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Solving Smooth Generalized Equations Using Modified Gauss-Type Proximal Point Method

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Abstract

Consider X and Y are two real or complex Banach spaces. We introduce and study a modified Gauss-type proximal point algorithm (in short modified G-PPA) for solving the generalized equations of the form $0 \in h(x) + H(x)$, where $h: X \to Y$ is a smooth function on $\Omega \subseteq X$ and $H: X \rightrightarrows 2^Y$ is a set valued mapping with closed graph. When H is metrically regular and under some sufficient conditions, we analyze both semi-local and local convergence of the modified G-PPA. Moreover, the convergence results of the modified G-PPA are justified by presenting a numerical example.

Keywords

Set-Valued Mapping, Metrically Regular Mapping, Smooth Function, Lipschitz-Like Mapping, Semi-Local Convergence

1. Introduction

We are concerned with the problem of finding a point $x \in \Omega \subseteq X$ satisfying

$$0 \in h(x) + H(x), \tag{1.1}$$

where $h: X \to Y$ is a smooth function, that is, $h: X \to Y$ is a Fréchet differentiable function and $H: X \rightrightarrows 2^Y$ is a set valued mapping with closed graph acting between two different Banach spaces X and Y.

The generalized equation problem (1.1), for $h = 0$, was introduced by Robinson [1] [2] as a general tool for describing, analyzing, and solving different problems in a unified manner. This kind of generalized equation problems has been studied extensively. Various examples are system of inequalities, variational inequalities, linear and nonlinear complementary problems, system of nonlinear equations, equilibrium problems, etc.; see for examples, [3] [4] [5].

For solving (1.1), several iterative methods have been discussed such as Newton-type method, proximal point method, etc.; see in [6] [7] [8]. To solve (1.1) for the case $h=0$ and $Y=X$ a Hilbert space, the proximal point algorithm (PPA) is a very popular method. The origin of the PPA can be traced back in the works of Martinet [9] for variational inequalities. This PPA has been further refined and extended in [10] [11] [12] to a more general setting, including convex programs, convex-concave saddle point problems and variational inequality problems. Rockafellar [11] earnestly analyzed the PPA in the general structure of maximal monotone inclusions.

Let $T(\zeta_k, x)$ denotes the subset of X for all $x \in X$ and for some sequence of positive numbers ζ_k, which is characterized as follows:

$$T(\zeta_k, x) := \{t \in X : 0 \in \zeta_k t + h(x+t) + H(x+t)\}. \tag{1.2}$$

Dontchev and Rockafellar [13] planned the following proximal point algorithm for solving (1.1):

Algorithm 1. (PPA)

Step 1. Let $x_0 \in X$, $\zeta > 0$ and put $k := 0$.

Step 2. If $0 \in T(\zeta_k, x_k)$, then stop; otherwise, go to Step 3.

Step 3. Put $\{\zeta_k\} \subseteq (0, \zeta)$ and if $0 \notin T(\zeta_k, x_k)$, choose d_k such that $t_k \in T(\zeta_k, x_k)$.

Step 4. Write $x_{k+1} := x_k + t_k$.

Step 5. Set k by $k+1$ and go to Step 2.

By using some suitable conditions, the sequences generated by Dontchev and Rockafellar [13] are not uniquely defined and not every sequence is convergent. The results, obtained in [13], guarantee the existence of one sequence, which is linearly convergent to the solution. Hence, from the aspect of mathematical estimations, this type of method is not agreeable in mathematical utilizations. This barrier inspires us to nominate a method "so called" modified Gauss-type proximal point algorithm (modified G-PPA). The difference between **Algorithm 1** and our proposed **Algorithm 2** is that the modified G-PPA generates sequences, whose every sequence is convergent, but this does not happen for **Algorithm 1**.

Algorithm 2. (modified G-PPA)

Step 1. Let $\eta \geq 1$, $x_0 \in X$, $\zeta > 0$ and put $k := 0$.

Step 2. If $0 \in T(\zeta_k, x_k)$, then stop; otherwise, go to Step 3.

Step 3. Put $\{\zeta_k\} \subseteq (0, \zeta)$ and if $0 \notin T(\zeta_k, x_k)$, choose t_k such that $t_k \in T(\zeta_k, x_k)$ and $\|t_k\| \leq \eta \operatorname{dist}(0, T(\zeta_k, x_k))$.

Step 4. Write $x_{k+1} := x_k + t_k$.

Step 5. Set k by $k+1$ and go to Step 2.

We observe from **Algorithm 2**, that:

1) if $\eta = 1$ and $T(\zeta_k, x_k)$ is singleton, **Algorithm 2** matches with **Algo-**

rithm 1.

2) if $h=0$, and $Y=X$ a Banach space, **Algorithm 2** is equivalent to the Gauss-type proximal point method, which has been introduced by Rashid *et al.* [14].

3) if $\zeta_k u = g_k(u)$ a sequence of Lipschitz continuous functions and $t=0$, **Algorithm 2** is comparable to the general version of Gauss-type proximal point algorithm, which has been introduced by Alom *et al.* [5].

There has been a large study on semi-local analysis for some special cases such as Newton method for nonlinear least square problems [7], the extended Newton-type method for solving variational inclusions [15] and the Gauss-Newton method for convex inclusion problems [16]. For finding the solution to (1.1), Rashid *et al.* [8] introduced the Gauss-Newton type method and achieved the semi-local and local convergence results. In his sequential paper [3], Rashid introduced the Gauss-type proximal point method for finding the solution to variational inequality problem and obtained the semi-local and local convergence results. Alom *et al.* [5] have presented the general version of Gauss-type proximal point algorithm for solving (1.1) in the case $h=0$ and analyzed the semi-local and local convergence results. In recent times, Khatun and Rashid [4] have presented the extended Newton-type method for generalized equations with Holderian assumptions and obtained both semi-local and local convergence results.

Gauss-type proximal point method (G-PPA) is presented for variational inequalities, metrically regular mappings, etc; see for examples [3] [14]. We present the G-PPA for smooth generalized equations with some modifications in the main theorem, which is presented by Alom and Rashid [17] to prove the semi-local and local convergence results. We show that our method is more effective than the previous method for solving the smooth generalized equations. To the best of our knowledge, there is no study on semi-local analysis for solving (1.1) by using the modified Gauss-type proximal point method. Thus we conclude that the contributions, presented in this study seem new.

In the present paper, our aim is to study the semi-local convergence of the modified G-PPA defined by **Algorithm 2**. The vital tools in our study are the metric regularity property, which was introduced by Dontchev and Rockafellar [18], and Lipschitz-like property for set-valued mappings, whose concept was introduced by Aubin [19] [20]. Our fundamental results are the convergence principle, entrenched in Section 3, which, based on the information around the initial point, provide some sufficient conditions to assure the convergence to a solution of any sequence generated by **Algorithm 2**. As a consequence, local convergence results for the modified G-PPA are obtained.

This paper is arranged as follows: In Section 2, we recall some significant notations, concepts, some preliminary results and also recall a fixed point theorem which has been proved by Dontchev and Hager (cf. [21]). This fixed-point theorem is the vital mechanism to prove the existence of any sequence generated by **Algorithm 2**. In Section 3, we consider the modified G-PPA, which is introduced in this section, as well as the concept of metric regularity property and the Lipschitz-like property for set valued mappings to show the existence and the convergence of the sequence generated by **Algorithm 2**. To verify the convergence results of the modified G-PPA, we present a numerical example in Section 4. In the last section, we give a summary of the main results obtained in this paper.

2. Notations and Preliminary Results

Suppose X and Y are two real or complex Banach spaces. In this section, we present some notations and collect some results that will be helpful to prove our necessary results. The closed ball with centered at a and radius r is denoted by $\mathbb{B}_r(a)$. All the norms are denoted by $\|\cdot\|$. For each $x \in X$, the distance from a point x to a set $C \subseteq X$ is defined by $\operatorname{dist}(x,C) := \inf\{\|x-y\| : y \in C\}$, while the excess from the set $E \subseteq X$ to the set C is defined by $e(E,C) := \sup\{\operatorname{dist}(x,C) : x \in E\}$.

Let $H : X \rightrightarrows 2^Y$ be a set-valued mapping. Here $\operatorname{gph} H := \{(x,y) \in X \times Y : y \in H(x)\}$ is the graph of H and $\operatorname{dom} H := \{x \in X : H(x) \neq \varnothing\}$ is the domain of H. The inverse of H is denoted by H^{-1} and is defined by $H^{-1}(y) := \{x \in X : y \in H(x)\}$ for each $y \in Y$.

We recall the concept of metric regularity for a set valued mapping from [14], and have been studied extensively; see for examples [3] [6].

Definition 2.1. *Let* $H : X \rightrightarrows 2^Y$ *be a set-valued mapping and* $(x',y') \in \operatorname{gph} H$. *Let* $r_{x'} > 0, r_{y'} > 0$ *and* $\kappa > 0$. *Then H is said to be*

1) *metrically regular at* (x',y') *on* $\mathbb{B}_{r_{x'}}(x') \times \mathbb{B}_{r_{y'}}(y')$ *with constant* κ *if*

$$\operatorname{dist}(x, H^{-1}(y)) \le \kappa \operatorname{dist}(y, H(x)) \text{ for all } x \in \mathbb{B}_{r_{x'}}(x'), y \in \mathbb{B}_{r_{y'}}(y').$$

2) *metrically regular at* (x',y') *if there exist constants* $r'_{x'} > 0, r'_{y'} > 0$ *and* $\kappa' > 0$ *such that H is metrically regular at* (x',y') *on* $\mathbb{B}_{r'_{x'}}(x') \times \mathbb{B}_{r'_{y'}}(y')$ *with constant* κ'.

The notion of Lipschitz-like continuity for set-valued mapping is taken from [15]. This concept was introduced by Aubin [20] and has been studied extensively; see for examples [5] [8] [13].

Definition 2.2. *Let* $\gamma : Y \rightrightarrows 2^X$ *be a set-valued mapping and let* $(\bar{y},\bar{x}) \in \operatorname{gph}\gamma$. *Let* $r_{\bar{x}} > 0, r_{\bar{y}} > 0$ *and* $m > 0$. *Then* γ *is said to be Lipschitz-like at* $(\bar{y},\bar{x})$ *on* $\mathbb{B}_{r_{\bar{y}}}(\bar{y}) \times \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ *with constant m if the following inequality hold*:

$$e\left(\gamma(y_1) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), \gamma(y_2)\right) \le m\|y_1 - y_2\| \text{ for any } y_1, y_2 \in \mathbb{B}_{r_{\bar{y}}}(\bar{y}).$$

From [18], we take the following result which establishes the equivalence relation between metric regularity of a mapping H at $(\bar{x},\bar{y})$ and the Lipschitz-like continuity of the inverse H^{-1} at $(\bar{y},\bar{x})$. This result obtained from the idea in [12] [14].

Lemma 2.1. *Let* $H : X \rightrightarrows 2^Y$ *be a set valued mapping and* $(\bar{x},\bar{y}) \in \operatorname{gph} H$. *Let* $r_{\bar{x}} > 0, r_{\bar{y}} > 0$. *Then H is metrically regular at* $(\bar{x},\bar{y})$ *on* $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ *with constant L if and only if its inverse* $H^{-1} : Y \rightrightarrows 2^X$ *is Lipschitz-like at* $(\bar{y},\bar{x})$

on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})\times\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ *with constant L, that is, for all* $y, y' \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$,

$$e\left(H^{-1}(y)\cap\mathbb{B}_{r_{\bar{x}}}(\bar{x}), H^{-1}(y')\right)\le L\|y-y'\|.$$

We end this section with the following lemma. This lemma is known as Banach fixed point theorem which has been proved by Dontchev and Hagger in [21]. This lemma is used to prove the existence of any sequence generated by **Algorithm 2**.

Lemma 2.2. *Let* $\Psi: X \rightrightarrows 2^X$ *be a set-valued mapping. Let* $\eta_0 \in X$, $r\in(0,\infty)$ *and* $\alpha\in(0,1)$ *be such that*

$$\operatorname{dist}\left(\eta_0,\Psi(\eta_0)\right)<r(1-\alpha) \tag{2.1}$$

and

$$e\left(\Psi(x_1)\cap\mathbb{B}_r(\eta_0),\Psi(x_2)\right)\le\alpha\|x_1-x_2\| \text{ for any } x_1,x_2\in\mathbb{B}_r(\eta_0). \tag{2.2}$$

Then Ψ *has a fixed point in* $\mathbb{B}_r(\eta_0)$, *that is, there exists* $x\in\mathbb{B}_r(\eta_0)$ *such that* $x\in\Psi(x)$. *If* Ψ *is additionally single-valued, then the fixed point of* Ψ *in* $\mathbb{B}_r(\eta_0)$ *is unique.*

3. Convergence Analysis of the Modified G-PPA

Let $h: X\to Y$ be a smooth function on $\Omega\subseteq X$, and let $H: X\rightrightarrows 2^Y$ be a set valued mapping with closed graph, where X and Y are Banach spaces. Let $r_{\bar{x}}>0$, $r_{\bar{y}}>0$, $\lambda>0$ and $\kappa>0$ be such that $\lambda\kappa<1$. We define

$$r^* := \max\left\{\frac{2r_{\bar{x}}+\kappa r_{\bar{y}}}{1-\lambda\kappa},\frac{2\lambda r_{\bar{x}}+r_{\bar{y}}}{1-\lambda\kappa}\right\}. \tag{3.1}$$

From (3.1), it is obvious that $r_{\bar{x}}<r^*$ and $r_{\bar{y}}<r^*$.

We need the following lemma to establish our main result:

Lemma 3.1. *Let* $H: X\rightrightarrows 2^Y$ *be a set valued mapping which has locally closed graph at* $(\bar{x},\bar{y})\in\operatorname{gph}H$. *Let* r^* *be defined by* (3.1). *Let H be metrically regular at* $(\bar{x},\bar{y})$ *on* $\mathbb{B}_{r^*}(\bar{x})\times\mathbb{B}_{r^*}(\bar{y})$ *with constant* κ. *Let* $h: X\to Y$ *be Lipschitz continuous on* $\mathbb{B}_{r^*}(\bar{x})$ *with Lipschitz constant* λ *and* $h(\bar{x})=0$. *Then the mapping* $h+H$ *is metrically regular at* $(\bar{x},\bar{y})$ *on* $\mathbb{B}_{r_{\bar{x}}}(\bar{x})\times\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ *with constant* $\frac{\kappa}{1-\lambda\kappa}$.

Proof. We obtain according to our assumption on H that

$$\operatorname{dist}\left(x,H^{-1}(y)\right)\le\kappa\operatorname{dist}\left(y,H(x)\right) \text{ for all } x\in\mathbb{B}_{r^*}(\bar{x}), y\in\mathbb{B}_{r^*}(\bar{y}).$$

For all $x\in\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ and $y\in\mathbb{B}_{r_{\bar{y}}}(\bar{y})$, we will show that

$$\operatorname{dist}\left(x,(h+H)^{-1}(y)\right)\le\frac{\kappa}{1-\lambda\kappa}\operatorname{dist}\left(y,(h+H)(x)\right).$$

To finish this, we will proceed by induction on k and verify that there exists a sequence $\{x_k\}\subseteq\mathbb{B}_{r^*}(\bar{x})$, with $x_0=x$, such that, for $k=0,1,2,\cdots$, satisfies the following assertions:

$$x_{k+1}\in H^{-1}\left(y-h(x_k)\right) \tag{3.2}$$

and

$$\|x_{k+1}-x_k\|\le(\lambda\kappa)^k\|x_1-x\|. \tag{3.3}$$

It is clear that (3.3) is hold for $k=0$. Using the second condition in (3.1), we get $2\lambda r_{\bar{x}}+r_{\bar{y}}\le r^*(1-\lambda\kappa)$ and since $\lambda\kappa<1$, so $(1-\lambda\kappa)$ is positive, and hence $2\lambda r_{\bar{x}}+r_{\bar{y}}\le r^*$. This implies that $\lambda r_{\bar{x}}+r_{\bar{y}}\le r^*$. Thus, for all $x\in\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ and $y\in\mathbb{B}_{r_{\bar{y}}}(\bar{y})$, we have

$$\begin{aligned}\|(y-h(x))-\bar{y}\|&=\|y-\bar{y}+h(\bar{x})-h(x)\|\le\|h(x)-h(\bar{x})\|+\|y-\bar{y}\|\\&\le\lambda\|x-\bar{x}\|+\|y-\bar{y}\|\le\lambda r_{\bar{x}}+r_{\bar{y}}\le r^*.\end{aligned} \tag{3.4}$$

This shows that $y-h(x)\in\mathbb{B}_{r^*}(\bar{y})$. Since H has locally closed graph, there exists $x_1\in H^{-1}(y-h(x))$ with $x_0=x$ and it shows that (3.2) holds for $k=0$. Also, for the metric regularity condition of *H*, we can write

$$\|x_1-x\|\le\operatorname{dist}\left(x,H^{-1}(y-h(x))\right)\le\kappa\operatorname{dist}\left(y,(h+H)(x)\right). \tag{3.5}$$

Also,

$$\begin{aligned}\|x_1-x\|&=\|x_1-\bar{x}+\bar{x}-x\|\le\|x-\bar{x}\|+\|\bar{x}-x_1\|\le r_{\bar{x}}+\operatorname{dist}\left(\bar{x},H^{-1}(y-h(x))\right)\\&\le r_{\bar{x}}+\kappa\operatorname{dist}\left(y-h(x),H(\bar{x})\right)\le r_{\bar{x}}+\kappa\|y-\bar{y}\|+\kappa\|h(x)-h(\bar{x})\|\\&\le r_{\bar{x}}+\kappa r_{\bar{y}}+\lambda\kappa r_{\bar{x}}=(1+\lambda\kappa)r_{\bar{x}}+\kappa r_{\bar{y}}.\end{aligned} \tag{3.6}$$

Hence

$$\|x_1-\bar{x}\|\le\|x_1-x\|+\|x-\bar{x}\|\le(1+\lambda\kappa)r_{\bar{x}}+\kappa r_{\bar{y}}+r_{\bar{x}}=(2+\lambda\kappa)r_{\bar{x}}+\kappa r_{\bar{y}}. \tag{3.7}$$

As $\lambda\kappa<1$, from the first condition in (3.1) we have that

$$(2+\lambda\kappa)r_{\bar{x}}+\kappa r_{\bar{y}}<\frac{2r_{\bar{x}}+\kappa r_{\bar{y}}}{1-\lambda\kappa}\le r^*.$$

Thus, from (3.7), we write

$$\|x_1-\bar{x}\|\le r^*.$$

It shows that $x_1\in\mathbb{B}_{r^*}(\bar{x})$. By using (3.7) we can write that

$$\begin{aligned}\|(y-h(x_1))-\bar{y}\|&=\|y-\bar{y}+h(\bar{x})-h(x_1)\|\le\|y-\bar{y}\|+\|h(x_1)-h(\bar{x})\|\\&\le\|y-\bar{y}\|+\lambda\|x_1-\bar{x}\|\le r_{\bar{y}}+\lambda\left[(2+\lambda\kappa)r_{\bar{x}}+\kappa r_{\bar{y}}\right]\\&=2\lambda r_{\bar{x}}+r_{\bar{y}}+\lambda\kappa(\lambda r_{\bar{x}}+r_{\bar{y}}).\end{aligned} \tag{3.8}$$

We obtain from the second condition in (3.1) that $2\lambda r_{\bar{x}}+r_{\bar{y}}\le r^*(1-\lambda\kappa)$ and since $(1-\lambda\kappa)$ is positive, so $2\lambda r_{\bar{x}}+r_{\bar{y}}\le r^*$ implies that $\lambda r_{\bar{x}}+r_{\bar{y}}\le r^*$. Thus, from (3.8) we get

$$\|(y-h(x_1))-\bar{y}\|\le r^*(1-\lambda\kappa)+\lambda\kappa r^*=r^*.$$

This implies that $y-h(x_1)\in\mathbb{B}_{r^*}(\bar{y})$. As H has locally closed graph, there exists $x_2\in H^{-1}(y-h(x_1))$ and it is clear that (3.2) is true for $k=1$. Now, by using $x_0=x$ and the metric regularity condition of *H*, we can write

$$\begin{aligned}\|x_2-x\|&\le\operatorname{dist}\left(x,H^{-1}(y-h(x_1))\right)\le\kappa\operatorname{dist}\left(y-h(x_1),H(x)\right)\\&\le\kappa\operatorname{dist}\left(y-h(x_1),y-h(x)\right)\le\lambda\kappa\|x_1-x\|.\end{aligned} \tag{3.9}$$

We obtain from (3.6) and (3.9) that

$$\begin{aligned}\|x_2-\bar{x}\| &\le \|x_2-x\|+\|x-\bar{x}\| \le \lambda\kappa\|x_1-x\|+r_{\bar{x}} \le \lambda\kappa\left[(1+\lambda\kappa)r_{\bar{x}}+\kappa r_{\bar{y}}\right]+r_{\bar{x}} \\ &= \left(1+\lambda\kappa+(\lambda\kappa)^2\right)r_{\bar{x}}+(\lambda\kappa)\kappa r_{\bar{y}} = \frac{1}{1-\lambda\kappa}r_{\bar{x}}+(\lambda\kappa)\kappa r_{\bar{y}}.\end{aligned} \tag{3.10}$$

We obtain from (3.10), by using $\frac{1}{1-\lambda\kappa}<\frac{2}{1-\lambda\kappa}$ and $\lambda\kappa<\frac{1}{1-\lambda\kappa}$ for all values of $\lambda\kappa$ such that $\lambda\kappa<1$ and the first condition in (3.1), that

$$\|x_2-\bar{x}\| < \frac{2}{1-\lambda\kappa}r_{\bar{x}}+\frac{1}{1-\lambda\kappa}\kappa r_{\bar{y}} = \frac{2r_{\bar{x}}+\kappa r_{\bar{y}}}{1-\lambda\kappa} \le r^*.$$

It shows that $x_2\in\mathbb{B}_{r^*}(\bar{x})$. We obtain by using the metric regularity condition on H that

$$\begin{aligned}\|x_2-x_1\| &\le \operatorname{dist}\left(x_1,H^{-1}\left(y-h(x_1)\right)\right) \le \kappa\operatorname{dist}\left(y-h(x_1),H(x_1)\right) \\ &\le \kappa\operatorname{dist}\left(y-h(x_1),y-h(x)\right) \le \lambda\kappa\|x_1-x\|.\end{aligned}$$

Hence (2.3) is true for $k=1$. This shows that (3.2) and (3.3) are valid for $k=0,1$ for the constructed points x_1,x_2. Suppose $x_1,x_2,\cdots,x_n$ are constructed such that (3.2) and (3.3) are valid for $k=0,1,2,\cdots,n-1$. By induction hypothesis, we have to create x_{n+1} such that (3.2) and (3.3) are valid for $k=n$.

First, we will show that $x_i\in\mathbb{B}_{r^*}(\bar{x})$ for all $i=1,2,\cdots,n$. Thus from (3.3), we obtain that

$$\|x_i-x\| \le \sum_{j=0}^{i-1}\|x_{j+1}-x_j\| \le \sum_{j=0}^{i-1}(\lambda\kappa)^j\|x_1-x\| \le \frac{1}{1-\lambda\kappa}\|x_1-x\|. \tag{3.11}$$

Also, by using (3.11), (3.6) and the first condition in (3.1), we write that

$$\begin{aligned}\|x_i-\bar{x}\| &\le \|x_i-x\|+\|x-\bar{x}\| \le \frac{1}{1-\lambda\kappa}\|x_1-x\|+\|x-\bar{x}\| \\ &\le \frac{1}{1-\lambda\kappa}\left[(1+\lambda\kappa)r_{\bar{x}}+\kappa r_{\bar{y}}\right]+r_{\bar{x}} = \frac{2r_{\bar{x}}+\kappa r_{\bar{y}}}{1-\lambda\kappa} \le r^*.\end{aligned} \tag{3.12}$$

It shows that $x_i\in\mathbb{B}_{r^*}(\bar{x})$ for all $i=1,2,\cdots,n$. From (3.12) for $i=n$ and by using the second condition in (3.1), we obtain

$$\begin{aligned}\left\|\left(y-h(x_n)\right)-\bar{y}\right\| &\le \|y-\bar{y}\|+\|h(\bar{x})-f(x_n)\| \le \|y-\bar{y}\|+\lambda\|x_n-\bar{x}\| \\ &\le r_{\bar{y}}+\lambda\left(\frac{2r_{\bar{x}}+\kappa r_{\bar{y}}}{1-\lambda\kappa}\right) = \frac{2\lambda r_{\bar{x}}+r_{\bar{y}}}{1-\lambda\kappa} \le r^*.\end{aligned}$$

Hence $y-h(x_n)\in\mathbb{B}_{r^*}(\bar{y})$. Thus, there exists $x_{n+1}\in H^{-1}\left(y-h(x_n)\right)$ as H has locally closed graph. This implies that (3.2) is valid for $k=n$. We obtain by using the metric regularity condition on H that

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \operatorname{dist}\left(x_n,H^{-1}\left(y-h(x_n)\right)\right) \le \kappa\operatorname{dist}\left(y-h(x_n),H(x_n)\right) \\ &\le \kappa\operatorname{dist}\left(y-h(x_n),y-h(x_{n-1})\right) \le \kappa\left\|h(x_n)-h(x_{n-1})\right\| \\ &\le \lambda\kappa\|x_n-x_{n-1}\| \le (\lambda\kappa)^n\|x_1-x\|.\end{aligned} \tag{3.13}$$

Thus the induction steps are finished and therefore (3.2) and (3.3) are true for all *k*. By using $x_0=x$, we obtain from (3.13) that

$$\|x_{n+1}-x\| \le \sum_{i=0}^{n}\|x_{i+1}-x_i\| \le \sum_{i=0}^{n}(\lambda\kappa)^i\|x_1-x\| \le \frac{1}{1-\lambda\kappa}\|x_1-x\|. \quad (3.14)$$

We obtain from (3.14) and by using the relation $\frac{1}{1-\lambda\kappa}\|x_1-x\|+\|x-\bar{x}\| \le r^*$ from (3.12) that

$$\|x_{n+1}-\bar{x}\| \le \|x_{n+1}-x\|+\|x-\bar{x}\| \le \frac{1}{1-\lambda\kappa}\|x_1-x\|+\|x-\bar{x}\| \le r^*.$$

Therefore $x_{n+1}\in\mathbb{B}_{r^*}(\bar{x})$. As $\lambda\kappa<1$, we observe from (3.13) that the sequence $\{x_k\}$ is a Cauchy sequence, and all its elements are in $\mathbb{B}_{r^*}(\bar{x})$. So, this sequence converges to some $\hat{x}\in\mathbb{B}_{r^*}(\bar{x})$, that is, $\hat{x}=\lim_{k\to\infty}x_k$. Then taking limit in (3.2) and the local closedness of $\operatorname{gph}H$, satisfies $\hat{x}\in H^{-1}(y-h(\hat{x}))$, that is, $\hat{x}\in(h+H)^{-1}(y)$.

We obtain by using (3.3) and (3.5) that

$$\begin{aligned}\operatorname{dist}\left(x,(h+H)^{-1}(y)\right) &\le \|\hat{x}-x\| = \lim_{k\to\infty}\|x_k-x\| \le \lim_{k\to\infty}\sum_{i=0}^{k}\|x_{i+1}-x_i\| \\ &\le \lim_{k\to\infty}\sum_{i=0}^{k}(\lambda\kappa)^i\|x_1-x\| \le \frac{1}{1-\lambda\kappa}\|x_1-x\| \\ &\le \frac{\kappa}{1-\lambda\kappa}\operatorname{dist}\left(y,(h+H)(x)\right).\end{aligned}$$

Hence the proof of the Lemma 3.1 is completed. □

To proof our main result, we consider a sequence of scalars $\{\zeta_k\}\subseteq(0,\zeta)$.

Define a mapping $E_{(\zeta_k,x)}:X\to Y$ by

$$E_{(\zeta_k,x)}(\cdot)=-\zeta_k(\cdot-x), \quad (3.15)$$

and a set valued mapping $\Psi_{(\zeta_k,x)}:X\rightrightarrows 2^X$ by

$$\Psi_{(\zeta_k,x)}(\cdot)=(h+H)^{-1}\left[E_{(\zeta_k,x)}(\cdot)\right], \quad (3.16)$$

for each $x\in X$.

Our main theorem, which ensures the convergence of the modified G-PPA by using some sufficient conditions with initial point $\bar{x}$, gives as follows:

Theorem 3.1. *Suppose* $\eta>1$ *and that* $(h+H)$ *is metrically regular at* $(\bar{x},\bar{y})$ *on* $\mathbb{B}_{r_{\bar{x}}}(\bar{x})\times\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ *with constant* $\frac{\kappa}{1-\lambda\kappa}$ *and* $\operatorname{gph}(h+H)\cap\left(\mathbb{B}_{r_{\bar{x}}}(\bar{x})\times\mathbb{B}_{r_{\bar{y}}}(\bar{y})\right)$ *is closed. Let* $\delta>0$ *be such that*

(a) $\delta\le\min\left\{\frac{r_{\bar{x}}}{2},\frac{r_{\bar{y}}}{(2\eta+1)\zeta},1\right\}$,

(b) $3\eta\kappa\zeta+\lambda\kappa\le1$,

(c) $\|\bar{y}\|<\zeta\delta$.

Suppose that

$$\lim_{x\to\bar{x}}\operatorname{dist}\left(\bar{y},h(x)+H(x)\right)=0. \quad (3.17)$$

Then, with initial point $\bar{x}$, *there exists some* $\hat{\delta}>0$ *such that* **Algorithm 1** *generates at least one sequence and any generated sequence* $\{x_k\}$ *converges to* *a solution* $x^* \in \mathbb{B}_{\hat{\delta}}(\bar{x})$ *of* (1.1), *that is,* x^* *satisfies that* $0 \in h(x^*) + H(x^*)$.

Proof. Consider $0 < \hat{\delta} \le \delta$ such that

$$\operatorname{dist}\left(0, h(x_0) + H(x_0)\right) \le \zeta\delta \text{ for each } x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x}) \tag{3.18}$$

(such $\hat{\delta}$ exists by (3.17) and assumption (c)). It is enough to show that **Algorithm 2** generates at least one sequence and any generated sequence $\{x_k\}$ satisfies

$$\|x_k - \bar{x}\| \le 2\delta, \tag{3.19}$$

and

$$\|x_{k+1} - x_k\| \le \left(\frac{1}{2}\right)^{k+1} \delta. \tag{3.20}$$

We will proceed by induction method. For each $x \in X$, define

$$r_{(\zeta,x)} := \frac{3\kappa}{2(1-\lambda\kappa)}\left(\|\bar{y}\| + \zeta\|x - \bar{x}\|\right). \tag{3.21}$$

Using assumptions (b) and (c) with $\eta > 1$, we have

$$r_{(\zeta,x)} \le \frac{3\kappa}{2(1-\lambda\kappa)} 3\zeta\delta \le \frac{3}{2\eta}\delta < \frac{3}{2}\delta < 2\delta \text{ for each } x \in \mathbb{B}_{2\delta}(\bar{x}). \tag{3.22}$$

It is clear that (3.19) is true for $k = 0$. For showing that (3.20) is true for $k = 0$, it is sufficient to prove that the point x_1 exists, that is, $D(\zeta_0, x_0) \ne \varnothing$. By applying Lemma 2.2 to the mapping $\Psi_{(\zeta_0,x_0)}$ with $\eta_0 = \bar{x}$, $r := r_{(\lambda,x_0)}$ and $\alpha := \frac{1}{3}$, we have to show that $D(\zeta_0, x_0) \ne \varnothing$. Let us check that assertions (2.1) and (2.2) of Lemma 2.2 are satisfied with $\eta_0 = \bar{x}$, $r := r_{(\zeta,x_0)}$ and $\alpha := \frac{1}{3}$. Granting this, Lemma 2.2 is applicable to conclude that there exists a fixed point $\hat{x}_1 \in \mathbb{B}_{r_{(\zeta,x_0)}}(\bar{x})$ such that $\hat{x}_1 \in \Psi_{(\zeta_0,x_0)}(\hat{x}_1)$, which implies that $E_{(\zeta_0,x_0)}(\hat{x}_1) \in (h+H)(\hat{x}_1)$, that is, $0 \in \zeta_0(\hat{x}_1 - x_0) + (h+H)(\hat{x}_1)$.

Note that $\bar{x} \in (h+H)^{-1}(\bar{y}) \cap \mathbb{B}_{r_{(\zeta,x_0)}}(\bar{x})$. By using the definition of excess e with the mapping $\Psi_{(\zeta_0,x_0)}$ in (3.16) and using the relations $\mathbb{B}_{r_{(\zeta,x_0)}}(\bar{x}) \subseteq \mathbb{B}_{2\delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$, we have

$$\begin{aligned} \operatorname{dist}\left(\bar{x}, \Psi_{(\zeta_0,x_0)}(\bar{x})\right) &\le e\left((h+H)^{-1}(\bar{y}) \cap \mathbb{B}_{r_{(\zeta,x_0)}}(\bar{x}), \Psi_{(\zeta_0,x_0)}(\bar{x})\right) \\ &\le e\left((h+H)^{-1}(\bar{y}) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), (h+H)^{-1}\left[E_{(\zeta_0,x_0)}(\bar{x})\right]\right). \end{aligned} \tag{3.23}$$

From the second relation in assumption (a), since $(2\eta+1)\zeta\delta \le r_{\bar{y}}$, so $3\zeta\delta \le r_{\bar{y}}$ (as $\eta > 1$). Again, using assumption (c) with the relation $x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x}) \subseteq \mathbb{B}_{\delta}(\bar{x})$, we see that,

$$\begin{aligned} \left\|E_{(\zeta_0,x_0)}(\bar{x}) - \bar{y}\right\| &= \left\|-\zeta_0(\bar{x} - x_0) - \bar{y}\right\| \le \zeta_0\|x_0 - \bar{x}\| + \|\bar{y}\| \\ &\le \zeta\|x_0 - \bar{x}\| + \|\bar{y}\| \le 3\zeta\delta \le r_{\bar{y}}. \end{aligned} \tag{3.24}$$

This becomes $E_{(\zeta_0,x_0)}(\bar{x}) \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. Thus from (3.23), by using (3.24), (3.21) and Lemma 2.1, we observe that

$$\begin{aligned}\operatorname{dist}\left(\bar{x},\Psi_{(\zeta_0,x_0)}(\bar{x})\right)&\le\frac{\kappa}{1-\lambda\kappa}\left\|\bar{y}-E_{(\zeta_0,x_0)}(\bar{x})\right\|\\&\le\frac{\kappa}{1-\lambda\kappa}\left(\zeta\|x_0-\bar{x}\|+\|\bar{y}\|\right)\\&=\left(1-\frac{1}{3}\right)r_{(\zeta,x_0)}=(1-\alpha)r.\end{aligned}$$

This implies that assertion (2.1) of Lemma 2.2 is satisfied. Next, we show that assertion (2.2) of Lemma 2.2 is also satisfied. Suppose $x',x''\in\mathbb{B}_{r_{(\zeta,x_0)}}(\bar{x})$. Thus, by the first relation $2\delta\le r_{\bar{x}}$ from assumption (a) and $r_{(\zeta,x_0)}\le 2\delta$ from (3.22), we have $x',x''\in\mathbb{B}_{r_{(\zeta,x_0)}}(\bar{x})\subseteq\mathbb{B}_{2\delta}(\bar{x})\subseteq\mathbb{B}_{r_{\bar{x}}}(\bar{x})$. By the second relation in assumption (a), we get $3\zeta\delta\le r_{\bar{y}}$ (as $\eta>1$) and by using the assumption (c) with the relation $x_0\in\mathbb{B}_{\hat{\delta}}(\bar{x})\subseteq\mathbb{B}_{\delta}(\bar{x})$, we obtain that

$$\begin{aligned}\left\|E_{(\zeta_0,x_0)}(x')-\bar{y}\right\|&=\left\|-\zeta_0(x'-x_0)-\bar{y}\right\|\le\zeta\|x'-x_0\|+\|\bar{y}\|\\&\le\zeta\|x'-\bar{x}\|+\zeta\|\bar{x}-x_0\|+\|\bar{y}\|\le 3\zeta\delta\le r_{\bar{y}}.\end{aligned}$$

Thus $E_{(\zeta_0,x_0)}(x')\in\mathbb{B}_{r_{\bar{y}}}(\bar{y})$. Similarly, $E_{(\zeta_0,x_0)}(x'')\in\mathbb{B}_{r_{\bar{y}}}(\bar{y})$. Hence, by Lemma 2.1, we see that

$$\begin{aligned}&e\left(\Psi_{(\zeta_0,x_0)}(x')\cap\mathbb{B}_{r_{(\zeta,x_0)}}(\bar{x}),\Psi_{(\zeta_0,x_0)}(x'')\right)\\&\le e\left(\Psi_{(\zeta_0,x_0)}(x')\cap\mathbb{B}_{r_{\bar{x}}}(\bar{x}),\Psi_{(\zeta_0,x_0)}(x'')\right)\\&=e\left((h+H)^{-1}\left[E_{(\zeta_0,x_0)}(x')\right]\cap\mathbb{B}_{r_{\bar{x}}}(\bar{x}),(h+H)^{-1}\left[E_{(\zeta_0,x_0)}(x'')\right]\right)\\&\le\frac{\kappa}{1-\lambda\kappa}\left\|E_{(\zeta_0,x_0)}(x')-E_{(\zeta_0,x_0)}(x'')\right\|\\&=\frac{\kappa}{1-\lambda\kappa}\zeta_0\|x'-x''\|\le\frac{\zeta\kappa}{1-\lambda\kappa}\|x'-x''\|.\end{aligned}\tag{3.25}$$

Since $\eta>1$, by assumption (b) (3.25) becomes

$$e\left(\Psi_{(\zeta_0,x_0)}(x')\cap\mathbb{B}_{r_{(\zeta,x_0)}}(\bar{x}),\Psi_{(\zeta_0,x_0)}(x'')\right)\le\frac{1}{3\eta}\|x'-x''\|<\frac{1}{3}\|x'-x''\|=\alpha\|x'-x''\|.$$

It shows that assertion (2.2) of Lemma 2.2 is also satisfied. Thus, both the assertions of fixed point lemma 2.2 are satisfied. Thus, we can conclude that there exists a fixed point $\hat{x}_1\in\mathbb{B}_{r_{(\zeta,x_0)}}(\bar{x})$ such that $\hat{x}_1\in\Psi_{(\zeta_0,x_0)}(\hat{x}_1)$. Hence, $T(\zeta_0,x_0)\neq\varnothing$, and consequently, we can choose $t_0\in T(\zeta_0,x_0)$ such that

$$\|t_0\|\le\eta\operatorname{dist}\left(0,T(\zeta_0,x_0)\right)\le\eta r_{(\zeta,x_0)}\le 2\eta\delta.\tag{3.26}$$

Now, $x_1:=x_0+d_0$ is defined by **Algorithm 2** and by the definition of $T(\zeta_0,x_0)$, we obtain that

$$\begin{aligned}T(\zeta_0,x_0)&:=\left\{t_0\in X:0\in\zeta_0t_0+h(x_0+t_0)+H(x_0+t_0)\right\}\\&=\left\{t_0\in X:x_0+t_0\in(h+H)^{-1}(-\zeta_0d_0)\right\}.\end{aligned}$$

Hence we get

$$\operatorname{dist}\left(0, T\left(\zeta_0, x_0\right)\right)=\operatorname{dist}\left(x_0,(h+H)^{-1}\left(-\zeta_0 t_0\right)\right). \tag{3.27}$$

Using the second relation $(2\eta+1)\zeta\delta \le r_{\bar{y}}$ in assumption (a) and assumption (c) with the choice of t_0, we see that

$$\left\|-\zeta_0 t_0-\bar{y}\right\| \le \zeta\left\|t_0\right\|+\|\bar{y}\| \le 2\zeta\eta\delta+\zeta\delta \le r_{\bar{y}},$$

and so $-\zeta_0 t_0 \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. We obtain from (3.26) and (3.27) by using the metric regularity condition of $(h+H)$ at $(\bar{x},\bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\frac{\kappa}{1-\lambda\kappa}$ that

$$\begin{aligned}
\left\|t_0\right\| & \le \eta \operatorname{dist}\left(x_0,(h+H)^{-1}\left(-\zeta_0 t_0\right)\right) \\
& \le \frac{\eta\kappa}{1-\lambda\kappa} \operatorname{dist}\left(-\zeta_0 t_0, h\left(x_0\right)+H\left(x_0\right)\right) \\
& \le \frac{\eta\kappa}{1-\lambda\kappa}\left\|-\zeta_0 t_0-0\right\|+\frac{\eta\kappa}{1-\lambda\kappa}\left\|0-\left(h\left(x_0\right)+H\left(x_0\right)\right)\right\| \\
& \le \frac{\eta\kappa\zeta}{1-\lambda\kappa}\left\|t_0\right\|+\frac{\eta\kappa}{1-\lambda\kappa} \operatorname{dist}\left(0, h\left(x_0\right)+H\left(x_0\right)\right).
\end{aligned} \tag{3.28}$$

We obtain from (3.28) by using (3.18) that

$$\left\|t_0\right\| \le \frac{\eta\kappa\zeta}{1-\lambda\kappa}\left\|t_0\right\|+\frac{\eta\kappa}{1-\lambda\kappa}\zeta\delta. \tag{3.29}$$

Also, from (3.29) by using assumption (b), we get

$$\left\|x_1-x_0\right\|=\left\|t_0\right\| \le \frac{\eta\kappa\zeta}{1-\lambda\kappa-\eta\kappa\zeta}\delta \le \frac{1}{2}\delta.$$

Hence (3.20) is true for $k=0$. Consider (3.19) and (3.20) are true for $k=0,1,2,\cdots,n-1$ and assume that $x_1,\cdots,x_n$ are generated by **Algorithm 2**. Thus, we obtain

$$\left\|x_n-\bar{x}\right\| \le \sum_{i=0}^{n-1}\left\|x_{i+1}-x_i\right\|+\left\|x_0-\bar{x}\right\| \le \delta \sum_{i=0}^{n-1}\left(\frac{1}{2}\right)^{i+1}+\delta \le 2\delta. \tag{3.30}$$

Hence, (3.19) is true for $k=n$. Now, we have to show that there exists a point x_{n+1} such that (3.20) is true for $k=n$. In the similar way, we obtain by using **Algorithm 2** as we did for the case of $k=0$, that

$$\begin{aligned}
\left\|x_{n+1}-x_n\right\|=\left\|t_n\right\| & \le \eta \operatorname{dist}\left(x_n,(h+H)^{-1}\left(-\zeta_n t_n\right)\right) \\
& \le \frac{\eta\kappa}{1-\lambda\kappa} \operatorname{dist}\left(-\zeta_n t_n, h\left(x_n\right)+H\left(x_n\right)\right) \\
& \le \frac{\eta\kappa}{1-\lambda\kappa}\left\|-\zeta_n t_n-\left(h\left(x_n\right)+H\left(x_n\right)\right)\right\| \\
& \le \frac{\eta\kappa\zeta_n}{1-\lambda\kappa}\left\|t_n\right\|+\frac{\eta\kappa}{1-\lambda\kappa}\left\|h\left(x_n\right)+H\left(x_n\right)\right\| \\
& \le \frac{\eta\kappa\zeta}{1-\lambda\kappa}\left\|t_n\right\|+\frac{\eta\kappa\zeta_{n-1}}{1-\lambda\kappa}\left\|-\zeta_{n-1}\left(x_n-x_{n-1}\right)\right\| \\
& \le \frac{\eta\kappa\zeta}{1-\lambda\kappa}\left\|t_n\right\|+\frac{\eta\kappa\zeta}{1-\lambda\kappa}\left\|x_n-x_{n-1}\right\|
\end{aligned}$$

$$\begin{aligned}&\le \frac{\eta\kappa\zeta}{1-\lambda\kappa-\eta\kappa\zeta}\|x_n - x_{n-1}\| \\ &\le \frac{1}{2}\cdot\left(\frac{1}{2}\right)^n \delta = \left(\frac{1}{2}\right)^{n+1}\delta.\end{aligned} \tag{3.31}$$

Therefore (3.20) is true for $k=n$ and so (3.19) and (3.20) are true for all *k*. It shows that $\{x_k\}$ is a Cauchy sequence and hence it is convergent, say, to x^*. So, we obtain a point $x^* \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ such that $x^* := \lim_{k\to\infty}(x_k)$. Thus $0 \in h(x^*) + H(x^*)$ by the closedness of $\operatorname{gph}(h+H) \cap \left(\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})\right)$. Hence, the proof of theorem 3.1 is completed. □

Consider the special case where $\bar{x}$ is a solution of (1.1), that is, $\bar{y}=0$, Theorem 3.1 is reduced to the following corollary, which describes the local convergence of the sequence generated by **Algorithm 2**.

Corollary 3.1. *Suppose that* $\eta > 1$, $\zeta > 0$ *and that* $\bar{x}$ *satisfies* $0 \in h(\bar{x}) + H(\bar{x})$. *Assume that* $(h+H)$ *is metrically regular at* $(\bar{x},0)$ *which have locally closed graph at* $(\bar{x},0)$ *with constant* $\frac{\kappa}{1-\lambda\kappa}$. *Suppose that*

$$\lim_{x\to\bar{x}} \operatorname{dist}\left(0, h(x) + H(x)\right) = 0. \tag{3.32}$$

Choose a sequence of scalars $\{\zeta_k\} \subseteq (0,\zeta)$. *Then there exists* $\hat{\delta} > 0$ *such that any sequence* $\{x_k\}$ *generated by* **Algorithm 2** *with initial point* $x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x})$ *converges to a solution* x^* *satisfies that* $0 \in h(x^*) + H(x^*)$.

Proof. By our assumption $(h+H)$ is metrically regular at $(\bar{x},0)$ which have locally closed graph at $(\bar{x},0)$ with constant $\frac{\kappa}{1-\lambda\kappa}$. Then there exist constants $r_{\bar{x}} > 0$ and $r_0 > 0$ such that $(h+H)$ is metrically regular at $(\bar{x},0)$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_0}(0)$ with constant $\frac{\kappa}{1-\lambda\kappa}$, that is, the following inequality holds

$$\operatorname{dist}\left(x, (h+H)^{-1}(y)\right) \le \frac{\kappa}{1-\lambda\kappa} \operatorname{dist}\left(y, (h+H)(x)\right) \text{ for all } x \in \mathbb{B}_{r_{\bar{x}}}(\bar{x}), y \in \mathbb{B}_{r_0}(0).$$

Consider $\sup_k \zeta_k := \zeta \in (0,1)$ be such that $3\eta\kappa\zeta + \lambda\kappa \le 1$ and let $x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x})$. Since x_0 is very close to $\bar{x}$, then, for every y_0 near 0 such that $\operatorname{gph}(h+H)$ is locally closed at (x_0, y_0). Then (3.32) allow us to take $0 < \hat{\delta} \le \delta$ so that

$$\operatorname{dist}\left(0, h(x_0) + H(x_0)\right) \le \zeta\delta \quad \text{for each } x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x}).$$

Thus, for each $0 < r \le r_{\bar{x}}$ and $0 < \tilde{r} \le r_0$, one has that

$$\operatorname{dist}\left(x, (h+H)^{-1}(y)\right) \le \frac{\kappa}{1-\lambda\kappa} \operatorname{dist}\left(y, (h+H)(x)\right) \text{ for all } x \in \mathbb{B}_r(\bar{x}), y \in \mathbb{B}_{\tilde{r}_{\bar{y}}}(\bar{y}),$$

that is, $(h+H)$ is metrically regular at $(\bar{x},\bar{y})$ on $\mathbb{B}_r(\bar{x}) \times \mathbb{B}_{\tilde{r}_{\bar{y}}}(\bar{y})$ with constant $\frac{\kappa}{1-\lambda\kappa}$. Choose $0 < r_1 < \frac{r_{\bar{x}}}{2}$ and $0 < r_2 < \frac{r_0}{2}$ be such that

$$\min\left\{\frac{r_1}{2}, \frac{r_2}{(2\eta+1)\zeta}\right\} > 0.$$

Thus, we can choose $0<\delta\le 1$ such that

$$\delta \le \min\left\{\frac{r_1}{2},\frac{r_2}{(2\eta+1)\zeta}\right\}.$$

Now, it is routine to verify that all the assumptions in Theorem 3.1 are verified. Therefore to complete the proof of the corollary, we can apply Theorem 3.1. □

4. Numerical Experiment

To verify the semi-local convergence results of the modified G-PPA, a numerical example is presented in this section.

Example 4.1. *Let* $X=Y=\mathbb{R}, x_0=0.1, \eta=1.5, \zeta=0.3, \lambda=0.4$ and $\kappa=0.25$. *Define a differentiable function* h *on* $\mathbb{R}$ *by* $h(x)=-4x+2$ *and a set-valued mapping* H *on* $\mathbb{R}$ *by* $H(x)=\{8x-3,7x-1\}$. *Then* $h+H$ *is a set-valued mapping on* $\mathbb{R}$ *defined by* $h(x)+H(x)=\{4x-1,3x+1\}$. *Then* **Algorithm 2** *generates a sequence which converges to* $x^*=0.25$.

Consider $h(x)+H(x)=4x-1$ and $\sup_k \zeta_k := \zeta = 0.3$. Then it is clear from the statement that $h+H$ is metrically regular at $(-0.1,-0.6)\in \mathrm{gph}(h+H)$. From (1.2), we obtain that

$$T(\zeta_k,x_k)=\{t_k\in\mathbb{R}:0\in\zeta_k(t_k)+h(x_k+t_k)+H(x_k+t_k)\}$$
$$=\left\{t_k\in\mathbb{R}:t_k=\frac{1}{43}(10-40x_k)\right\}.$$

On the other hand, if $T(\zeta_k,x_k)\neq\varnothing$ we obtain that

$$0\in\zeta_k(x_{k+1}-x_k)+h(x_{k+1})+H(x_{k+1})\Rightarrow x_{k+1}=\frac{10+3x_k}{43}.$$

Thus from (3.31), we obtain that

$$\|t_k\|\le\frac{\eta\kappa\zeta}{1-\lambda\kappa-\eta\kappa\zeta}\|t_{k-1}\|.$$

The following **Figure 1** is the graphical representation of $h(x)+H(x)$.

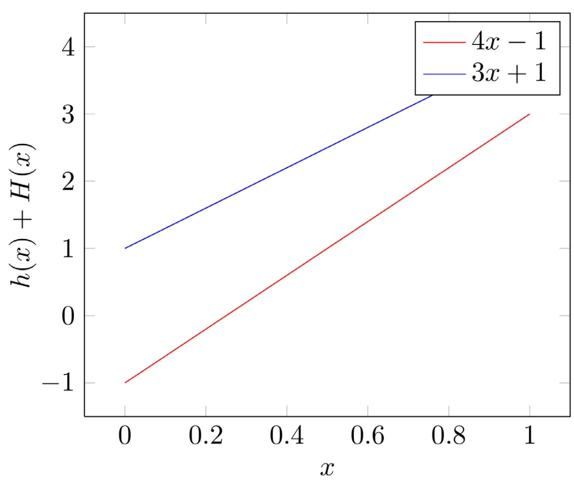

Figure 1. The graph of $h(x)+H(x)$.

Since $\frac{\eta\kappa\zeta}{1-\lambda\kappa-\eta\kappa\zeta}=\frac{0.1125}{0.8375}=0.13432836<1$ for the given values of η,ζ,κ and λ, thus we conclude that the sequence generated by **Algorithm 2** converges linearly, which confirms the semi-local convergence result of our algorithm. The following Table 1, obtained by using Matlab program, indicates that the solution of the generalized equation is 0.25 when $k=5$.

Table 1. Finding a solution of generalized equation.

x	$h(x)+H(x)$
−0.1000	−0.6000
0.2256	−0.0977
0.2483	−0.0068
0.2499	−0.0005
0.2500	0.0000
0.2500	−0.0000

5. Concluding Remarks

We have established semi-local and local convergence result for the modified G-PPA defined by **Algorithm 2** under the assumptions that h is a smooth function and H is metrically regular with $\eta>1$. We have presented a numerical example to justify the semi-local convergence of the modified G-PPA. If $\eta=1$ and $T(\zeta_k,x_k)$ is singleton, **Algorithm 2** is identical with the **Algorithm 1** introduced by Dontchev and Rockafellar [13]. This result extends and improves the result obtained in [3] [13].

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Robinson, S.M. (1979) Generalized Equations and Their Solutions, Part I: Basic Theory. *Mathematical Programming Study*, **10**, 128-141. https://doi.org/10.1007/BFb0120850

[2] Robinson, S.M. (1982) Generalized Equations and Their Solutions, Part II: Applications to Nonlinear Programming. *Mathematical Programming Study*, **19**, 200-221. https://doi.org/10.1007/BFb0120989

[3] Rashid, M.H. (2014) Convergence Analysis of Gauss-Type Proximal Point Method for Variational Inequalities. *Open Science Journal of Mathematics and Application*, **2**, 5-14.

[4] Khaton, M.Z. and Rashid, M.H. (2021) Extended Newton-Type Method for Generalized Equations with Hölderian Assumptions. *Journal of Communications in Advanced Mathematical Sciences*, **4**, 1-13. https://doi.org/10.33434/cams.738324

[5] Alom, M.A., Rashid, M.H. and Dey, K.K. (2016) Convergence Analysis of the General Version of Gauss-Type Proximal Point Method for Metrically Regular Mappings. *Journal of Applied Mathematics*, **7**, 1248-1259.

https://doi.org/10.4236/am.2016.711110

[6] Alom, M.A., Rashid, M.H. and Dey, K.K. (2017) Uniform Convergence of the General Version of Gauss-type Proximal Point Method for Metrically Regular Mappings. *British Journal of Mathematics and Computer Science*, **20**, 1-13. https://doi.org/10.9734/BJMCS/2017/31193

[7] He, J.S., Wang, J.H. and Li, C. (2007) Newton's Method for Undetermined Systems of Equations under the Modified γ-Condition. *Numerical Functional Analysis and Optimization*, **28**, 663-679. https://doi.org/10.1080/01630560701348509

[8] Rashid, M.H., Yu, S.H., Li, C. & S.Y. Wu, (2013) Convergence Analysis of the Gauss-Newton-Type Method for Lipschitz-Like Mappings. *Journal of Optimization Theory and Applications*, **158**, 216-233. https://doi.org/10.1007/s10957-012-0206-3

[9] Martinet, B. (1970) Régularisation d'inéquations variationnelles par approximations successives. *Revue Française d'Automatique*, *Informatique*, *Recherche Opérationnelle*, **4**, 154-158. https://doi.org/10.1051/m2an/197004R301541

[10] Aragón Artacho, F.J., Dontchev, A.L. and Geoffroy, M.H. (2007) Convergence of the Proximal Point Method for Metrically Regular Mappings. *ESAIM*: *Proceedings*, **17**, 1-8. https://doi.org/10.1051/proc:071701

[11] Rockafellar, R.T. (1976) Monotone Operators and the Proximal Point Algorithm. *SIAM Journal on Control and Optimization*, **14**, 877-898. https://doi.org/10.1137/0314056

[12] Rockafellar, R.T. and Wets, R.J.B. (1997) Variational Analysis. Springer, Berlin. https://doi.org/10.1007/978-3-642-02431-3

[13] Dontchev, A.L. and Rockafellar, R.T. (2009) Implicit Functions and Solution Mappings: A View from Variational Analysis. Springer, New York. https://doi.org/10.1007/978-0-387-87821-8

[14] Rashid, M.H., Wang, J.H. and Li, C. (2013) Convergence Analysis of Gauss-Type Proximal Point Method for Metrically Regular Mappings. *Journal of Nonlinear and Convex Analysis*, **14**, 627-635.

[15] Rashid, M.H. (2014) On the Convergence of Extended Newton-Type Method for Solving Variational Inclusions. *Cogent Mathematics*, **1**, Article ID: 980600. https://doi.org/10.1080/23311835.2014.980600

[16] Li, C. and Ng, K.F. (2007) Majorizing Functions and Convergence of the Gauss-Newton Method for Convex Composite Optimization. *SIAM Journal on Optimization*, **18**, 613-642. https://doi.org/10.1137/06065622X

[17] Alom, M.A. and Rashid, M.H. (2017) On the Convergence of Gauss-Type Proximal Point Method for Smooth Generalized Equations. *Asian Research Journal of Mathematics*, **2**, 1-15. https://doi.org/10.9734/ARJOM/2017/31288

[18] Dontchev, A.L. and Rockafellar, R.T. (2004) Regularity and Conditioning of Solution Mappings in Variational Analysis. *Set-Valued Analysis*, **12**, 79-109. https://doi.org/10.1023/B:SVAN.0000023394.19482.30

[19] Aubin, J.P. (1984) Lipschitz Behavior of Solutions to Convex Minimization Problems. *Mathematics of Operations Research*, **9**, 87-111. https://doi.org/10.1287/moor.9.1.87

[20] Aubin, J.P. and Frankowska, H. (1990) Set-Valued Analysis. Birkhäuser, Boston.

[21] Dontchev, A.L. and Hager, W.W. (1994) An Inverse Mapping Theorem for Set-Valued Maps. *Proceedings of the American Mathematical Society*, **121**, 481-498. https://doi.org/10.1090/S0002-9939-1994-1215027-7