

$$\left[{}^{t}\bar{S}_l\right]=\left[\bar{\Phi}\right]\left[\sqrt{{}^{t}\bar{\lambda}}\right]\left[\bar{\Phi}\right]^{\mathrm{T}} \tag{6}$$

$$\left[{}^{t}\bar{R}\right]=\left[{}^{t}\bar{S}_l\right]^{-1}\left[\bar{L}\right] \tag{7}$$

where $\left[{}^{t}\bar{R}\right]$ defined by (4) and (7) is unique. We note that in this approach $\left[{}^{t}\bar{R}\right]$ is a rotation rate transformation matrix, hence does not contain rotation angle rates. Alternatively, we can consider decomposition of $\left[\bar{L}\right]$ into symmetric ($\left[\bar{D}\right]$) and skew-symmetric ($\left[\bar{W}\right]$) tensors.

$$\left[\bar{L}\right]=\left[\frac{\partial\{\bar{v}\}}{\partial\{\bar{x}\}}\right]=\left[\bar{D}\right]+\left[\bar{W}\right] \tag{8}$$

$$\left[\bar{D}\right]=\frac{1}{2}\left(\left[\bar{L}\right]+\left[\bar{L}\right]^{\mathrm{T}}\right);\quad \left[\bar{W}\right]=\frac{1}{2}\left(\left[\bar{L}\right]-\left[\bar{L}\right]^{\mathrm{T}}\right) \tag{9}$$

or

$$\bar{D}_{ij}=\frac{1}{2}\left(\bar{v}_{i,j}+\bar{v}_{j,i}\right);\quad \bar{W}_{ij}=\frac{1}{2}\left(\bar{v}_{i,j}-\bar{v}_{j,i}\right) \tag{10}$$

We define positive rotation rates ${}_{i}^{t}\bar{\Theta}$ using

$$\bar{\boldsymbol{\nabla}}\times\bar{\boldsymbol{v}}=\boldsymbol{e}_i\times\boldsymbol{e}_j\frac{\partial\bar{v}_j}{\partial\bar{x}_i}=\epsilon_{ijk}\boldsymbol{e}_k\frac{\partial\bar{v}_j}{\partial\bar{x}_i} \tag{11}$$

or

$$\bar{\boldsymbol{\nabla}}\times\bar{\boldsymbol{v}}=\boldsymbol{e}_1\left(\frac{\partial\bar{v}_3}{\partial\bar{x}_2}-\frac{\partial\bar{v}_2}{\partial\bar{x}_3}\right)+\boldsymbol{e}_2\left(\frac{\partial\bar{v}_1}{\partial\bar{x}_3}-\frac{\partial\bar{v}_3}{\partial\bar{x}_1}\right)+\boldsymbol{e}_3\left(\frac{\partial\bar{v}_2}{\partial\bar{x}_1}-\frac{\partial\bar{v}_1}{\partial\bar{x}_2}\right) \tag{12}$$

or

$$\bar{\boldsymbol{\nabla}}\times\bar{\boldsymbol{v}}=\boldsymbol{e}_1\left({}_{i}^{t}\bar{\Theta}_{x_1}\right)+\boldsymbol{e}_2\left({}_{i}^{t}\bar{\Theta}_{x_2}\right)+\boldsymbol{e}_3\left({}_{i}^{t}\bar{\Theta}_{x_3}\right) \tag{13}$$

We note that

$$\left\{{}_i^t\bar{\Theta}\right\} = \left\{{}_i\bar{\omega}\right\} \quad \text{and} \quad \frac{D}{Dt}\left({}_i\bar{\omega}\right) = {}_i\dot{\bar{\omega}} = {}_i^t\dot{\bar{\Theta}} = \left\{{}_i\bar{a}\right\}$$

in which ${}_i\bar{\omega}$ are angular velocities and ${}_i\bar{a}$ are angular accelerations using the rotation rates in (13). We can write the expanded form of $\left[\bar{W}\right]$

$$\left[\bar{W}\right] = \begin{bmatrix} 0 & -\frac{1}{2}\left({}_i^t\Theta_{x_3}\right) & \frac{1}{2}\left({}_i^t\Theta_{x_2}\right) \\ \frac{1}{2}\left({}_i^t\Theta_{x_3}\right) & 0 & -\frac{1}{2}\left({}_i^t\Theta_{x_1}\right) \\ -\frac{1}{2}\left({}_i^t\Theta_{x_2}\right) & \frac{1}{2}\left({}_i^t\Theta_{x_1}\right) & 0 \end{bmatrix} \tag{14}$$

where ${}_i^t\Theta_{x_1}$, ${}_i^t\Theta_{x_2}$, ${}_i^t\Theta_{x_3}$ are rotation rates related to total 90 degree angle and are positive counterclockwise and $\left[\bar{W}\right]$ contains half of the total rotation rate, *i.e.*, related to half of the rate of change of 90 degree angle. It is obvious that $\bar{\boldsymbol{W}}$ is a tensor of rank two, whereas the rotation rates defined in (13) are clearly a tensor of rank one. In other words, rotation rates in (13) constitute a tensor of rank one, but the components of this tensor arranged in the form in which they appear in $\left[\bar{W}\right]$ constitute a tensor of rank two. We determine gradients of the rotation rate tensor (13). Let

$$\left\{{}_i^t\bar{\Theta}\right\}^{\mathrm{T}} = \left[{}_i^t\Theta_{x_1}, {}_i^t\Theta_{x_2}, {}_i^t\Theta_{x_3}\right] \tag{15}$$

be a vector representation of (13), then the gradient of ${}_i^t\Theta$ can be defined by

$$\left[{}^{{}_i^t\Theta}\bar{J}\right] = \left[\frac{\partial\left\{{}_i^t\Theta\right\}}{\partial\left\{\bar{x}\right\}}\right] \quad \text{or} \quad {}^{{}_i^t\Theta}J_{jk} = \frac{\partial\left({}_i^t\Theta_j\right)}{\partial\bar{x}_k} \tag{16}$$

The gradient tensor $\left[{}^{{}_i^t\Theta}\bar{J}\right]$ of the internal rotation rates defined by (16) can be decomposed into symmetric and antisymmetric tensors $\left[{}^{{}_i^t\Theta}_{s}\bar{J}\right]$ and $\left[{}^{{}_i^t\Theta}_{a}\bar{J}\right]$.

$$\left[{}^{{}_i^t\Theta}\bar{J}\right] = \left[{}^{{}_i^t\Theta}_{s}\bar{J}\right] + \left[{}^{{}_i^t\Theta}_{a}\bar{J}\right] \tag{17}$$

$$\begin{aligned} \left[{}^{{}_i^t\Theta}_{s}\bar{J}\right] &= \frac{1}{2}\left(\left[{}^{{}_i^t\Theta}\bar{J}\right] + \left[{}^{{}_i^t\Theta}\bar{J}\right]^{\mathrm{T}}\right) \\ \left[{}^{{}_i^t\Theta}_{a}\bar{J}\right] &= \frac{1}{2}\left(\left[{}^{{}_i^t\Theta}\bar{J}\right] - \left[{}^{{}_i^t\Theta}\bar{J}\right]^{\mathrm{T}}\right) \end{aligned} \tag{18}$$

when the velocity gradient tensor varies between the neighboring material points so do the internal rotation rates ${}_i^t\boldsymbol{\Theta}$ (or $\left[\bar{W}\right]$), their rates as well as their gradients and their rates. Varying ${}_i^t\boldsymbol{\Theta}$ and ${}^{{}_i^t\Theta}\boldsymbol{J}$, when resisted by deforming fluent continua, results in moments, angular momenta and angular inertial effects as a consequence. Thus, on the oblique plane of the tetrahedron defining part of $\partial\bar{V}(t)$ or defining a part of the bounding surface due to cut principle of cauchy, resultant moment can exist.

2.3. Stress, Moment and Strain Rate Tensors

Consider a volume of matter $\underset{\sim}{V}$ in the reference configuration with closed boundary $\partial\underset{\sim}{V}$. Volume V is isolated from $\underset{\sim}{V}$ by a hypothetical surface ∂V as in the cut principle of Cauchy. Consider a tetrahedron T_1 such that its oblique plane is part of ∂V and its other three planes are orthogonal to each other parallel to the planes of the x-frame. Upon deformation, $\underset{\sim}{V}$ and $\partial \underset{\sim}{V}$ occupy $\underset{\sim}{\bar{V}}$ and $\partial \underset{\sim}{\bar{V}}$ and likewise V and ∂V deform into $\bar{V}$ and $\partial \bar{V}$. The tetrahedron T_1 deforms into $\bar{T}_1$ whose edges (under finite deformation) are non-orthogonal covariant base vectors $\tilde{g}_i$. The planes of the tetrahedron formed by the covariant base vectors are flat but obviously non-orthogonal to each other. We assume the tetrahedron to be the small neighborhood of material point $\bar{o}$ so that the assumption of the oblique plane $\bar{A}\bar{B}\bar{C}$ being flat but still part of $\partial \bar{V}$ is valid. When the deformed tetrahedron is isolated from volume $\bar{V}$ it must be in equilibrium under the action of disturbance on surface $\bar{A}\bar{B}\bar{C}$ from the volume surrounding $\bar{V}$ and the internal fields that act on the flat faces which equilibrium with the mating faces in volume $\bar{V}$ when the tetrahedron $\bar{T}_1$ is 4 the volume $\bar{V}$.

Consider the deformed tetrahedron $\bar{T}_1$. Let $\bar{\boldsymbol{P}}$ be the average stress per unit area on plane $\bar{A}\bar{B}\bar{C}$, $\bar{\boldsymbol{M}}$ be the average moment per unit area on plane $\bar{A}\bar{B}\bar{C}$ (henceforth referred to as moment for short), and $\bar{\boldsymbol{n}}$ be the unit exterior normal to the face $\bar{A}\bar{B}\bar{C}$. $\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{M}}$, and $\bar{\boldsymbol{n}}$ all have different directions when the deformation is finite. The edges of the deformation tetrahedron are covariant base vectors $\tilde{\boldsymbol{g}}_i$ that are tangent to the deformed curvilinear material lines.

$$\tilde{\boldsymbol{g}}_i = \boldsymbol{e}_k \frac{\partial \bar{x}_k}{\partial x_i}; \quad J_{ij} = \frac{\partial \bar{x}_i}{\partial x_j} \tag{19}$$

Columns of $\boldsymbol{J}$ are covariant base vectors $\tilde{\boldsymbol{g}}_i$ that form non-orthogonal covariant basis. Contravariant base vectors of $\tilde{\boldsymbol{g}}^j$ are normal to the faces of the tetrahedron formed by the covariant base vectors

$$\tilde{\boldsymbol{g}}^j = \boldsymbol{e}_l \frac{\partial x_j}{\partial \bar{x}_l}; \quad \bar{J}_{ij} = \frac{\partial x_i}{\partial \bar{x}_j} \tag{20}$$

The rows of $\bar{\boldsymbol{J}}$ are contravariant base vectors $\tilde{\boldsymbol{g}}^j$. These form a non-orthogonal contravariant basis. Covariant and contravariant bases are reciprocal to each other [4]. If $\underset{\sim}{\bar{\boldsymbol{\sigma}}}^{(0)}$ or $\underset{\sim}{\boldsymbol{\sigma}}^{(0)}$ is the contravariant stress tensor with components $\underset{\sim}{\bar{\sigma}}_{ij}^{(0)}$ or $\underset{\sim}{\sigma}_{ij}^{(0)}$ with dyads $\tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j$, then using dyads $\tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j$ or contravariant laws of transformation we can define contravariant Cauchy stress tensors $\boldsymbol{\sigma}^{(0)}$ in Lagrangian description

$$\boldsymbol{\sigma}^{(0)} = \tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j \underset{\sim}{\sigma}_{ij}^{(0)} \tag{21}$$

using (19)-(21), we can write

$$\boldsymbol{\sigma}^{(0)} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \sigma_{ij}^{(0)}; \quad \sigma_{ij}^{(0)} = J_{ik} \left(\underset{\sim}{\sigma}_{kl}^{(0)} \right) J_{jl} \tag{22}$$

or

$$\left[\sigma^{(0)} \right] = [J] \left[\underset{\sim}{\sigma}^{(0)} \right] [J]^{\mathrm{T}} \tag{23}$$

where $\bar{\boldsymbol{\sigma}}^{(0)}$ is Eulerian description of $\boldsymbol{\sigma}^{(0)}$ which is obtained from (23) by replacing $[J]$ with $\left[\bar{J}\right]^{-1}$ and $\boldsymbol{\sigma}^{(0)}$ with $\bar{\boldsymbol{\sigma}}^{(0)}$. Since dyads of $\bar{\boldsymbol{\sigma}}^{(0)}$ and $\boldsymbol{\sigma}^{(0)}$ are $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, Cauchy principle holds between $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{\sigma}}^{(0)}$.

$$\bar{\boldsymbol{P}} = \left(\bar{\boldsymbol{\sigma}}^{(0)}\right)^{\mathrm{T}} \cdot \bar{\boldsymbol{n}} \tag{24}$$

Similarly we can define covariant Cauchy stress tensors $\boldsymbol{\sigma}_{(0)}$ or $\bar{\boldsymbol{\sigma}}_{(0)}$ and Cauchy principle between $\bar{\boldsymbol{\sigma}}_{(0)}$ and $\bar{\boldsymbol{P}}$.

$$\bar{\boldsymbol{\sigma}}_{(0)} = \tilde{\boldsymbol{g}}^i \otimes \tilde{\boldsymbol{g}}^j \left(\underset{\sim}{\sigma}_{(0)}\right)_{ij} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \left(\bar{\sigma}_{(0)}\right)_{ij};\ \left(\bar{\sigma}_{(0)}\right)_{ij} = \bar{J}_{ki} \left(\underset{\sim}{\sigma}_{(0)}\right)_{kl} \bar{J}_{lj} \tag{25}$$

or

$$\left[\bar{\sigma}_{(0)}\right] = \left[\bar{J}\right]^{\mathrm{T}} \left[\underset{\sim}{\sigma}_{(0)}\right]\left[\bar{J}\right] \tag{26}$$

and

$$\bar{\boldsymbol{P}} = \left(\bar{\boldsymbol{\sigma}}_{(0)}\right)^{\mathrm{T}} \cdot \bar{\boldsymbol{n}} \tag{27}$$

We define the contravariant and covariant Cauchy moment tensor in similar fashion and the corresponding Cauchy principle

$$\boldsymbol{m}^{(0)} = \tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j \underset{\sim}{m}_{ij}^{(0)} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j m_{ij}^{(0)};\quad m_{ij}^{(0)} = J_{ik}\left(\underset{\sim}{m}_{kl}^{(0)}\right) J_{jl} \tag{28}$$

or

$$\left[m^{(0)}\right] = [J]\left[\underset{\sim}{m}^{(0)}\right][J]^{\mathrm{T}};\quad \left[\bar{m}^{(0)}\right] = \left[\bar{J}\right]^{-1}\left[\underset{\sim}{m}^{(0)}\right]\left[\left[\bar{J}\right]^{-1}\right]^{\mathrm{T}} \tag{29}$$

$$\bar{\boldsymbol{M}} = \left(\bar{\boldsymbol{m}}^{(0)}\right)^{\mathrm{T}} \cdot \bar{\boldsymbol{n}} \tag{30}$$

and

$$\bar{\boldsymbol{m}}_{(0)} = \tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j \left(\underset{\sim}{m}_{(0)}\right)_{ij} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \left(\bar{m}_{(0)}\right)_{ij};\ \left(m_{(0)}\right)_{ij} = \bar{J}_{ki} \left(\underset{\sim}{m}_{(0)}\right)_{kl} \bar{J}_{lj} \tag{31}$$

or

$$\left[\bar{m}_{(0)}\right] = \left[\bar{J}\right]^{\mathrm{T}}\left[\underset{\sim}{m}_{(0)}\right]\left[\bar{J}\right];\quad \left[m_{(0)}\right] = \left[[J]^{-1}\right]^{\mathrm{T}}\left[\underset{\sim}{m}_{(0)}\right][J]^{-1} \tag{32}$$

and

$$\bar{\boldsymbol{M}} = \left(\bar{\boldsymbol{m}}_{(0)}\right)^{\mathrm{T}} \cdot \bar{\boldsymbol{n}} \tag{33}$$

At this state $\bar{\boldsymbol{\sigma}}^{(0)}$, $\boldsymbol{\sigma}^{(0)}$, $\bar{\boldsymbol{\sigma}}_{(0)}$, $\boldsymbol{\sigma}_{(0)}$, $\boldsymbol{m}^{(0)}$, $\bar{\boldsymbol{m}}^{(0)}$, $\bar{\boldsymbol{m}}_{(0)}$, and $\boldsymbol{m}_{(0)}$ are all nonsymmetric tensors of rank two. Thus, we note that the Cauchy stress tensors and the Cauchy moment tensors are basis dependent. It has been shown that [4] for finite strain rates the contravariant measures are meritorious. However, in deriving conservation and balance laws and the constitutive theories either measure yields a covariant mathematical model. We introduce stress measure ${}^{(0)}\bar{\boldsymbol{\sigma}}$ that could represent $\bar{\boldsymbol{\sigma}}^{(0)}$ or $\bar{\boldsymbol{\sigma}}_{(0)}$ and the moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$ that could represent $\bar{\boldsymbol{m}}^{(0)}$ or $\bar{\boldsymbol{m}}_{(0)}$ depending upon our choice. We present derivation of the balance laws and constitutive theories using ${}^{(0)}\bar{\boldsymbol{\sigma}}$ and ${}^{(0)}\bar{\boldsymbol{m}}$, thus making the derivations basis independent. Basis dependent mathematical model is recoverable from the derivation by specific choice of ${}^{(0)}\bar{\boldsymbol{\sigma}}$ and ${}^{(0)}\bar{\boldsymbol{m}}$.

3. Conservation and Balance Laws

In the following we present conservation and balance laws in Eulerian description for non-classical fluent continua incorporating internal rotation rates and their spatial and temporal gradients. The fluent continua is assumed homogeneous and isotropic.

3.1. Conservation of Mass: CM

The continuity equation resulting from the principle of conservation of mass remains the same in the non-classical continuum theory considered here as in case of classical continuum mechanics. The differential form of the continuity equation in Eulerian description for compressible matter is given by

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{\nabla} \cdot \left(\overline{\rho \boldsymbol{v}} \right) \qquad \text{(CM)} \tag{34}$$

or

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho}\,\mathrm{div}\left(\bar{\boldsymbol{v}} \right) = 0 \tag{35}$$

For incompressible matter $\rho_0 = \bar{\rho}$; hence (34) or (35) reduce to

$$\mathrm{div}\left(\bar{\boldsymbol{v}} \right) = 0 \tag{36}$$

3.2. Balance of Linear Momenta: BLM

For a deforming volume of matter, the rate of change of linear momentum must be equal to the sum of all other forces acting on it. This is Newton's second law applied to a volume of matter. The derivation of the balance laws is exactly same as in case of CCM [4] and we can write the following (Using ${}^{(0)}\bar{\boldsymbol{\sigma}}$ as Cauchy stress measure) in Eulerian description.

$$\bar{\rho}\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\rho}\bar{\boldsymbol{F}}^b - \bar{\nabla} \cdot \left({}^{(0)}\bar{\boldsymbol{\sigma}} \right) = 0 \qquad \text{(BLM)} \tag{37}$$

or

$$\bar{\rho}\frac{\partial \bar{v}_i}{\partial t} + \bar{\rho}\bar{v}_j \frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \bar{\rho}\bar{F}_i^b - \frac{\partial \left({}^{(0)}\bar{\sigma}_{ji} \right)}{\partial \bar{x}_j} = 0 \tag{38}$$

in which $\bar{\boldsymbol{F}}^b$ is body force per unit mass.

3.3. Balance of Angular Momenta: BAM

The principle of balance of angular momenta for non-classical continuum mechanics (NCCM) incorporating internal rotation rates, their spatial and temporal derivatives and inertial effects can be stated as: The time rate of change of moment of moments is equal to the sum of moments of the forces and the moments in the current configuration at any time *t*. Let ${}^{\theta}\bar{I}$ be the rotational inertia per unit mass of the deforming fluent continua then ${}^{\theta}\bar{I}\bar{\rho}\left({}_i\bar{\boldsymbol{\varpi}} \right)\mathrm{d}\bar{V}$ is the angular momenta per unit mass of the fluent continua for the elemental volume $\mathrm{d}\bar{V}$ due to ${}^{\theta}\bar{I}$ and angular velocity ${}_i\bar{\boldsymbol{\varpi}}$. The moment of linear momenta for the same volume $\mathrm{d}\bar{V}$ is $\bar{\boldsymbol{x}} \times \overline{\rho v}\mathrm{d}\bar{V}$. Then, according to this balance law:

rate of change of angular momenta
$= \left(\text{moments due to } \bar{P} + \bar{M}\right) + \text{moments of the body forces } \bar{\rho}\bar{\boldsymbol{F}}^b \text{ acting on } \mathrm{d}\bar{V}.$

Thus, for the deformed volume $\bar{V}$ bounded by $\partial\bar{V}$ we can write:

$$\frac{D}{Dt}\int_{\bar{V}}\left({}^{\theta}\bar{I}\left({}_i\bar{\boldsymbol{\omega}}\right)\bar{\rho} + \bar{\boldsymbol{x}}\times\rho\bar{\boldsymbol{v}}\right)\mathrm{d}\bar{V} = \int_{\partial\bar{V}}\left(\bar{\boldsymbol{x}}\times\bar{\boldsymbol{P}} + \bar{\boldsymbol{M}}\right)\mathrm{d}\bar{A} + \int_{\bar{V}}\left(\bar{\boldsymbol{x}}\times\bar{\rho}\bar{\boldsymbol{F}}^b\right)\mathrm{d}\bar{V} \tag{39}$$

We consider each term of (39)

$$\begin{aligned}\frac{D}{Dt}\int_{\bar{V}}{}^{\theta}\bar{I}\left({}_i\bar{\boldsymbol{\omega}}\right)\bar{\rho}\,\mathrm{d}\bar{V} &= \frac{D}{Dt}\int_{V}{}^{\theta}I\left({}_i\boldsymbol{\omega}\right)\rho_0\,\mathrm{d}V\\ &= \int_{V}\frac{D}{Dt}\left({}^{\theta}I\left({}_i\boldsymbol{\omega}\right)\right)\rho_0\,\mathrm{d}V\\ &= \int_{\bar{V}}\frac{D}{Dt}\left({}^{\theta}\bar{I}\left({}_i\bar{\boldsymbol{\omega}}\right)\right)\bar{\rho}\,\mathrm{d}\bar{V}\end{aligned} \tag{40}$$

if ${}^{\theta}\bar{I}$ is constant, then (40) reduces to

$$\frac{D}{Dt}\int_{\bar{V}}{}^{\theta}\bar{I}\left({}_i\bar{\boldsymbol{\omega}}\right)\bar{\rho}\,\mathrm{d}\bar{V} = \int_{\bar{V}}{}^{\theta}\bar{I}\,\bar{\rho}\frac{D}{Dt}\left({}_i\bar{\boldsymbol{\omega}}\right)\mathrm{d}\bar{V} \tag{41}$$

and

$$\begin{aligned}\frac{D}{Dt}\int_{\bar{V}}\bar{\boldsymbol{x}}\times\rho\bar{\boldsymbol{v}}\,\mathrm{d}\bar{V} &= \frac{D}{Dt}\int_{\bar{V}}\epsilon_{ijk}\bar{x}_i\bar{v}_j\bar{\rho}\,\mathrm{d}\bar{V}\\ &= \frac{D}{Dt}\int_{V}\epsilon_{ijk}x_i v_j\rho_0\,\mathrm{d}V\\ &= \int_{\bar{V}}\frac{D}{Dt}\left(\epsilon_{ijk}\bar{x}_i\bar{v}_j\right)\bar{\rho}\,\mathrm{d}\bar{V}\\ &= \int_{\bar{V}}\epsilon_{ijk}\left(\bar{v}_i\bar{v}_j + \bar{x}_i\frac{D\bar{v}_j}{Dt}\right)\bar{\rho}\,\mathrm{d}\bar{V}\end{aligned} \tag{42}$$

since $\epsilon_{ijk}\bar{v}_i\bar{v}_j = 0$, (42) reduces to

$$\frac{D}{Dt}\int_{\bar{V}}\bar{\boldsymbol{x}}\times\rho\bar{\boldsymbol{v}}\,\mathrm{d}\bar{V} = \int_{\bar{V}}\epsilon_{ijk}\bar{x}_i\frac{D\bar{v}_j}{Dt}\mathrm{d}\bar{V} \tag{43}$$

and

$$\begin{aligned}\int_{\partial\bar{V}}\left(\bar{\boldsymbol{x}}\times\bar{\boldsymbol{P}} + \bar{\boldsymbol{M}}\right)\mathrm{d}\bar{A} &= \int_{\partial\bar{V}}\left(\bar{\boldsymbol{x}}\times{}^{(0)}\bar{\boldsymbol{\sigma}}^{\mathrm{T}}\cdot\bar{\boldsymbol{n}} + {}^{(0)}\bar{\boldsymbol{m}}^{\mathrm{T}}\cdot\bar{\boldsymbol{n}}\right)\mathrm{d}\bar{A}\\ &= \int_{\partial\bar{V}}\left(\epsilon_{ijk}\bar{x}_i\left({}^{(0)}\bar{\sigma}_{mj}\right)\bar{n}_m + {}^{(0)}\bar{m}_{mj}\bar{n}_m\right)\mathrm{d}\bar{A}\end{aligned} \tag{44}$$

using Divergence Theorem

$$\int_{\partial\bar{V}}\left(\bar{\boldsymbol{x}}\times\bar{\boldsymbol{P}} + \bar{\boldsymbol{M}}\right)\mathrm{d}\bar{V} = \int_{\bar{V}}\left(\epsilon_{ijk}\left(\bar{x}_i\left({}^{(0)}\bar{\sigma}_{mj}\right)\right)_{,m} + \left({}^{(0)}\bar{m}_{mj}\right)_{,m}\right)\mathrm{d}\bar{V} \tag{45}$$

we note the following

$$\epsilon_{ijk}\left(\bar{x}_i\left({}^{(0)}\bar{\sigma}_{mj}\right)\right)_{,m} = \epsilon_{ijk}\left(\delta_{im}{}^{(0)}\bar{\sigma}_{mj} + \bar{x}_i{}^{(0)}\bar{\sigma}_{mj,m}\right) \tag{46}$$

using (46) in (45) we can write

$$\int_{\partial\bar{V}}\left(\bar{\boldsymbol{x}}\times\bar{\boldsymbol{P}} + \bar{\boldsymbol{M}}\right)\mathrm{d}\bar{V} = \int_{\bar{V}}\left(\epsilon_{ijk}\left(\bar{x}_i\left({}^{(0)}\bar{\sigma}_{mj}\right)\right)_{,m} + \left({}^{(0)}\bar{m}_{mj}\right)_{,m}\right)\mathrm{d}\bar{V} \tag{47}$$

and

$$\int_{\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{F}}^b \mathrm{d}\bar{V} = \int_{\bar{V}} \epsilon_{ijk} \bar{x}_i \bar{F}_j^b \bar{\rho} \mathrm{d}\bar{V} \tag{48}$$

substituting from (41), (43), (47) and (48) in (39) and rearranging terms, we obtain

$$\begin{aligned} &\int_{\bar{V}} {}^{\theta}\bar{I}\bar{\rho}\frac{D}{Dt}\left({}_i\bar{\omega}_k\right)\mathrm{d}\bar{V} + \int_{\bar{V}} \epsilon_{ijk}\bar{x}_i\left(\bar{\rho}\frac{D\bar{v}_j}{Dt} - {}^{(0)}\bar{\sigma}_{mj,m} - \bar{\rho}\bar{F}_j^b\right)\mathrm{d}\bar{V} \\ &-\int_{\bar{V}}\left(\epsilon_{ijk}\,{}^{(0)}\bar{\sigma}_{ij} + {}^{(0)}\bar{m}_{mk,m}\right)\mathrm{d}\bar{V} = 0 \end{aligned} \tag{49}$$

The coefficient of $\epsilon_{ijk}\bar{x}_i$ in the second term in (49) is zero due to balance of linear momenta, hence (49) reduces to

$$\int_{\bar{V}}\left({}^{\theta}\bar{I}\bar{\rho}\frac{D}{Dt}\left({}_i\bar{\omega}_k\right) - \epsilon_{ijk}\,{}^{(0)}\bar{\sigma}_{ij} - {}^{(0)}\bar{m}_{mk,m}\right)\mathrm{d}\bar{V} = 0 \tag{50}$$

For isotropic homogeneous matter $\bar{V}$ is arbitrary hence we can obtain differential form of (50)

$${}^{\theta}\bar{I}\bar{\rho}\frac{D}{Dt}\left({}_i\bar{\omega}_k\right) - \epsilon_{ijk}\,{}^{(0)}\bar{\sigma}_{ij} - {}^{(0)}\bar{m}_{mk,m} = 0 \qquad \text{(BAM)} \tag{51}$$

Remarks

1. If we set the first and the last term in (51) to zero, then we recover balance of angular momenta for classical continuum mechanics in Eulerian description.

2. If we set the first term in (51) to zero but retain second and third order terms, then we have balance of angular momenta for NCCM incorporating internal rotation rates without the rotational inertial physics.

3. Appearance of the first term in (51) is due to consideration of time varying rotation rates and rotational inertia ${}^{\theta}\bar{I}$. This is new physics considered in the present work that neither appears in CCM nor NCCM published works.

4. Equation (51) is the final form of balance of angular momenta.

3.4. Balance of Moment of Moments: BMM

This is a new balance law originally proposed by Yang *et al.* [6] for NCCM. This balance law was derived based on static considerations (hence cannot be referred to as a balance law). Later, Surana *et al.* explained the rationale for this balance law and pointed out that a balance law must be derived using rate considerations. In references [1] [7] [33] they presented derivation of the "balance of moment of moments" balance law for NCCM for fluent and solid continua in the presence of internal rotation rates and internal rotations. In the work presented in this paper, the physics considered is different than in reference [1], hence a rederivation of this balance law is necessary. According to this balance law the rate of change of moment of angular momenta due to rotation rates in a deformed volume $\bar{V}$ must be equal to the sum of the moment of moments due to the antisymmetric components of the Cauchy stress tensor over the same deformed volume $\bar{V}$ and the moment of $\bar{\boldsymbol{M}}$ acting on boundary $\partial\bar{V}$ of $\bar{V}$.

$$\frac{D}{Dt}\int_{\bar{V}} \bar{\boldsymbol{x}} \times {}^{\theta}\bar{I}\left({}_i\bar{\boldsymbol{\omega}}\right)\bar{\rho}\mathrm{d}\bar{V} = \int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\right)\mathrm{d}\bar{V} + \int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\mathrm{d}\bar{A} \tag{52}$$

we expand each term of (52) in the following

$$
\begin{aligned}
\frac{D}{Dt}\int_{\bar{V}} \bar{\boldsymbol{x}} \times {}^{\theta}\bar{I}\left({}_{i}\bar{\boldsymbol{\omega}}\right)\bar{\rho}\,\mathrm{d}\bar{V} &= \frac{D}{Dt}\int_{\bar{V}} {}^{\theta}\bar{I}\,\epsilon_{jkl}\bar{x}_j\left({}_{i}\bar{\omega}_k\right)\bar{\rho}\,\mathrm{d}\bar{V} \\
&= \frac{D}{Dt}\int_{V} {}^{\theta}I\,\epsilon_{jkl}x_j\left({}_{i}\omega_k\right)\rho_0\,\mathrm{d}V \\
&= \int_{\bar{V}} \frac{D}{Dt}\left({}^{\theta}\bar{I}\,\epsilon_{jkl}\bar{x}_j\left({}_{i}\bar{\omega}_k\right)\bar{\rho}\right)\mathrm{d}\bar{V}
\end{aligned} \tag{53}
$$

Assuming ${}^{\theta}\bar{I}$ to be constant

$$
\frac{D}{Dt}\int_{\bar{V}} \bar{\boldsymbol{x}} \times {}^{\theta}\bar{I}\left({}_{i}\bar{\boldsymbol{\omega}}\right)\bar{\rho}\,\mathrm{d}\bar{V} = \int_{\bar{V}} {}^{\theta}\bar{I}\,\epsilon_{jkl}\left(\bar{v}_j\left({}_{i}\bar{\omega}_k\right) + \bar{x}_j\frac{D}{Dt}\left({}_{i}\bar{\omega}_k\right)\right)\bar{\rho}\,\mathrm{d}\bar{V} \tag{54}
$$

$$
\int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\,\mathrm{d}\bar{A} = \int_{\partial\bar{V}} \epsilon_{jkl}\bar{x}_j\bar{M}_k\,\mathrm{d}\bar{A} \tag{55}
$$

using Cauchy principle for $\bar{\boldsymbol{M}}$

$$
\int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\,\mathrm{d}\bar{A} = \int_{\partial\bar{V}} \epsilon_{jkl}\bar{x}_j\left({}^{(0)}\bar{m}_{mk}\right)\bar{n}_m\,\mathrm{d}\bar{A} \tag{56}
$$

using Divergence Theorem

$$
\begin{aligned}
\int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\,\mathrm{d}\bar{A} &= \int_{\bar{V}} \epsilon_{jkl}\left(\bar{x}_j\left({}^{(0)}\bar{m}_{mk}\right)\right)_{,m}\mathrm{d}\bar{V} \\
&= \int_{\bar{V}} \epsilon_{jkl}\left(\delta_{jm}\,{}^{(0)}\bar{m}_{mk} + \bar{x}_j\left({}^{(0)}\bar{m}_{mk,m}\right)\right)\mathrm{d}\bar{V} \\
&= \int_{\bar{V}} \epsilon_{jkl}\left({}^{(0)}\bar{m}_{jk} + \bar{x}_j\,{}^{(0)}\bar{m}_{mk,m}\right)\mathrm{d}\bar{V}
\end{aligned} \tag{57}
$$

substituting from (54) and (57) in (52)

$$
\begin{aligned}
&\int_{\bar{V}} {}^{\theta}\bar{I}\,\bar{\rho}\epsilon_{jkl}\left(\bar{v}_j\left({}_{i}\bar{\omega}_k\right) + \bar{x}_j\frac{D\left({}_{i}\bar{\omega}_k\right)}{Dt}\right)\mathrm{d}\bar{V} \\
&= \int_{\bar{V}} \bar{\boldsymbol{x}} \times \boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\,\mathrm{d}\bar{V} + \int_{\bar{V}} \epsilon_{jkl}\left({}^{(0)}\bar{m}_{jk} + \bar{x}_j\left({}^{(0)}\bar{m}_{mk,m}\right)\right)\mathrm{d}\bar{V}
\end{aligned} \tag{58}
$$

we note that

$$
\begin{aligned}
&{}^{(0)}\bar{m}_{mk,m} = \bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}} \\
&\epsilon_{jkl}\left(\bar{x}_j\left({}^{(0)}\bar{m}_{mk,m}\right)\right) = \bar{\boldsymbol{x}} \times \bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}} \\
&{}^{\theta}\bar{I}\,\bar{\rho}\epsilon_{jkl}\frac{D\left({}_{i}\bar{\omega}_k\right)}{Dt} = {}^{\theta}\bar{I}\,\bar{\rho}\,\bar{\boldsymbol{x}} \times \frac{D\left({}_{i}\bar{\omega}\right)}{Dt}
\end{aligned} \tag{59}
$$

using (59) in (58) and regrouping terms

$$
\begin{aligned}
&\int_{\bar{V}} {}^{\theta}\bar{I}\,\bar{\rho}\epsilon_{jkl}\bar{v}_j\left({}_{i}\omega_k\right)\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{\boldsymbol{x}} \times \left({}^{\theta}\bar{I}\,\bar{\rho}\frac{D\left({}_{i}\bar{\boldsymbol{\omega}}\right)}{Dt} - \boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right)\mathrm{d}\bar{V} \\
&= \int_{\bar{V}} \epsilon_{jkl}\,{}^{(0)}\bar{m}_{jk}\,\mathrm{d}\bar{V}
\end{aligned} \tag{60}
$$

using balance of angular momenta (51) in (60), we obtain

$$
\int_{\bar{V}} \epsilon_{jkl}\left({}^{\theta}\bar{I}\,\bar{\rho}\bar{v}_j\left({}_{i}\bar{\omega}_k\right) - {}^{(0)}\bar{m}_{jk}\right)\mathrm{d}\bar{V} = 0 \tag{61}
$$

For homogeneous, isotropic continua, $\bar{V}$ is arbitrary, hence we obtain the following from (61)

$$
\epsilon_{jkl}\left({}^{\theta}\bar{I}\,\bar{\rho}\bar{v}_j\left({}_{i}\bar{\omega}_k\right) - {}^{(0)}\bar{m}_{jk}\right) = 0 \qquad \text{(BMM)} \tag{62}
$$

Equation (62) is the final form resulting from the balance of moment of moments balance law.

Remarks

1. We note that in the absence of rotational inertia ${}^{\theta}\bar{I}$ (new physics considered in this paper), *i.e.*, when ${}^{\theta}\bar{I} = 0$, (62) reduces to

$$\epsilon_{jkl}\,{}^{(0)}\bar{m}_{jk} = 0 \tag{63}$$

This is same as the BMM balance law introduced in references [1] [6] [7].

2. When ${}^{\theta}\bar{I}$ is not zero, (62) yields three equations defining antisymmetric parts of the Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$ in terms of velocities and the rotation rates (angular velocities) and the properties $\bar{\rho}$ and ${}^{\theta}\bar{I}$ of the continua.

3.5. First Law of Thermodynamics: FLT

The sum of work and heat added to a volume of matter must result in increase of the energy of the volume. This can be expressed as a rate equation in Eulerian description.

$$\frac{D\bar{E}_t}{Dt} = \frac{D\bar{Q}}{dt} + \frac{D\bar{W}}{Dt} \tag{64}$$

where $\bar{E}_t$, $\bar{Q}$ and $\bar{W}$ are total energy, heat added and work done. Their rates can be written as

$$\frac{D\bar{E}_t}{Dt} = \frac{D}{Dt}\int_{\bar{V}} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} + \frac{1}{2}{}^{\theta}\bar{I}\left({}_{i}\bar{\boldsymbol{\omega}}\cdot{}_{i}\bar{\boldsymbol{\omega}}\right) - \bar{\boldsymbol{F}}^{b}\cdot\bar{\boldsymbol{v}}\right)\mathrm{d}\bar{V} \tag{65}$$

$$\frac{D\bar{Q}}{Dt} = -\int_{\partial\bar{V}} \bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{n}}\,\mathrm{d}\bar{A} \tag{66}$$

$$\frac{D\bar{W}}{Dt} = \int_{\partial\bar{V}}\left(\bar{\boldsymbol{P}}\cdot\bar{\boldsymbol{v}} + \bar{\boldsymbol{M}}\cdot{}_{i}^{t}\boldsymbol{\Theta}\right)\mathrm{d}\bar{A} \tag{67}$$

where $\bar{e}$ is specific internal energy, $\bar{\boldsymbol{F}}^{b}$ are body forces per unit mass and $\bar{\boldsymbol{g}}$ is heat vector. The second term in the integrand is due to additional rate of work due to rotation rates. We expand integrals in (65)-(67). Following reference [4] we can show

$$\begin{aligned}\frac{D\bar{E}_t}{Dt} &= \frac{D}{Dt}\int_{\bar{V}} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} + \frac{1}{2}{}^{\theta}\bar{I}\left({}_{i}\bar{\boldsymbol{\omega}}\cdot{}_{i}\bar{\boldsymbol{\omega}}\right) - \bar{\boldsymbol{F}}^{b}\cdot\bar{\boldsymbol{v}}\right)\mathrm{d}\bar{V} \\ &= \int_{V}\left(\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\rho}\bar{\boldsymbol{v}}\frac{D\bar{\boldsymbol{v}}}{Dt} + \bar{\rho}\,{}^{\theta}\bar{I}\left({}_{i}\bar{\boldsymbol{\omega}}\right)\cdot\frac{D_{i}\bar{\boldsymbol{\omega}}}{Dt} - \bar{\rho}\bar{\boldsymbol{F}}^{b}\cdot\bar{\boldsymbol{v}}\right)\mathrm{d}\bar{V}\end{aligned} \tag{68}$$

using Divergence Theorem (66) can be written as

$$\frac{D\bar{Q}}{Dt} = -\int_{\partial\bar{V}} \bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{n}}\,\mathrm{d}\bar{A} = \int_{\bar{V}} \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}}\,\mathrm{d}\bar{V} \tag{69}$$

using Cauchy principle for $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{M}}$ we can show that

$$\begin{aligned}\frac{D\bar{W}}{Dt} &= \int_{\partial\bar{V}}\left(\bar{\boldsymbol{P}}\cdot\bar{\boldsymbol{v}} + \bar{\boldsymbol{M}}\cdot{}_{i}^{t}\boldsymbol{\Theta}\right)\mathrm{d}\bar{A} \\ &= \int_{\partial\bar{V}}\left(\bar{\boldsymbol{v}}\cdot\left({}^{(0)}\bar{\boldsymbol{\sigma}}\right)^{\mathrm{T}}\cdot\bar{\boldsymbol{n}} + {}_{i}^{t}\boldsymbol{\Theta}\cdot\left({}^{(0)}\bar{\boldsymbol{m}}\right)\cdot\bar{\boldsymbol{n}}\right)\mathrm{d}\bar{A} \\ &= \int_{\partial\bar{V}}\left(\bar{\boldsymbol{v}}\cdot\left({}^{(0)}\bar{\boldsymbol{\sigma}}\right)^{\mathrm{T}} + {}_{i}^{t}\boldsymbol{\Theta}\cdot\left({}^{(0)}\bar{\boldsymbol{m}}\right)\right)\mathrm{d}\bar{\boldsymbol{A}}\end{aligned} \tag{70}$$

using Divergence Theorem

$$\frac{D\bar{W}}{Dt} = \int_{\bar{V}} \left(\bar{\nabla} \cdot \left(\bar{\boldsymbol{v}} \cdot \left({}^{(0)}\bar{\boldsymbol{\sigma}} \right)^{\mathrm{T}} \right) + \bar{\nabla} \cdot \left({}_i^t\bar{\boldsymbol{\Theta}} \cdot \left({}^{(0)}\bar{\boldsymbol{m}} \right)^{\mathrm{T}} \right) \right) \mathrm{d}\bar{V} \tag{71}$$

following reference [4], we can show

$$\bar{\nabla} \cdot \left(\bar{\boldsymbol{v}} \cdot \left({}^{(0)}\bar{\boldsymbol{\sigma}} \right)^{\mathrm{T}} \right) = \bar{\boldsymbol{v}} \cdot \left(\bar{\nabla} \cdot {}^{(0)}\bar{\boldsymbol{\sigma}} \right) + {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} \tag{72}$$

$$\bar{\nabla} \cdot \left({}_i^t\bar{\boldsymbol{\Theta}} \cdot \left({}^{(0)}\bar{\boldsymbol{m}} \right)^{\mathrm{T}} \right) = {}_i^t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\nabla} \cdot {}^{(0)}\bar{\boldsymbol{m}} \right) + {}^{(0)}\bar{\boldsymbol{m}} : {}_i^t{}^{\Theta}\bar{\boldsymbol{J}} \tag{73}$$

substituting from (72) and (73) in (71)

$$\frac{D\bar{W}}{Dt} = \int_{\bar{V}} \left(\bar{\boldsymbol{v}} \cdot \left(\bar{\nabla} \cdot {}^{(0)}\bar{\boldsymbol{\sigma}} \right) + {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} + {}_i^t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\nabla} \cdot {}^{(0)}\bar{\boldsymbol{m}} \right) + {}^{(0)}\bar{\boldsymbol{m}} : {}_i^t{}^{\Theta}\bar{\boldsymbol{J}} \right) \mathrm{d}\bar{V} \tag{74}$$

Substituting from (65), (66) and (74) in (64)

$$\begin{aligned} &\int_{\bar{V}} \bar{\boldsymbol{v}} \cdot \left(\bar{\rho} \frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\rho} \bar{\boldsymbol{F}}^b - \bar{\nabla} \cdot {}^{(0)}\bar{\boldsymbol{\sigma}} \right) \mathrm{d}\bar{V} + \int_{\bar{V}} \left(\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot \bar{\boldsymbol{q}} - {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} \right. \\ &\left. - {}^{(0)}\bar{\boldsymbol{m}} : {}_i^t{}^{\Theta}\bar{\boldsymbol{J}} - {}_i^t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\nabla} \cdot {}^{(0)}\bar{\boldsymbol{m}} \right) + \bar{\rho} \left({}^{\theta}\bar{I} \right) \left({}_i\bar{\boldsymbol{\varpi}} \cdot \frac{D_i\bar{\boldsymbol{\varpi}}}{Dt} \right) \right) \mathrm{d}\bar{V} = 0 \end{aligned} \tag{75}$$

Using balance of linear momenta (37) in (75) and grouping last two terms in the integrand we obtain (noting that ${}_i^t\bar{\boldsymbol{\Theta}} = {}_i\bar{\boldsymbol{\varpi}}$)

$$\begin{aligned} &\int_{\bar{V}} \left(\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot \bar{\boldsymbol{q}} - {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} - {}^{(0)}\bar{\boldsymbol{m}} : {}_i^t{}^{\Theta}\bar{\boldsymbol{J}} \right. \\ &\left. + {}_i\bar{\boldsymbol{\varpi}} \cdot \left(\bar{\rho} \left({}^{\theta}\bar{I} \right) \frac{D_i\bar{\boldsymbol{\varpi}}}{Dt} - \bar{\nabla} \cdot \left({}^{(0)}\bar{\boldsymbol{m}} \right) \right) \right) \mathrm{d}\bar{V} = 0 \end{aligned} \tag{76}$$

For isotropic, homogenous continua, $\bar{V}$ is arbitrary, hence we can set the integrand in (76) to zero.

$$\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot \bar{\boldsymbol{q}} - {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} - {}^{(0)}\bar{\boldsymbol{m}} : {}_i^t{}^{\Theta}\bar{\boldsymbol{J}} + {}_i\bar{\boldsymbol{\varpi}} \cdot \left(\bar{\rho} \left({}^{\theta}\bar{I} \right) \frac{D_i\bar{\boldsymbol{\varpi}}}{Dt} - \bar{\nabla} \cdot \left({}^{(0)}\bar{\boldsymbol{m}} \right) \right) = 0 \tag{77}$$

From balance of angular momenta

$$\bar{\rho} \left({}^{\theta}\bar{I} \right) \frac{D_i\bar{\boldsymbol{\varpi}}}{Dt} - \bar{\nabla} \cdot \left({}^{(0)}\bar{\boldsymbol{m}} \right) = \boldsymbol{\epsilon} : \left({}^{(0)}\bar{\boldsymbol{m}} \right) \tag{78}$$

Substituting from (78) into (77)

$$\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot \bar{\boldsymbol{q}} - {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} - {}^{(0)}\bar{\boldsymbol{m}} : {}_i^t{}^{\Theta}\bar{\boldsymbol{J}} + {}_i\bar{\boldsymbol{\varpi}} \cdot \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} \right) = 0 \tag{79}$$

Let

$$\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} = {}^{(0)}\bar{\boldsymbol{\tau}} \tag{80}$$

in which ${}^{(0)}\bar{\boldsymbol{\tau}}$ is a vector, containing three components, and noting that

$${}_i\bar{\boldsymbol{\varpi}} \cdot {}^{(0)}\bar{\boldsymbol{\tau}} = {}^{(0)}\bar{\boldsymbol{\tau}} \cdot {}_i\bar{\boldsymbol{\varpi}} \tag{81}$$

using (80) and (81) in (79) we obtain

$$\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot \bar{\boldsymbol{q}} - {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} - {}^{(0)}\bar{\boldsymbol{m}} : {}_i^t{}^{\Theta}\bar{\boldsymbol{J}} + {}^{(0)}\bar{\boldsymbol{\tau}} \cdot {}_i\bar{\boldsymbol{\varpi}} = 0 \tag{82}$$

the energy Equation (82) resulting from the first law of thermodynamics can be further simplified (shown below). We note the following,

$$\bar{\boldsymbol{L}} = \bar{\boldsymbol{D}} + \bar{\boldsymbol{W}} \tag{83}$$

$${}_{i}^{t}\Theta\bar{\boldsymbol{J}} = {}_{i}^{t}{}_{s}^{\Theta}\bar{\boldsymbol{J}} + {}_{i}^{t}{}_{a}^{\Theta}\bar{\boldsymbol{J}} \tag{84}$$

We consider decomposition of ${}^{(0)}\bar{\boldsymbol{\sigma}}$ and ${}^{(0)}\bar{\boldsymbol{m}}$ into symmetric and anti-symmetric parts

$$\begin{aligned} {}^{(0)}\bar{\boldsymbol{\sigma}} &= {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} + {}_{a}^{(0)}\bar{\boldsymbol{\sigma}} \\ {}^{(0)}\bar{\boldsymbol{m}} &= {}_{s}^{(0)}\bar{\boldsymbol{m}} + {}_{a}^{(0)}\bar{\boldsymbol{m}} \end{aligned} \tag{85}$$

$$\begin{aligned} {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} &= {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} + {}_{a}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{W}} \\ {}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{t}\Theta\bar{\boldsymbol{J}} &= {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{t}{}_{s}^{\Theta}\bar{\boldsymbol{J}} + {}_{a}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{t}{}_{a}^{\Theta}\bar{\boldsymbol{J}} \end{aligned} \tag{86}$$

Substituting (85)-(88) in (83) we can obtain

$$\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\nabla}\cdot\bar{\boldsymbol{q}} - {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}_{a}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{W}} - {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{t}{}_{s}^{\Theta}\bar{\boldsymbol{J}} - {}_{a}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{t}{}_{a}^{\Theta}\bar{\boldsymbol{J}} + {}^{(0)}\bar{\boldsymbol{\tau}} \cdot {}_{i}\bar{\boldsymbol{\varpi}} = 0 \tag{87}$$

We can show that

$${}^{(0)}\bar{\boldsymbol{\tau}} \cdot {}_{i}\bar{\boldsymbol{\varpi}} = {}_{a}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{W}} \tag{88}$$

using (89) in (88)

$$\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\nabla}\cdot\bar{\boldsymbol{q}} - {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{t}{}_{s}^{\Theta}\bar{\boldsymbol{J}} - {}_{a}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{t}{}_{a}^{\Theta}\bar{\boldsymbol{J}} = 0 \qquad \text{(FLT)} \tag{89}$$

This is the final form of the energy equation resulting from the first law of thermodynamics.

3.6. Second Law of Thermodynamics: SLT

If $\bar{\eta}$ is the entropy density in the volume $\bar{V}$, $\bar{h}$ is the entropy flux between $\bar{V}$ and the volume of matter surrounding it and $\bar{s}$ is the source of entropy in $\bar{V}$ due to non contacting sources (bodies), then the rate of increase of entropy in volume $\bar{V}$ is at least equal to that applied to $\bar{V}$ from all contacting and non-contacting sources [4]. Thus

$$\frac{D}{Dt}\int_{\bar{V}} \bar{\eta}\bar{\rho}\,\mathrm{d}\bar{V} \geq \int_{\partial\bar{V}} \bar{h}\,\mathrm{d}\bar{A} + \int_{\bar{V}} \bar{s}\,\bar{\rho}\,\mathrm{d}\bar{V} \tag{90}$$

using Cauchy's postulate for $\bar{h}$

$$\bar{h} = -\bar{\boldsymbol{\psi}}\cdot\bar{\boldsymbol{n}} \tag{91}$$

using (91) in (90)

$$\frac{D}{Dt}\int_{\bar{V}} \bar{\eta}\bar{\rho}\,\mathrm{d}\bar{V} \geq -\int_{\partial\bar{V}} \bar{\boldsymbol{\psi}}\cdot\bar{\boldsymbol{n}}\,\mathrm{d}\bar{A} + \int_{\bar{V}} \bar{s}\,\bar{\rho}\,\mathrm{d}\bar{V} \tag{92}$$

using Gauss' Divergence Theorem for the terms over $\partial\bar{V}$ gives (noting that $\bar{\boldsymbol{\psi}}$ is a tensor of rank one)

$$\frac{D}{Dt}\int_{\partial\bar{V}} \bar{\eta}\bar{\rho}\,\mathrm{d}\bar{V} \geq -\int_{\bar{V}} \bar{\nabla}\cdot\bar{\boldsymbol{\psi}}\,\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{s}\,\bar{\rho}\,\mathrm{d}\bar{V} \tag{93}$$

we note that

$$\frac{D}{Dt}\int_{\bar{V}}\bar{\eta}\left(\bar{\rho}\mathrm{d}\bar{V}\right)=\frac{D}{Dt}\int_{V}\eta\rho_0\mathrm{d}V=\int_{V}\frac{D\eta}{Dt}\rho_0\mathrm{d}V=\int_{\bar{V}}\frac{D\bar{\eta}}{Dt}\bar{\rho}\mathrm{d}\bar{V} \tag{94}$$

using (94) in (93) we obtain

$$\int_{\bar{V}}\left(\bar{\rho}\frac{D\bar{\eta}}{Dt}+\bar{\nabla}\cdot\bar{\boldsymbol{\psi}}-\bar{s}\,\bar{\rho}\right)\mathrm{d}\bar{V}\geq 0 \tag{95}$$

For homogeneous. isotropic matter volume $\bar{V}$ is arbitrary hence we can write the following from (95)

$$\bar{\rho}\frac{D\bar{\eta}}{Dt}+\bar{\nabla}\cdot\bar{\boldsymbol{\psi}}-\bar{s}\,\bar{\rho}\geq 0 \tag{96}$$

Equation (96) is the most fundamental form of the SLT or entropy inequality (Clausius Duhem inequality). We note that entropy inequality is strictly a statement that contains entropy terms, hence contains no information regarding reversible deformation physics. In this form (96) the entropy inequality provides no mechanism(s) for deriving constitutive theories. Only when the mechanical rate of work that results in rate of entropy production is introduced in the entropy inequality, will the entropy inequality contain information regarding conjugate pairs resulting in rate of entropy production. We also note entropy inequality (96) does not provide any information regarding constitutive theory for heat vector $\bar{\boldsymbol{q}}$. In the following we derive another form of the entropy inequality using a relationship between $\bar{\boldsymbol{\psi}}$ and $\bar{\boldsymbol{q}}$ and relationship between $\bar{\Phi}$, $\bar{e}$ and $\bar{\eta}$. Since the energy equation has all possible mechanisms that result in energy storage and dissipation, the form of entropy inequality derived using energy equation is expected to be helpful in the derivation of the constitutive theories. Using

$$\bar{\boldsymbol{\psi}}=\frac{\bar{\boldsymbol{q}}}{\bar{\theta}},\quad \bar{s}=\frac{\bar{r}}{\bar{\theta}} \tag{97}$$

where $\bar{\theta}$ is absolute temperature and $\bar{r}$ is a suitable potential

$$\bar{\nabla}\cdot\bar{\boldsymbol{\psi}}=\bar{\psi}_{i,i}=\frac{\bar{q}_{i,i}}{\bar{\theta}}-\frac{\bar{q}_i\bar{\theta}_{,i}}{\bar{\theta}^2}=\frac{\bar{\nabla}\cdot\bar{\boldsymbol{q}}}{\bar{\theta}}-\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}^2} \tag{98}$$

substituting from (98) into (96) and multiplying through by $\bar{\theta}$

$$\bar{\rho}\bar{\theta}\frac{D\bar{\eta}}{Dt}+\left(\bar{\nabla}\cdot\bar{\boldsymbol{q}}-\bar{\rho}\bar{r}\right)-\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}}\geq 0 \tag{99}$$

From energy Equation (89) (after including $\bar{s}\,\bar{\rho}$) term)

$$\bar{\nabla}\cdot\bar{\boldsymbol{q}}-\bar{s}\,\bar{\rho}=-\bar{\rho}\frac{D\bar{e}}{Dt}+{}_{s}^{(0)}\bar{\boldsymbol{\sigma}}:\bar{\boldsymbol{D}}+{}_{s}^{(0)}\bar{\boldsymbol{m}}:{}_{i}^{t}{}_{s}^{\Theta}\bar{\boldsymbol{J}}+{}_{a}^{(0)}\bar{\boldsymbol{m}}:{}_{i}^{t}{}_{a}^{\Theta}\bar{\boldsymbol{J}} \tag{100}$$

substituting (100) into (99) and using $\bar{\Phi}=\bar{e}-\bar{\eta}\bar{\theta}$ and regrouping terms

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt}+\bar{\eta}\frac{D\bar{\theta}}{Dt}\right)+\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}}-{}_{s}^{(0)}\bar{\boldsymbol{\sigma}}:\bar{\boldsymbol{D}}-{}_{s}^{(0)}\bar{\boldsymbol{m}}:{}_{i}^{t}{}_{s}^{\Theta}\bar{\boldsymbol{J}}-{}_{a}^{(0)}\bar{\boldsymbol{m}}:{}_{i}^{t}{}_{a}^{\Theta}\bar{\boldsymbol{J}}\leq 0\quad\text{(SLT)} \tag{101}$$

Equation (101) is the final form of the entropy inequality resulting from the second law of thermodynamics.

4. Complete Mathematical Model Resulting from CBL of NCCM

The system of partial differential equations and algebraic equations resulting from the conservation and balance laws of NCCM incorporating internal rotation rates and their material derivatives and rotational inertial effects are given by: conservation of mass (CM), balance of linear momenta (BLM), balance of angular momenta (BAM), balance of moment of moments (BMM), first law of thermodynamics (FLT) and the second law of thermodynamics (SLT). These are listed in the following using:

$$ {}_i\bar{\omega}_k = {}_i^t\bar{\boldsymbol{\Theta}}, \quad \frac{D\left({}_i\bar{\omega}_k\right)}{Dt} = \frac{D}{Dt}\left({}_i^t\bar{\Theta}_k\right), \quad \frac{D\left({}_i\bar{\omega}_k\right)}{Dt} = {}_i^t\dot{\bar{\boldsymbol{\Theta}}} \tag{102}$$

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho}\,\mathrm{div}\left(\bar{\boldsymbol{v}}\right) = 0 \qquad \text{(CM)} \tag{103}$$

$$\bar{\rho}\frac{\partial\bar{\boldsymbol{v}}}{\partial t} + \bar{\rho}\left(\bar{\boldsymbol{v}}\cdot\bar{\nabla}\right)\bar{\boldsymbol{v}} - \bar{\rho}\bar{\boldsymbol{F}}^b - \bar{\nabla}\cdot{}^{(0)}\bar{\boldsymbol{\sigma}} = 0 \qquad \text{(BLM)} \tag{104}$$

$${}^{\theta}\bar{I}\bar{\rho}\frac{D}{Dt}\left({}_i\bar{\omega}_k\right) - \epsilon_{ijk}\,{}^{(0)}\bar{\sigma}_{ij} - {}^{(0)}\bar{m}_{mk,m} = 0 \qquad \text{(BAM)} \tag{105}$$

$$\epsilon_{jkl}\left({}^{\theta}\bar{I}\bar{\rho}\bar{v}_j\left({}_i\bar{\omega}_k\right) - {}^{(0)}\bar{m}_{jk}\right) = 0 \qquad \text{(BMM)} \tag{106}$$

$$\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\nabla}\cdot\bar{\boldsymbol{q}} - {}_s^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}_s^{(0)}\bar{\boldsymbol{m}} : {}_i^t{}_s^{\Theta}\bar{\boldsymbol{J}} - {}_a^{(0)}\bar{\boldsymbol{m}} : {}_i^t{}_a^{\Theta}\bar{\boldsymbol{J}} = 0 \qquad \text{(FLT)} \tag{107}$$

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}} - {}_s^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}_s^{(0)}\bar{\boldsymbol{m}} : {}_i^t{}_s^{\Theta}\bar{\boldsymbol{J}} - {}_a^{(0)}\bar{\boldsymbol{m}} : {}_i^t{}_a^{\Theta}\bar{\boldsymbol{J}} \le 0 \quad \text{(SLT)} \tag{108}$$

Remarks

1. The mathematical model consists of eleven equations: CM (1), BLM (3), BAM (3), BMM(3), FLT (1) in twenty six dependent variables: $\bar{\rho}$ (1), $\bar{\boldsymbol{v}}$ (3), ${}^{(0)}\bar{\boldsymbol{\sigma}}$ (9), ${}^{(0)}\bar{\boldsymbol{m}}$ (9), $\bar{\boldsymbol{q}}$ (3), $\bar{\theta}$ (1), thus we need additional fifteen equations for the mathematical model to have closure. These additional equations are obtained from the constitutive theories.

2. We shall see that $\bar{\Phi}$, $\bar{\eta}$ and $\bar{e}$ are not dependent variables in the mathematical model as these can be expressed in terms of other dependent variables in remark (1).

3. From entropy inequality we can conclude the following.

(a) From the term $\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}}$, we conclude that $\bar{\boldsymbol{q}}, \bar{\boldsymbol{g}}$ is a conjugate pair.

(b) The term ${}_s^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}}$, suggests that ${}_s^{(0)}\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{D}}$ are rate of work (mechanical) conjugate pair. This is obviously due to classical continuum mechanics.

(c) The term ${}_s^{(0)}\bar{\boldsymbol{m}} : {}_i^t{}_s^{\Theta}\bar{\boldsymbol{J}}$ suggests that ${}_s^{(0)}\bar{\boldsymbol{m}}$ and ${}_i^t{}_s^{\Theta}\bar{\boldsymbol{J}}$ are also rate of work (mechanical) conjugate pair. This is the contribution of non-classical continuum mechanics incorporating internal rotation rates.

(d) From the term ${}_a^{(0)}\bar{\boldsymbol{m}} : {}_i^t{}_a^{\Theta}\bar{\boldsymbol{J}}$ it can be concluded that ${}_a^{(0)}\bar{\boldsymbol{m}}$, ${}_i^t{}_a^{\Theta}\bar{\boldsymbol{J}}$ are a rate of work (mechanical) conjugate pair. However, based on Surana *et al.* [1] in non-classical continuum mechanics the constitutive theory for ${}^{(0)}_{a}\bar{\boldsymbol{m}}$ (when ${}^{(0)}_{a}\bar{\boldsymbol{m}}$ is a possible choice of constitutive variable) leads to deformation physics that is non-physical. In reference [1] authors present constitutive theory for ${}^{(0)}_{a}\bar{\boldsymbol{m}}$ (in the absence of BMM balance law) as well as for ${}^{(0)}_{s}\bar{\boldsymbol{m}}$ and model problem studies to substantiate this issue. Based on reference [1], ${}^{(0)}_{a}\bar{\boldsymbol{m}}$ and ${}^{t}_{i}{}^{\Theta}_{a}\bar{\boldsymbol{J}}$ are not a conjugate pair, therefore ${}^{(0)}_{a}\bar{\boldsymbol{m}}$ is not a constitutive tensor. Thus, ${}^{(0)}_{a}\bar{\boldsymbol{m}} : {}^{t}_{i}{}^{\Theta}_{a}\bar{\boldsymbol{J}} = 0$ must be used as a constraint equation in the mathematical model.

4. From remark (3) we can conclude that it is possible to obtain the following additional equations through constitutive theories

(a) Constitutive theory for ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ (6);

(b) Constitutive theory for ${}^{(0)}_{s}\bar{\boldsymbol{m}}$ (6);

(c) Constitutive theory for $\bar{\boldsymbol{q}}$ (3).

This provides us with additional fifteen equations needed to provide closure to the mathematical model consisting of Equations (103)-(107).

5. In this paper, we consider compressible as well as incompressible thermoviscous fluent continua.

6. In a recent paper Surana *et al.* [9] presented non-classical continuum theory for thermoelastic solid continua (small deformation, small strain physics for homogeneous and isotropic) incorporating internal rotations with rotational inertial effects. Authors showed the existence of rotational waves similar to translational waves due to BAM when rotational inertial effects are considered. In this derivation the kinetic energy due to ${}_{i}\bar{\boldsymbol{\varpi}}$ was not considered *i.e.*, the term $\frac{1}{2}{}^{\theta}I_{0}\,{}_{i}\bar{\boldsymbol{\varpi}} \cdot {}_{i}\bar{\boldsymbol{\varpi}}$ (no sum over i) was neglected in the consideration of total energy per unit mass. A consequence of this was appearance of additional term ${}^{\theta}I_{0}\rho_{0}\left({}_{i}\bar{\boldsymbol{\varpi}} \cdot {}_{i}\bar{\boldsymbol{\varpi}}\right)$ in the energy equation and in the entropy inequality (see ref [9] Equations (75) and (76)). Thus, to satisfy SLT in the presence of this term, we must set ${}^{\theta}I_{0}\rho_{0}\left({}_{i}\bar{\boldsymbol{\varpi}} \cdot {}_{i}\bar{\boldsymbol{\varpi}}\right) = 0$ as additional constraint equation in the mathematical model.

7. In the derivation presented here for fluent continua, the kinetic energy due to angular velocities is accounted for in the total energy per unit mass (Equation (65)). This is obviously more complete physics describing kinetic energy. A consequence of this is the absence of the term ${}^{\theta}\bar{I}\bar{\rho}\left({}_{i}\bar{\boldsymbol{\varpi}} \cdot {}_{i}\bar{\boldsymbol{\varpi}}\right)$ (no sum over i) in the energy equation and the entropy inequality. This adjustment is beneficial in terms of more complete physics and is suggested and can be easily made in the works of reference [9] by simply neglecting the ${}^{\theta}I_{0}\rho_{0}\left({}_{i}\bar{\boldsymbol{\varpi}} \cdot {}_{i}\bar{\boldsymbol{\varpi}}\right)$ (no sum over i) term in the energy equation and the entropy inequality.

5. Constitutive Theories

The conjugate pairs in the entropy inequality (101) expressed in terms of Helmholtz free energy density are instrumental in determining the constitutive variables, their argument tensors as well as derivation of some constitutive theories. Choice of $\bar{\Phi}$, $\bar{\eta}$, ${}_s^{(0)}\bar{\boldsymbol{\sigma}}$, ${}_s^{(0)}\bar{\boldsymbol{m}}$ and $\bar{\boldsymbol{q}}$ as constitutive variables based on axioms of constitutive theories [4] [34], entropy inequality as well as the other balance laws is straightforward. The choice of some argument tensors of ${}_s^{(0)}\bar{\boldsymbol{\sigma}}$, ${}_s^{(0)}\bar{\boldsymbol{m}}$ and $\bar{\boldsymbol{q}}$ can be made based on conjugate pairs in the SLT. Additionally, temperature $\bar{\theta}$ is also required to be an argument tensor of all constitutive variables due to non-isothermal physics.

For compressible continua, density varies during evolution. Based on conservation of mass in Lagrangian description, changing density is defined by changing $[J]$, deformation gradient tensor.

$$|\bar{J}| = \frac{\rho_0}{\rho(\boldsymbol{x},t)}$$

Thus, $|J|$ or $\rho_0/\rho(\boldsymbol{x},t)$ or $1/\rho(\boldsymbol{x},t)$ must be argument tensor of the constitutive variables in Lagrangian description. In Eulerian description choice of $1/\rho(\boldsymbol{x},t)$ is replaced by $1/\bar{\rho}(\bar{\boldsymbol{x}},t)$ hence at the onset we begin with

$${}_s^{(0)}\bar{\boldsymbol{\sigma}} = {}_s^{(0)}\bar{\boldsymbol{\sigma}}\left(\frac{1}{\bar{\rho}}, \bar{\boldsymbol{D}}, \bar{\theta}\right) \tag{109}$$

$${}_s^{(0)}\bar{\boldsymbol{m}} = {}_s^{(0)}\bar{\boldsymbol{m}}\left(\frac{1}{\bar{\rho}}, {}_s^{t}{}_i^{\Theta}\bar{\boldsymbol{J}}, \bar{\theta}\right) \tag{110}$$

$$\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}\left(\frac{1}{\bar{\rho}}, \bar{\boldsymbol{g}}, \bar{\theta}\right) \tag{111}$$

The argument tensors of $\bar{\Phi}$ and $\bar{\eta}$ at this stage can be chosen using principle of equipresence [4] [34], we remark that principle of equipresence is not used in (110)-(112) as the conjugate pairs in entropy inequality specifically dictate the choice of argument tensors used and additionally $1/\bar{\rho}$ and $\bar{\theta}$.

$$\bar{\Phi} = \bar{\Phi}\left(\frac{1}{\bar{\rho}}, \bar{\boldsymbol{D}}, {}_s^{t}{}_i^{\Theta}\bar{\boldsymbol{J}}, \bar{\boldsymbol{g}}, \bar{\theta}\right) \tag{112}$$

$$\bar{\eta} = \bar{\eta}\left(\frac{1}{\bar{\rho}}, \bar{\boldsymbol{D}}, {}_s^{t}{}_i^{\Theta}\bar{\boldsymbol{J}}, \bar{\boldsymbol{g}}, \bar{\theta}\right) \tag{113}$$

The argument tensors of ${}_s^{(0)}\bar{\boldsymbol{\sigma}}$ can be enhanced to permit more comprehensive physics. Let $\gamma_{(i)}$; $i = 1,2,\cdots,n$ be the convected time derivatives of the Green's strain tensor $\boldsymbol{\varepsilon}_{(0)}$ (covariant basis) up to order n and let $\gamma^{(i)}$; $i = 1,2,\cdots,n$ be the convected time derivatives of the Almansi strain tensor $\bar{\boldsymbol{\varepsilon}}^{[0]}$ (contravariant basis) up to order n (see reference [4] for details). Then, we find that

$$\gamma_{(1)} = \gamma^{(1)} = \bar{\boldsymbol{D}} \tag{114}$$

i.e., $\bar{\boldsymbol{D}}$ is basis independent, however $\gamma_{(i)}$; $i = 2,3,\cdots,n$ and $\gamma^{(i)}$; $i = 2,3,\cdots,n$ are in covariant and in contravariant basis. Thus, we note that the first convected time derivative of $\boldsymbol{\varepsilon}_{[0]}$, *i.e.* $\bar{\boldsymbol{D}}$ or $\gamma_{(1)}$, is argument tensor of ${}_s^{(0)}\bar{\boldsymbol{\sigma}}$. The first convected time derivative of $\bar{\boldsymbol{\varepsilon}}^{[0]}$ *i.e.*, $\gamma_{(1)}$ is also equal to $\bar{\boldsymbol{D}}$. This suggests that perhaps a constitutive theory that considers convected time derivatives of $\boldsymbol{\varepsilon}_{[0]}$ or $\bar{\boldsymbol{\varepsilon}}^{[0]}$ up to order n is worthy of consideration. Thus $\bar{\boldsymbol{D}}$ can also be replaced by $\gamma^{(i)}$; $i = 1,2,\cdots,n$ or $\gamma_{(i)}$; $i = 1,2,\cdots,n$. The choice of

$\boldsymbol{\gamma}_{(i)}$; $i=1,2,\cdots,n$ (covariant basis) or $\boldsymbol{\gamma}^{(i)}$; $i=1,2,\cdots,n$ (contravariant basis) depends upon whether ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ is chosen to be ${}_{s}\bar{\boldsymbol{\sigma}}^{(0)}$ (contravariant measure) or ${}_{s}\bar{\boldsymbol{\sigma}}_{(0)}$ (covariant measure). To make the derivation basis independent we replace $\bar{\boldsymbol{D}}$ by ${}^{(i)}\boldsymbol{\gamma}$; $i=1,2,\cdots,n$ convected time derivative of the desired strain tensor. More specifically when

$$ {}^{(0)}_{s}\bar{\boldsymbol{\sigma}} = {}_{s}\bar{\boldsymbol{\sigma}}^{(0)}; \quad {}^{(i)}\boldsymbol{\gamma} = \boldsymbol{\gamma}_{(i)};\ i=1,2,\cdots,n \tag{115} $$

and when

$$ {}^{(0)}_{s}\bar{\boldsymbol{\sigma}} = {}_{s}\bar{\boldsymbol{\sigma}}_{(0)}; \quad {}^{(i)}\boldsymbol{\gamma} = \boldsymbol{\gamma}^{(i)};\ i=1,2,\cdots,n \tag{116} $$

when we replace $\bar{\boldsymbol{D}}$ in (110), (113) and (114) by ${}^{(i)}\boldsymbol{\gamma}$; $i=1,2,\cdots,n$ the resulting constitutive theory for ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ is referred to as ordered rate constitutive theory of order n. Thus now we have

$$ {}^{(0)}_{s}\bar{\boldsymbol{\sigma}} = {}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\left(\frac{1}{\bar{\rho}}, {}^{(i)}\boldsymbol{\gamma}, \bar{\theta}\right);\ i=1,2,\cdots,n \tag{117} $$

$$ {}^{(0)}_{s}\bar{\boldsymbol{m}} = {}^{(0)}_{s}\bar{\boldsymbol{m}}\left(\frac{1}{\bar{\rho}}, {}^{t}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}}, \bar{\theta}\right) \tag{118} $$

$$ \bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}\left(\frac{1}{\bar{\rho}}, \bar{\boldsymbol{g}}, \bar{\theta}\right) \tag{119} $$

$$ \bar{\Phi} = \bar{\Phi}\left(\frac{1}{\bar{\rho}}, {}^{(j)}\boldsymbol{\gamma}, {}^{t}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}}, \bar{\boldsymbol{g}}, \bar{\theta}\right);\ j=1,2,\cdots,n \tag{120} $$

$$ \bar{\eta} = \bar{\eta}\left(\frac{1}{\bar{\rho}}, {}^{(j)}\boldsymbol{\gamma}, {}^{t}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}}, \bar{\boldsymbol{g}}, \bar{\theta}\right);\ j=1,2,\cdots,n \tag{121} $$

In (117)-(121) we have the final choice of argument tensors of the constitutive variables. We can now obtain the material derivative of $\bar{\Phi}$ using (120)

$$ \begin{aligned} \frac{D\bar{\Phi}}{Dt} = \dot{\bar{\Phi}} = {} & \frac{\partial\bar{\Phi}}{\partial(1/\bar{\rho})}\frac{-1}{\bar{\rho}^2}\dot{\bar{\rho}} + \sum_{j=1}^{n}\frac{\partial\bar{\Phi}}{\partial\left({}^{(j)}\boldsymbol{\gamma}\right)} : \left({}^{(j)}\dot{\boldsymbol{\gamma}}\right) \\ & + \frac{\partial\bar{\Phi}}{\partial\left({}^{t}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}}\right)} : \left({}^{t}_{i}{}^{\Theta}_{s}\dot{\bar{\boldsymbol{J}}}\right) + \frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{g}}}\cdot\dot{\bar{\boldsymbol{g}}} + \frac{\partial\bar{\Phi}}{\partial\bar{\theta}}\dot{\bar{\theta}} \end{aligned} \tag{122} $$

From continuity Equation (35)

$$ \dot{\bar{\rho}} = -\bar{\rho}\bar{\nabla}\cdot\bar{\boldsymbol{v}} = -\bar{\rho}\bar{D}_{kk} = -\bar{\rho}\bar{D}_{kl}\delta_{lk} = -\bar{\rho}\boldsymbol{\delta} : \bar{\boldsymbol{D}} \tag{123} $$

Substituting from (123) in (122)

$$ \begin{aligned} \frac{D\bar{\Phi}}{Dt} = {} & \frac{\partial\bar{\Phi}}{\partial(1/\bar{\rho})}\boldsymbol{\delta} : \bar{\boldsymbol{D}} + \sum_{j=1}^{n}\frac{\partial\bar{\Phi}}{\partial\left({}^{(j)}\boldsymbol{\gamma}\right)} : \left({}^{(j)}\dot{\boldsymbol{\gamma}}\right) \\ & + \frac{\partial\bar{\Phi}}{\partial\left({}^{t}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}}\right)} : \left({}^{t}_{i}{}^{\Theta}_{s}\dot{\bar{\boldsymbol{J}}}\right) + \frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{g}}}\cdot\dot{\bar{\boldsymbol{g}}} + \frac{\partial\bar{\Phi}}{\partial\bar{\theta}}\dot{\bar{\theta}} \end{aligned} \tag{124} $$

we note that

$$ \frac{\partial\bar{\Phi}\left(\frac{1}{\bar{\rho}}\right)}{\partial\left(\frac{1}{\bar{\rho}}\right)} = \frac{\partial\bar{\Phi}(\bar{\rho})}{\partial\bar{\rho}}\left(-\bar{\rho}^2\right) \tag{125} $$

we can also make substitution from (125) in (124). After this substitution $\bar{\Phi} = \bar{\Phi}(\bar{\rho}, \cdots)$ and so are the remaining constitutive variables.

$$\frac{D\bar{\Phi}}{Dt} = -\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} : \bar{\boldsymbol{D}} + \sum_{j=1}^{n} \frac{\partial \bar{\Phi}}{\partial \left({}^{(j)}\boldsymbol{\gamma} \right)} : \left({}^{(j)}\dot{\boldsymbol{\gamma}} \right) + \frac{\partial \bar{\Phi}}{\partial \left({}^{t}_{s}{}^{\Theta}_{i} \bar{\boldsymbol{J}} \right)} : \left({}^{t}_{s}{}^{\Theta}_{i} \dot{\bar{\boldsymbol{J}}} \right) + \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{g}}} \cdot \dot{\bar{\boldsymbol{g}}} + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \dot{\bar{\theta}} \tag{126}$$

Substituting (126) in the entropy inequality (108) and regrouping terms

$$\left(-\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} - {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} \right) : \bar{\boldsymbol{D}} + \bar{\rho} \sum_{j=1}^{n} \frac{\partial \bar{\Phi}}{\partial \left({}^{(j)}\boldsymbol{\gamma} \right)} : \left({}^{(j)}\dot{\boldsymbol{\gamma}} \right) + \frac{\partial \bar{\Phi}}{\partial \left({}^{t}_{s}{}^{\Theta}_{i} \bar{\boldsymbol{J}} \right)} : \left({}^{t}_{s}{}^{\Theta}_{i} \dot{\bar{\boldsymbol{J}}} \right) + \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{g}}} \cdot \dot{\bar{\boldsymbol{g}}} + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \dot{\bar{\theta}} + \bar{\rho} \left(\frac{\partial \bar{\Phi}}{\partial \bar{\theta}} + \bar{\eta} \right) \dot{\bar{\theta}} - {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}^{t}_{s}{}^{\Theta}_{i} \bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \le 0 \tag{127}$$

For arbitrary but admissible ${}^{(j)}\dot{\gamma}$; $j = 1, 2, \cdots, n$, ${}^{t}_{s}{}^{\Theta}_{i} \dot{\bar{\boldsymbol{J}}}$, $\dot{\bar{\boldsymbol{g}}}$ and $\dot{\bar{\theta}}$ the entropy inequality (128) is satisfied if the following hold (*i.e.* their coefficients are set to zero).

$$\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \left({}^{(j)}\boldsymbol{\gamma} \right)} = 0 \Rightarrow \frac{\partial \bar{\Phi}}{\partial \left({}^{(j)}\boldsymbol{\gamma} \right)} = 0 \Rightarrow \bar{\Phi} \ne \bar{\Phi} \left({}^{(j)}\boldsymbol{\gamma} \right); \; j = 1, 2, \cdots, n \tag{128}$$

$$\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \left({}^{t}_{s}{}^{\Theta}_{i} \bar{\boldsymbol{J}} \right)} = 0 \Rightarrow \frac{\bar{\Phi}}{\partial \left({}^{t}_{s}{}^{\Theta}_{i} \bar{\boldsymbol{J}} \right)} = 0 \Rightarrow \bar{\Phi} \ne \bar{\Phi} \left({}^{t}_{s}{}^{\Theta}_{i} \bar{\boldsymbol{J}} \right) \tag{129}$$

$$\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{g}}} = 0 \Rightarrow \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{g}}} = 0 \Rightarrow \bar{\Phi} \ne \bar{\Phi} \left(\bar{\boldsymbol{g}} \right) \tag{130}$$

$$\bar{\rho} \left(\frac{\partial \bar{\Phi}}{\partial \bar{\theta}} + \bar{\eta} \right) = 0 \Rightarrow \frac{\partial \bar{\Phi}}{\partial \bar{\theta}} + \bar{\eta} = 0 \Rightarrow \bar{\eta} = -\frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \tag{131}$$

From (128)-(130) we can conclude that $\bar{\rho}$ and $\bar{\theta}$ are the only argument tensors of $\bar{\Phi}$. From (131) we conclude that $\bar{\eta}$ is not a constitutive variable as it is deterministic using $\partial \bar{\Phi} / \partial \bar{\theta}$. Using (128)-(131) the entropy inequality (127) reduces to

$$\left(-\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} - {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} \right) : \bar{\boldsymbol{D}} - {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}^{t}_{s}{}^{\Theta}_{i} \bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \le 0 \tag{132}$$

We remark that setting coefficient of $\bar{\boldsymbol{D}}$ in (132) to zero and obtaining

$$-\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} - {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} = 0 \tag{133}$$

$$-{}_{s}^{(0)}\bar{\boldsymbol{m}} : {}^{t}_{s}{}^{\Theta}_{i} \bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \le 0 \tag{134}$$

are inappropriate due to the fact that (133) implies that ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}$ is not a function of ${}^{(j)}\gamma$; $j = 1, 2, \cdots, n$ as $\bar{\Phi}$ is not a function of these which is invalid based on (117). Thus, at this stage, we must maintain entropy inequality in the form stated in (132). In order to proceed further, we consider decomposition of

Cauchy stress tensor ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ into equilibrium ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$ and deviatoric tensor ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ where ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$ causes change of volume without distortion and ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ causes distortion of volume without change of volume.

$$ {}^{(0)}_{s}\bar{\boldsymbol{\sigma}} = {}^{(0)}_{es}\bar{\boldsymbol{\sigma}} + {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} \tag{135} $$

Thus, we consider

$$ \begin{aligned} &{}^{(0)}_{es}\bar{\boldsymbol{\sigma}} = {}^{(0)}_{es}\bar{\boldsymbol{\sigma}}\left(\bar{\rho}, 0, \bar{\theta}\right) \\ &{}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} = {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}\left(\bar{\rho}, {}^{(j)}\boldsymbol{\gamma}, \bar{\theta}\right);\ j = 1, 2, \cdots, n \\ &\text{and } {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} = {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}\left(\bar{\rho}, 0, \bar{\theta}\right) = 0 \end{aligned} \tag{136} $$

The remaining constitutive variables and their argument tensors remain the same

$$ {}^{(0)}_{s}\bar{\boldsymbol{m}} = {}^{(0)}_{s}\bar{\boldsymbol{m}}\left(\frac{1}{\bar{\rho}}, {}^{t}_{s}{}^{\Theta}\bar{\boldsymbol{J}}, \bar{\theta}\right) \tag{137} $$

$$ \bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}\left(\frac{1}{\bar{\rho}}, \bar{\boldsymbol{g}}, \bar{\theta}\right) \tag{138} $$

$$ \bar{\Phi} = \bar{\Phi}\left(\bar{\rho}, \bar{\theta}\right) \tag{139} $$

5.1. Constitutive Theory for ${}^{(0)}_{es}\bar{\sigma}$: Compressible Continua

Substituting (136) in entropy inequality and regrouping terms

$$ \left(-\bar{\rho}^{2}\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\boldsymbol{\delta} - {}^{(0)}_{es}\bar{\boldsymbol{\sigma}}\right) : \bar{\boldsymbol{D}} - {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}^{(0)}_{s}\bar{\boldsymbol{m}} : {}^{t}_{s}{}^{\Theta}\bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}} \le 0 \tag{140} $$

Since $\bar{\Phi}$ is a function of $\bar{\rho}$ and $\bar{\theta}$ so is ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$. Thus ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$ can be determined by setting the coefficient of $\bar{D}_{kl}$ in the first term of (140) to zero

$$ {}^{(0)}_{es}\bar{\boldsymbol{\sigma}} = -\bar{\rho}^{2}\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\boldsymbol{\delta} \quad \text{or} \quad \left[{}^{(0)}_{es}\bar{\sigma}\right] = \bar{p}\left(\bar{\rho}, \bar{\theta}\right)[I] \tag{141} $$

in which

$$ \bar{p}\left(\bar{\rho}, \bar{\theta}\right) = -\bar{\rho}^{2}\frac{\partial\bar{\Phi}}{\partial\bar{\rho}} \tag{142} $$

where $\bar{p}\left(\bar{\rho}, \bar{\theta}\right)$ is thermodynamic pressure for compressible fluent continua and is defined by equation of state. The entropy inequality (140) reduces to

$$ -{}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}^{(0)}_{s}\bar{\boldsymbol{m}} : {}^{t}_{s}{}^{\Theta}\bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}} \le 0 \tag{143} $$

Entropy inequality (143) is satisfied if

$$ {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} \ge 0 \tag{144} $$

$$ {}^{(0)}_{s}\bar{\boldsymbol{m}} : {}^{t}_{s}{}^{\Theta}\bar{\boldsymbol{J}} \ge 0 \tag{145} $$

and

$$ \frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}} \le 0 \tag{146} $$

Inequalities (144) and (145) require that rate of work due to ${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}}$ and ${}_{s}^{(0)}\bar{\boldsymbol{m}}$ be positive and (146) serves as restriction on the constitutive theory for $\bar{\boldsymbol{q}}$.

5.2. Constitutive Theory for ${}_{es}^{(0)}\bar{\sigma}$: Incompressible Continua

For incompressible matter $\bar{\rho} = \rho_0$, constant, hence $\dfrac{\partial \bar{\Phi}}{\partial \bar{\rho}} = 0$. Thus the Constitutive theory for ${}_{es}^{(0)}\bar{\boldsymbol{\sigma}}$ cannot be derived using (141). The incompressibility condition must be enforced in the derivation of the constitutive theory for ${}_{es}^{(0)}\bar{\boldsymbol{\sigma}}$ by incorporating it in the entropy inequality. The incompressibility condition is given by continuity equation.

$$\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{v}} = \operatorname{tr}\left(\bar{\boldsymbol{D}}\right) = \bar{D}_{kk} = \bar{D}_{kl}\delta_{lk} = \boldsymbol{\delta} : \bar{\boldsymbol{D}} = 0 \tag{147}$$

Thus, we can add the following to the entropy inequality (140)

$$\bar{p}\left(\bar{\theta}\right)\boldsymbol{\delta} : \bar{\boldsymbol{D}} = 0 \tag{148}$$

$$\left(\bar{p}\left(\bar{\theta}\right)\boldsymbol{\delta} - {}_{es}^{(0)}\bar{\boldsymbol{\sigma}}\right) : \bar{\boldsymbol{D}} - {}_{ds}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{s}^{t}{}^{\Theta}\bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \leq 0 \tag{149}$$

Setting the coefficient of $\bar{\boldsymbol{D}}$ to zero in the first term of (149)

$${}_{es}^{(0)}\bar{\boldsymbol{\sigma}} = \bar{p}\left(\bar{\theta}\right)\boldsymbol{\delta} \quad \text{or} \quad \left[{}_{es}^{(0)}\bar{\sigma}\right] = \bar{p}\left(\bar{\theta}\right)[I] \tag{150}$$

where $\bar{p}\left(\bar{\theta}\right)$ is mechanical pressure. Since $\bar{p}\left(\bar{\theta}\right)$ is an arbitrary Lagrange multiplier, it is independent of the deformation field. The entropy inequality (149) reduces to (143) with conditions (144)-(146) that must be satisfied by the constitutive theories for ${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}_{s}^{(0)}\bar{\boldsymbol{m}}$ and $\bar{\boldsymbol{q}}$.

5.3. Constitutive Theory for ${}_{ds}^{(0)}\bar{\sigma}$

We consider (136)

$${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}} = {}_{ds}^{(0)}\bar{\boldsymbol{\sigma}}\left(\bar{\rho}, {}^{(j)}\boldsymbol{\gamma}, \bar{\theta}\right);\ j = 1, 2, \cdots, n \tag{151}$$

Pairs in (144) from entropy inequality confirm that ${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}}$ and ${}^{(j)}\boldsymbol{\gamma}$; $j = 1, 2, \cdots, n$ are rate of work conjugate. We derive constitutive theory for ${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}}$ using representation theorem [35]-[51]. Let ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}$; $i = 1, 2, \cdots, N_\sigma$ be the combined generators of the argument tensors of ${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}}$ that are symmetric tensors of rank two, then ${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}}$ can be expressed using linear combination of $\boldsymbol{I}$ and ${}^{(j)}\boldsymbol{\gamma}$; $i = 1, 2, \cdots, N_\sigma$ in the current configuration.

$${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}} = {}^{\sigma}\alpha^{0}\boldsymbol{I} + \sum_{i=1}^{N_\sigma} {}^{\sigma}\alpha^{i}\left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}\right) \tag{152}$$

In the linear combination (152), coefficients ${}^{\sigma}\alpha^{i}$; $i = 0, 1, \cdots, N_\sigma$ are functions of the combined invariants ${}^{\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \cdots, M_\sigma$ of the same argument tensors of ${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}}$ in (151), $\bar{\rho}$ and $\bar{\theta}$.

$${}^{\sigma}\alpha^{i} = {}^{\sigma}\alpha^{i}\left(\bar{\rho}, {}^{\sigma}\underset{\sim}{I}^{j}, \bar{\theta}\right); j = 1, 2, \cdots, M_\sigma; i = 0, 1, \cdots, N_\sigma \tag{153}$$

The material coefficients in the constitutive theory for ${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}}$ given by (151) are determined by considering Taylor series expansion of ${}^{\sigma}\alpha^i$; $i=0,1,\cdots,N_\sigma$ in ${}^{\sigma}\underset{\sim}{I}^j$; $j=1,2,\cdots,M_\sigma$ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${}^{\sigma}\underset{\sim}{I}^j$; $j=1,2,\cdots,M_\sigma$ (for simplicity). Taylor series expansion in $\bar{\theta}$ is not considered as the influence of thermal field on stress tensor has already been considered in the constitutive theory for ${}_{es}^{(0)}\bar{\boldsymbol{\sigma}}$ stress tensor.

$$ {}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{M_\sigma} \frac{\partial\, {}^{\sigma}\alpha^i}{\partial\left({}^{\sigma}\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j - {}^{\sigma}\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right);\ i=0,1,\cdots,N_\sigma \tag{154}$$

Substituting ${}^{\sigma}\alpha^i$; $i=0,1,\cdots,N_\sigma$ into (152) and collecting coefficients of the terms defined in the current configuration and introducing new notations for the coefficients.

$$ {}_{ds}^{(0)}\bar{\boldsymbol{\sigma}} = \underset{\sim}{\sigma}^{(0)}\Big|_{\underline{\Omega}} \boldsymbol{I} + \sum_{j=1}^{M_\sigma} {}^{\sigma}\underset{\sim}{a}_j \left({}^{\sigma}\underset{\sim}{I}^j\right)\boldsymbol{I} + \sum_{i=1}^{N_\sigma} {}^{\sigma}\underset{\sim}{b}_i \left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N_\sigma}\sum_{j=1}^{M_\sigma} {}^{\sigma}\underset{\sim}{c}_{ij}\left({}^{\sigma}\underset{\sim}{I}^j\right){}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i \tag{155}$$

Coefficients ${}^{\sigma}\underset{\sim}{a}_j$, ${}^{\sigma}b_i$ and ${}^{\sigma}\underset{\sim}{c}_{ij}$; $i=1,2,\cdots,N_\sigma$, $j=1,2,\cdots,M_\sigma$ and functions of $\bar{\rho}\big|_{\underline{\Omega}}$, ${}^{\sigma}\underset{\sim}{I}^j\big|_{\underline{\Omega}}$ and $\bar{\theta}\big|_{\underline{\Omega}}$; $j=1,2,\cdots,M_\sigma$. These are material coefficients.

Remarks

1. This constitutive theory for ${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}}$ given by (155) contains $(N+M+NM)$ material coefficients. This is non-linear ordered rate constitutive theory of order n for ${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}}$ and is based on integrity.

2. A simple linear constitutive theory in which products of ${}^{\sigma}\underset{\sim}{I}^j$, ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i$ and ($\bar{\theta}-\bar{\theta}\big|_{\underline{\Omega}}$) are neglected is given by

$$ {}_{ds}^{(0)}\bar{\boldsymbol{\sigma}} = \underset{\sim}{\sigma}^0\Big|_{\underline{\Omega}}\boldsymbol{I} + \sum_{i=1}^{n} 2\mu_i\, {}^{(i)}\boldsymbol{\gamma} + \sum_{i=1}^{n}\lambda_i\, {}^{(i)}\boldsymbol{\gamma} : \boldsymbol{I} \tag{156}$$

μ_i and λ_i are material coefficients for convected time derivative ${}^{(i)}\boldsymbol{\gamma}$ of the corresponding strain tensor. The constitutive theory (156) is also ordered rate constitutive theory of order n, but is linear in the components of $\left[{}^{i}\gamma\right]$; $i=1,2,\cdots,n$.

3. From (157) we can obtain the most simplified constitutive theory for ${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}}$ if we choose $n=1$ (rate constitutive theory of order one)

$$ {}_{ds}^{(0)}\bar{\boldsymbol{\sigma}} = \underset{\sim}{\sigma}^0\Big|_{\underline{\Omega}}\boldsymbol{I} + 2\mu_1\, {}^{(1)}\boldsymbol{\gamma} + \lambda_1\, {}^{(1)}\boldsymbol{\gamma} : \boldsymbol{I} \tag{157}$$

The constitutive theory (157) is Newton's law of viscosity for thermoviscous compressible fluids. We note that ${}^{(1)}\boldsymbol{\gamma} = \bar{\boldsymbol{D}}$, symmetric part of the velocity gradient tensor. $\sigma^{(0)}[I]$ initial stress field and the last term is due to thermal expansion or contraction. μ_1 and λ_1 are first and second viscosities. For incompressible fluent continua for ${}^{(1)}\boldsymbol{\gamma}=0$, due to continuity, hence the third term in (157) becomes zero.

5.4. Constitutive Theory for ${}_{s}^{(0)}\bar{\boldsymbol{m}}$

We consider (138)

$$ {}_{s}^{(0)}\bar{\boldsymbol{m}} = {}_{s}^{(0)}\bar{\boldsymbol{m}}\left(\bar{\rho},\, {}_{s}^{i\Theta}\bar{\boldsymbol{J}}, \bar{\theta}\right) \tag{158}$$

Let ${}^{m}\underset{\sim}{\boldsymbol{G}}^{i}$; $i=1,2,\cdots,N_m$ be the combined generators of the argument tensors of ${}^{(0)}\bar{\boldsymbol{m}}$ in (158) that are symmetric tensors of rank two. Then, based on representation theorem [35]-[51] we can express ${}^{(0)}_{s}\bar{\boldsymbol{m}}$ as a linear combination of $\boldsymbol{I}$ and ${}^{m}\underset{\sim}{\boldsymbol{G}}^{i}$; $i=1,2,\cdots,N_m$ in the current configuration.

$$ {}^{(0)}_{s}\bar{\boldsymbol{m}} = {}^{m}\alpha^{0}\boldsymbol{I} + \sum_{i=1}^{N_m} {}^{m}\alpha^{i}\,{}^{m}\underset{\sim}{\boldsymbol{G}}^{i} \tag{159} $$

The coefficients in the linear combination (159) are functions of $\bar{\rho}$, $\bar{\theta}$ and ${}^{m}\underset{\sim}{I}^{j}$; $j=1,2,\cdots,M_m$, the combined invariants of the same argument tensors of ${}^{(0)}_{s}\bar{\boldsymbol{m}}$ in (159).

In this particular case $N_m=2$ and $M_m=3$

$$ {}^{m}\underset{\sim}{\boldsymbol{G}}^{1} = {}^{t}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}};\quad {}^{m}\underset{\sim}{\boldsymbol{G}}^{2} = {}^{t}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}}^{2} \tag{160} $$

and

$$ {}^{m}\underset{\sim}{I}^{1} = I_{\left({}^{t}_{i}{}^{\Theta}_{s}\bar{J}\right)},\quad {}^{m}\underset{\sim}{I}^{2} = II_{\left({}^{t}_{i}{}^{\Theta}_{s}\bar{J}\right)},\quad {}^{m}\underset{\sim}{I}^{3} = III_{\left({}^{t}_{i}{}^{\Theta}_{s}\bar{J}\right)} \tag{161} $$

The material coefficients in the constitutive theory (159) for ${}^{(0)}_{s}\bar{\boldsymbol{m}}$ are determined by considering Taylor series expansion of ${}^{m}\alpha^{i}$; $i=0,1,\cdots,N_m$ in ${}^{m}\underset{\sim}{I}^{j}$; $j=1,2,\cdots,M_m$ about a known configuration $\underset{\sim}{\Omega}$ and retaining only up to linear terms in ${}^{m}\underset{\sim}{I}^{j}$; $j=1,2,\cdots,M_m$.

$$ {}^{m}\alpha^{i} = {}^{m}\alpha^{i}\Big|_{\underset{\sim}{\Omega}} + \sum_{j=1}^{M_m} \left.\frac{\partial\left({}^{m}\alpha^{i}\right)}{\partial\left({}^{t}_{i}{}^{\Theta}_{s}\bar{J}\right)}\right|_{\underset{\sim}{\Omega}} \left({}^{t}_{i}{}^{\Theta}_{s}\bar{J} - {}^{t}_{i}{}^{\Theta}_{s}\bar{J}\Big|_{\underset{\sim}{\Omega}} \right);\ i=0,1,\cdots,N_m \tag{162} $$

Substituting (162) in (159) and collecting coefficients of the terms defined in the current configuration and introducing new notation for the coefficients, we can write

$$ {}^{(0)}_{s}\bar{\boldsymbol{m}} = \underset{\sim}{m}^{0}\Big|_{\underset{\sim}{\Omega}}\boldsymbol{I} + \sum_{j=1}^{M_m} {}^{m}\underset{\sim}{\alpha}_{j}\left({}^{m}\underset{\sim}{I}^{j}\right)\boldsymbol{I} + \sum_{i=1}^{N_m} {}^{m}\underset{\sim}{b}_{i}\left({}^{m}\underset{\sim}{\boldsymbol{G}}^{i}\right) + \sum_{i=1}^{N_m}\sum_{j=1}^{M_m} {}^{m}\underset{\sim}{c}_{ij}\left({}^{m}\underset{\sim}{I}^{j}\right){}^{m}\underset{\sim}{\boldsymbol{G}}^{i} \tag{163} $$

Coefficients ${}^{m}\underset{\sim}{\alpha}_{j}$, ${}^{m}\underset{\sim}{b}_{i}$ and ${}^{m}\underset{\sim}{c}_{ij}$ are $(N+M+NM)$ material coefficients. These can be function of $\bar{\rho}\big|_{\underset{\sim}{\Omega}}$, ${}^{m}\underset{\sim}{I}^{j}$ and $\bar{\theta}\big|_{\underset{\sim}{\Omega}}$; $j=1,2,\cdots,M_m$.

Remarks

1. This constitutive theory (163) is obviously a non-linear constitutive theory based on integrity.

2. Since $N_m=2$ and $M_m=3$, this constitutive theory requires eleven material coefficients.

3. This constitutive theory contains up to fifth degree terms of the components of ${}^{t}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}}$.

4. A linear constitutive theory in the components of ${}^{t}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}}$ in which products of ${}^{m}\underset{\sim}{I}^{j}$ and ${}^{m}\underset{\sim}{\boldsymbol{G}}^{i}$ are neglected is given by

$$ {}^{(i)}_{s}\bar{\boldsymbol{m}} = \underset{\sim}{m}^{0}\Big|_{\underset{\sim}{\Omega}}\boldsymbol{I} + {}^{m}\underset{\sim}{\alpha}^{1}\left({}^{m}\underset{\sim}{I}^{j}\right)\boldsymbol{I} + {}^{m}\underset{\sim}{b}^{1}\left({}^{m}\underset{\sim}{G}^{1}\right) \tag{164} $$

Since

$$ {}^{m}\underset{\sim}{I}^{1} = \mathrm{tr}\left({}^{t}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}}\right) = 0 \tag{165} $$

the constitutive theory (163) reduces to

$$ {}^{(0)}_{s}\bar{\boldsymbol{m}} = \underset{\sim}{m}^{0}\Big|_{\underline{\Omega}} \boldsymbol{I} + {}^{m}\underset{\sim}{b}^{1}\left({}^{t}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}}\right) \tag{166}$$

A further simplified theory in which first term in (166) is neglected is given by (defining $\mu_m = {}^{m}\underset{\sim}{b}_{i}$)

$$ {}^{(0)}_{s}\bar{\boldsymbol{m}} = \mu_m \, {}^{t}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}} \tag{167}$$

in which

$$\mu_m = \mu_m \left(\bar{\rho}\big|_{\underline{\Omega}}, I_{\left({}^{t}_{i}{}^{\Theta}_{s}\bar{J}\right)}\Big|_{\underline{\Omega}}, II_{\left({}^{t}_{i}{}^{\Theta}_{s}\bar{J}\right)}\Big|_{\underline{\Omega}}, III_{\left({}^{t}_{i}{}^{\Theta}_{s}\bar{J}\right)}\Big|_{\underline{\Omega}}, \bar{\theta}\big|_{\underline{\Omega}} \right) \tag{168}$$

5.5. Constitutive Theory for $\bar{\boldsymbol{q}}$

We consider $\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}\left(\bar{\boldsymbol{g}}, \bar{\theta}\right)$ and use representation theorem [35]-[51]. The combined generators of the argument tensors $\bar{\boldsymbol{g}}$ and $\bar{\theta}$ that are tensors of rank one is just $\bar{\boldsymbol{g}}$ and the combined invariant is $\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}}$ (or ${}^{q}I$). Thus, the constitutive theory for $\bar{\boldsymbol{q}}$ in the current configuration can be written as

$$\bar{\boldsymbol{q}} = -{}^{q}\alpha \bar{\boldsymbol{g}} \tag{169}$$

in which

$${}^{q}\alpha = {}^{q}\alpha\left(\bar{\rho}, {}^{q}I, \theta\right) \tag{170}$$

The material coefficients in the constitutive theory for $\bar{\boldsymbol{q}}$ given by (169) are obtained by considering Taylor series expansion ${}^{q}\alpha$ in ${}^{q}I$ and $\bar{\theta}$ in a known configuration $\underline{\Omega}$ and retaining up to linear terms in ${}^{q}I$ and $\bar{\theta}$

$${}^{q}\alpha = {}^{q}\alpha\big|_{\underline{\Omega}} + \frac{\partial\, {}^{q}\alpha}{\partial\left({}^{q}I\right)}\bigg|_{\underline{\Omega}} \left({}^{q}I - {}^{q}I\big|_{\underline{\Omega}}\right) + \frac{\partial\left({}^{q}\alpha\right)}{\partial \bar{\theta}}\bigg|_{\underline{\Omega}} \left(\bar{\theta} - \bar{\theta}\big|_{\underline{\Omega}}\right) \tag{171}$$

Substituting (171) in (169) and collecting coefficients of the terms defined in current configuration gives the following (after introducing new coefficients)

$$\bar{\boldsymbol{q}} = -k\big|_{\underline{\Omega}} \bar{\boldsymbol{g}} - k_1\big|_{\underline{\Omega}} \left(\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}}\right)\bar{\boldsymbol{g}} - k_2\big|_{\underline{\Omega}} \left(\bar{\theta} - \bar{\theta}\big|_{\underline{\Omega}}\right)\bar{\boldsymbol{g}} \tag{172}$$

the materials coefficients k, k_1 and k_2 can be functions of $\bar{\rho}\big|_{\underline{\Omega}}$, ${}^{q}I\big|_{\underline{\Omega}}$ and $\bar{\theta}\big|_{\underline{\Omega}}$ This constitutive theory (172) based on integrity is non-linear constitutive theory in temperature gradient (contains up to cubic terms of temperature gradients). A linear constitutive theory for $\bar{\boldsymbol{q}}$ is given by

$$\bar{\boldsymbol{q}} = k\big|_{\underline{\Omega}} \bar{\boldsymbol{g}} \tag{173}$$

This is Fourier heat conduction law in which $k = k\left(\bar{\rho}\big|_{\underline{\Omega}}, {}^{q}I\big|_{\underline{\Omega}}, \bar{\theta}\big|_{\underline{\Omega}}\right)$ still holds.

6. Significance and Influence of Internal Rotation Rates and Rotational Inertial Effects

In this section, we discuss the influence of internal rotation rates and rotational inertial effects on the deformation physics of thermoviscous incompressible and compressible fluent continua. In thermoviscous fluent continua (both incompressible and compressible) fluid particles experience motion (displacements) but strains are negligible, hence such fluids are considered to have no elasticity. Thus, thermoviscous fluent continua cannot support propagation of waves of deviatoric Cauchy stress tensor in a similar fashion as solid continua does as this physics requires elasticity and mass ($\sqrt{E/\rho}$ is the wave speed in solids, E is elastic modulus and ρ is mass density). We discuss details of the deformation physics in the following for incompressible and compressible fluent continua in view of the present work.

6.1. Incompressible Thermoviscous Fluent Continua

It is well known that speed of sound in incompressible classical thermoviscous fluent continua (CCM) is infinity. In such fluids equilibrium Cauchy stress is mechanical pressure and/or thermal pressure field (is Lagrange multiplier) that cannot be determined from the deformation but its presence influences the flow physics. Deviatoric Cauchy stress tensor causes distortion of the volume of fluid as well as dissipation (as ${}^{(0)}\bar{\boldsymbol{\sigma}}$ is conjugate with $\bar{\boldsymbol{D}}$) that results in entropy production which in term influences thermal field.

Surana *et al.* [2] [3] have shown when CBL of NCCM with internal rotation rate physics (but without rotational internal effects) are employed, the presence of Cauchy moment tensor (symmetric based on BMM balance law) that is conjugate with the symmetric part of the gradients of rotation rate tensor results in added resistance to flow and additional entropy production that alters the thermal field due to the CBL of CCM. In the presence of internal rotations and inertial physics considered in this paper a part of the applied rate of work gets converted into kinetic energy due to angular velocities, thus effecting the rate of production of entropy which influences thermal field. Thus, rate of entropy production differs in the absence and in the presence of rotational inertial effects when using CBL of NNCM based on internal rotation rates. Model problem studies are in progress to compare with the results reported by Surana *et al.* (in the absence of rotational inertial effects) with those obtained using CBL of NCCM with rotational inertial effects considered in this paper.

Thus, in incompressible thermoviscous fluent continua we do not have translational or rotational waves due to Cauchy deviatoric stress tensor and Cauchy moment tensor (as in elastic solid continua [9]), instead the entropy production is affected by the additional rotational inertial physics. The entropy production due to CBL of NCCM is expected to be different depending upon the consideration of absence of rotational inertial effects. Interdependence of the different sources of entropy production and final total entropy productions will be reported in the model problem studies in a follow up paper.

6.2. Compressible Thermoviscous Fluent Continua

In compressible thermoviscous fluent continua (CCM) we also decompose symmetric Cauchy stress tensor ${}^{(0)}\bar{\boldsymbol{\sigma}}$ into equilibrium Cauchy stress tensor ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$ and deviatoric Cauchy stress tensor ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$. Equilibrium stress tensor ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$ is thermodynamic pressure $\bar{p}(\bar{\rho},\bar{\theta})$, the equation of state that is known for a given compressible fluid. The physics of change in volume *i.e.*, compressibility, is due to ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$ whereas the change in shape or distortion of the fluid volume and dissipation mechanism resulting in entropy production is due to ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$. In this physics ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{D}}$ are conjugate pairs. That is ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}:\bar{\boldsymbol{D}}$ is rate of work that causes change of shape and entropy production.

When a disturbance is applied to a compressible thermoviscous fluent continua, due to compressibility of the fluid, local compression of the medium occurs resulting in localized higher density. In other words, a localized compression wave is generated purely due to ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$ or $\bar{p}(\bar{\rho},\bar{\theta})$. The deviatoric stress merely causes localized entropy production. If the disturbance is weak, the resulting compression wave or pressure wave is also weak (small pressure disturbance) resulting in insignificant changes in local density. Thus, the compression wave behind the current compression wave, although moving in a slightly compressed medium, will move almost at the same speed as the wave ahead of it due to insignificant changes in density in the weak compression wave. In other words, in this physics progressively generated compression waves propagate at almost the same speed, hence no "piling up" of the compression waves occurs. This physics of compressible thermoviscous medium is generally referred to as sound waves. Since sound waves are weak compression waves that exist and move only because of compressibility of the medium their weak nature suggests that density changes and the entropy production are almost insignificant. In this physics, CBL of CCM are sufficient and there is hardly any need for CBL of NCCM with or without rotational inertial effects as in this physics entropy production is not significant to consider.

If the disturbance applied to a thermoviscous compressible fluent continua (CCM) is of significant strength such as the two compartments of a shock tube containing compressed gases with higher pressure ratio [52] separated by a diaphragm or high Mach number external flows, then the physics of evolution is quite different from sound waves and may require different considerations. We use this shock tube as an example to illustrate the significance of CBL of NCCM with internal rotation rate physics with or without rotational inertia. Let the shock tube be divided in two compartments of equal length by a diaphragm in the middle. Let ρ_1, p_1, θ_1 and ρ_2, p_2, θ_2 be the state of the gas in the left and the right compartments and let $p_2 \gg p_1$ and $\rho_2 \gg \rho_1$. We can assume $\theta_1 = \theta_2 = 0$ *i.e.*, both compartments at the same temperature at time $t = 0$. When the diaphragm is ruptured at time $t = 0$, a compression wave (pressure wave) is created to the left of the diaphragm with pressure values of p_1 and p_2 across the pressure wave. The base of the pressure wave is the order of viscosity of the medium ($O(10^{-6})$ meters for air). A consequence of this is the density wave with values of ρ_1 and ρ_2 with the same base as the pressure wave. The compression wave behind this wave when it reaches the compression zone will travel at a faster speed, hence will "pile up" on the waves ahead of it. This process of compression waves "piling up" on the waves ahead of them eventually creates a steady wave that no longer changes in time and propagates to the left of the diaphragm. This is a shock wave. In the compressed zone high velocity gradients $\bar{\boldsymbol{D}}$ and ${}_{d}^{(0)}\bar{\boldsymbol{\sigma}}$ (in CCM) results in entropy production which stabilizes once the shock wave is fully formed and remains constant during propagation. Reflection of the shock waves from the impermeable boundaries and the details of the physics can be found in reference [52]. The purpose of describing this problem in detail is to point out that this problem contains high pressure, high temperature physics with large changes in density in which determination of correct entropy production during the entire evolution is extremely important as it allows us to determine if the shocks are sustaining (entropy production remaining constant) or diffusing, indicated by diminishing entropy production. Secondly, rate of entropy production controls the evolution and formation of the shock wave.

When we consider CBL of NCCM with internal rotation physics we have additional mechanism of rate of entropy production due to (${}^{(0)}\bar{\boldsymbol{m}} : {}_{s}^{t}\dot{\boldsymbol{J}}$) which undoubtedly will influence the flow physics in the entire spatial domain over time. The physics of compression waves, hence the shock waves are expected to be influenced the most as the entropy production is most significant in the compression zone. Consideration of rotational inertial physics in conjunction with CBL of NCCM (as presented in this paper) will further influence rate of entropy production. This problem illustrates that in high pressure, high temperature compressible physics in thermoviscous fluent continua such as high Mach number flows, the use of CBL of NCCM with internal rotation rates with and without rotational inertial physics may be more realistic for describing the deformation physics compared to CBL of CCM used currently. Model problem studies in progress will be presented in a follow up paper.

7. Summary and Conclusions

In this paper conservation and balance laws of non-classical continuum mechanics with internal rotation rate physics [2] [3] and the constitutive theories for thermoviscous fluent continua are rederived by incorporating rotational inertia effects. In the evolution of deforming fluent continua, when the time varying rotation rates (angular velocities) and angular accelerations are resisted by the deforming continua, moments, angular momentum and angular inertial effects are realized.

The paper presents complete derivation of CBL and the constitutive theories in the presence of internal rotation rates due to $\bar{\boldsymbol{L}}$ and the rotational inertial effects. The paper considers homogeneous and isotropic thermoviscous fluent continua. We summarize the work and draw some conclusions in the following.

1. As in most non-classical continuum theories, the Cauchy stress tensor is

not symmetric in this work also.

2. In the non-classical continuum theories for fluent continua incorporating internal rotation rates [2] [3], the Cauchy moment tensor is symmetric as a consequence of the balance of moment of moments balance law [1]. In the present work BMM balance law does not establish symmetry of the Cauchy moment tensor, but yields three additional equations in ${}_{a}^{(0)}\bar{\boldsymbol{m}}$.

3. In the CBL presented here for NCCM with internal rotation rates and rotational inertial effects, BAM balance law is not just a relationship between the gradients of the Cauchy moment tensor and the skew symmetric Cauchy stress tensor, but additional contains rotational inertial effects.

4. Constitutive variables are established using SLT (in conjunction with other balance laws) and their argument tensors are determined using the conjugate pairs in the entropy inequality and the principle of equipresence.

5. It is shown that the constitutive theories are needed only for ${}_{e}^{(0)}\bar{\boldsymbol{\sigma}}, {}_{d}^{(0)}\bar{\boldsymbol{\sigma}}, {}_{s}^{(0)}\bar{\boldsymbol{m}}$ and $\bar{\boldsymbol{q}}$. Based on Surana *et al.* [1] there cannot be a constitutive theory for ${}_{a}^{(0)}\bar{\boldsymbol{m}}$. Thus, ${}_{a}^{(0)}\bar{\boldsymbol{m}} : {}_{a}^{t}{}_{i}^{\Theta}\dot{\bar{\boldsymbol{J}}} = 0$ must serve as a constrain equation in the mathematical model consisting of CBL and the constitutive theories to satisfy the entropy inequality for all arbitrary but admissible ${}_{a}^{t}{}_{i}^{\Theta}\dot{\bar{\boldsymbol{J}}}$.

6. Constitutive theory for ${}_{e}^{(0)}\bar{\boldsymbol{\sigma}}$, the equilibrium Cauchy stress tensor is derived using Helmholtz free energy density $\bar{\Phi}$ for compressible thermoviscous fluent continua. The constitutive theories for ${}_{d}^{(0)}\bar{\boldsymbol{\sigma}}$ and ${}_{s}^{(0)}\bar{\boldsymbol{m}}$ are derived using representation theorem. It is shown that the constitutive theory for $\bar{\boldsymbol{q}}$ based on integrity is cubic in the temperature gradient $\bar{\boldsymbol{g}}$.

7. Unlike non-classical solid continua, in fluent continua translational stress waves and rotational moment waves [1] can not exist as the fluent continua has no elasticity (translational or rotational). Thus, in fluent continua only the pressure waves can be realized.

8. It is shown that NCCM with internal rotation rate physics also results in rate of entropy production due to ${}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{s}^{t}{}_{i}^{\Theta}\dot{\bar{\boldsymbol{J}}}$ that differs in the absence and presence of rotational inertial effects. We also have rate of entropy production due to ${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}}$. Both mechanisms of entropy production exist in compressible as well as incompressible fluent continua. In high pressure, high temperature compressible flow physics (with or without shocks) accurate determination of rate of entropy production is important as it controls shock formation, shock structure and shock relations (in general, isolated high gradient physics of dependent variables).

9. The NCCM work proposed here with internal rotation rates and rotational inertial physics may be more realistic approach to describing the flow physics at high pressures and high temperatures that may result in a severe change in state of matter that is critically influenced by the rate of entropy production.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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