Research Article

Characterization of Lagrangian Submanifolds by Geometric Inequalities in Complex Space Forms

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In this paper, we give an estimate of the first eigenvalue of the Laplace operator on a Lagrangian submanifold M^n minimally immersed in a complex space form. We provide sufficient conditions for a Lagrangian minimal submanifold in a complex space form with Ricci curvature bound to be isometric to a standard sphere $\mathbb{S}^n$. We also obtain Simons-type inequality for same ambient space form.

1. Introduction

In the last few years, there has been attentioned to the classification of Lagrangian submanifolds. Lagrangian submanifolds give an impression being of foliations in the cotangent bundle, and Hamilton-Jacobi type leads to the classification via partial differential equation. In differential geometry of submanifolds, theorems which connect the intrinsic and extrinsic curvatures have significant role in physics [1]. Moreover, the notion of second order differential equations (PDEs) has built essential contribution in the analyze problems in fluid mechanics, heat conduction in solids, diffusive transport of chemicals in porous media, and wave propagation in strings and in mechanics or solids. The eigenvalue problems are trying to obtain all possible real λ such that there exists a nontrivial solution φ to the second order partial differential equation (PDEs) $\Delta\varphi + \lambda\varphi = 0$ [2, 3]. On the other hand, the Ricci tensor is involving in the curvature spacetime, which finds the degree where matter will incline to converge or diverge in time (via the Raychaudhuri equation). By means of the Einstein field equation, it is also correlated to the matter content of the universe. In Riemannian geometry, on a Riemannian manifold, lower bounds of the Ricci tensor grant one to right geometric and topological understanding with the notion of a constant curvature space form. In Einstein manifold, the Ricci tensor verifies the vacuum Einstein

equation, which have been broadly studied in [4]. In this relation, the Ricci flow equation supervises the working out of a given metric to an Einstein metric. Similarly, the eigenvalue problems are fascinating topics in differential geometry which has physical background. Therefore, a distinguished problem in Riemannian geometry is to find isometrics on a given manifold. One of the most interesting geometries of Riemannian manifolds is to characterize complex space form in the framework of Lagrangian submanifold geometry among the classes of compact, connected Riemannian manifolds. Beginning from the originate work of Obata [5], differential equation has become an influential tool in the investigation of geometric analysis. Obata [5] tested characterizing theorem for the standard sphere. A complete manifold (M^n, g) yields function φ which is nonconstant and gratifying the ordinary differential equation

$$\nabla^2\varphi + \varphi g = 0, \tag{1}$$

if and only if (M^n, g) is isometric the sphere $\mathbb{S}^n$. A large scale of observations has been dedicated to this subject, and therefore, characterization of spaces, the Euclidean space $\mathbb{R}^n$, the Euclidean sphere $\mathbb{S}^n$, and the complex projective space $\mathbb{C}P^n$ are esteemed fields in differential geometry and are studied by a number of authors [2, 6–26]. Similarly, Tashiro [27] has proved that the Euclidean space $\mathbb{R}^n$ is designated through

the differential equation $\nabla^2 \varphi = cg$, where c is a positive constant. In [28], Lichnerowicz has been classified that the first nonzero eigenvalue μ_1 of the Laplacian on a compact manifold (M^n, g) with Ric $\geq n - 1$ is not less than n, while $\mu_1 = n$, then (M^n, g) is isometric to the sphere $\mathbb{S}^n$. This means that the Obata's rigidity theorem could be used to analyze the equality case of Lichnerowicz's eigenvalue estimates in [28]. Motivated from previously studied and historical development on such characterizations, we give our first result as the following.

Theorem 1. *Let $\Psi : M^n \longrightarrow \tilde{M}^n(4c)$ be a minimal immersion of a compact Lagrangian submanifold into complex space form $\tilde{M}^n(4c)$. If the Laplacian of M^n endowed to the first eigenvalue μ_1 corresponding eigenfunction φ, then the following inequality holds*

$$\int_M \sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV + c \int_M |\nabla\varphi|^2 dV \geq \int_M |\nabla^2\varphi|^2 dV, \quad (2)$$

where $|\nabla^2\varphi|^2$ denotes the norm of the Hessian of φ and $\{e_1, \cdots, e_n\}$ is frame on M^n which is orthonormal. The equality holds if and only if $\mu_1 = nc$. Besides, if the inequality holds

$$\int_M |\nabla^2\varphi|^2 dV \geq \int_M \sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV. \quad (3)$$

Then, we have $\mu_1 \geq c(n-1)$. In particular, if the following inequality satisfying

$$\int_M |\nabla\varphi|^2 dV \geq \frac{nc}{\mu_1} \int_M \sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV. \quad (4)$$

Then, eigenvalue is satisfied $\mu_1 \geq c(n-1)$.

By considered that compact submanifold M^n immersed in the Euclidean sphere $\mathbb{S}^{n+p}$ or Euclidean space $\mathbb{R}^{n+p}$, Jiancheng and Zhang [29] derived the Simons-type [30] inequalities about the first eigenvalue μ_1 and the squared norm of the second fundamental form S without using the condition that submanifold M is minimal. They also established a lower bound for S if it is constant. Similar results can be found in [4, 31]. Simon's inequalities and its corollary motivate the mathematicians try to improve the estimate the upper bound of S and study the rigidity of associated submanifolds. As a generalization of Euclidean sphere and Euclidean spaces, we consider a Lagrangian submanifold which minimally immersed into complex space form with constant holomorphic sectional curvature $4c$; we obtain our next result as the following.

Theorem 2. *Let $\Psi : M^n \longrightarrow \tilde{M}^n(4c)$ be a minimal immersion of a compact Lagrangian submanifold into the complex space form $\tilde{M}^n(4c)$. If dimKerh $= k$, then we have*

$$\int_M S|\nabla^2\varphi|^2 dV \geq \left\{ \frac{(n-k)(nc-1)(nc-\mu_1)}{(n-k-1)nc} \right\} \int_M |\nabla\varphi|^2 dV. \quad (5)$$

In circumstantial, if S is constant, then it is equal to

$$S \geq \frac{(n-k)(nc-1)}{nc(n-k-1))(nc-\mu_1)}. \quad (6)$$

A greatly motivated idea of Obata is associated to characterizing sphere $\mathbb{S}^n(c)$ through the second-order differential equation (1). By using the techniques of conformal vector field which have prominent appearance in deriving characterizations of spaces but also have high-level geometry in the theory of relativity and mechanics, Deshmukh and Al-Solamy [32] proved that an n-dimensional compact connected Riemannian manifold whose Ricci curvature satisfied the bound $0 < \text{Ric} \leq (n-1)(2 - nc/\mu_1)c$ for a constant c and μ_1 is the first nonzero eigenvalue of the Laplace operator; then, M^n is isometric to $\mathbb{S}^n(c)$ if M^n admitted a nonzero conformal gradient vector field. They also proved that if M^n is Einstein manifold such that Einstein constant $\mu = (n-1)c$, then M^n is isometric to $\mathbb{S}^n(c)$ with $c > 0$ if it is admitted conformal gradient vector field. Taking account of Obata equation (1), Barros et al. [20] shows that a compact gradient almost Ricci soliton $(M^n, g, \nabla\varphi, \lambda)$ is isometric to a Euclidean sphere whose Ricci tensor is Codazzi and has constant sectional curvature. For more terminology of Obata equation, see [14]. In the sequel, inspired by ideas are developed in [4, 29, 30]. So we give our result.

Theorem 3. *Let $\Psi : M^n \longrightarrow \tilde{M}^n(4c)$ be a minimal immersion of a compact Lagrangian submanifold into the complex space form $\tilde{M}^n(4c)$ and φ a first eigenfunction associated to the Laplacian of M^n. Then, we have the following:*

(i) *If $\nabla\varphi$ on Ker h, then $\Psi(M^n)$ is locally geodesic sphere $\mathbb{S}^n$, or $\Psi(M^n)$ is isometric to standard sphere $\mathbb{S}^n$*

(ii) *If $Ric_{M^n}(\nabla\varphi, \nabla\varphi) \geq c(n-1)|\nabla\varphi|^2$, then $\Psi(M^n)$ is isometric to a sphere $\mathbb{S}^n$*

The paper is organize as follows: In Section 2, we recall some preliminary formulas related to our study. Moreover, we also prove a proposition in this section which helps to derive our main results. In Section 3, we give the proofs of our theorems which we proposed in the first section. Finally, in Section 4, we provided some consequences of main results.

2. Preliminaries and Notations

Let $\tilde{M}(4c)$ be a complex space form of constant holomorphic sectional curvature $4c$ and of complex dimension m. Then, the curvature tensor R of $\tilde{M}^m(4c)$ can be expressed as:

$$R(U, V)Z = c(g(U, Z)V - g(V, Z)U + g(U, JZ)JV \\ - g(V, JZ)U + 2g(U, JV)JZ), \tag{7}$$

for any $U, V, Z \in \Gamma(T\tilde{M})$ [7, 33]. An n-dimensional Riemannian submanifold M^n of $\tilde{M}(4c)$ is classified as totally real if the standard complex structure J of $\tilde{M}(4c)$ maps any tangent space of M^n into the corresponding normal space [34]. In particular, a totally real submanifold is said to be a Lagrangian submanifold if $n = m$ (maximum dimension). Let $\{e_1, \cdots, e_{n+p}\}$ becoming an orthogonal frame to M^n; the second fundamental from h to M^n is given by

$$h(e_i, e_j) = \sum_{\alpha=1}^{n} \sigma_{ij}^{\alpha} e_{\alpha}, \tag{8}$$

where $\sigma_{ij}^{\alpha} = \langle A_{\alpha e_i}, e_j \rangle$ and A_{α} denote the shape operator. The Gauss equation for Lagrangian submanifold M^n in a complex space form $\tilde{M}^{n+p}(4c)$ in the form of local coordinates is given by

$$R_{jkl}^{i} = (\delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji})c + \sum_{\alpha=1}^{n} (\sigma_{ik}^{\alpha}\sigma_{jl}^{\alpha} - \sigma_{il}^{\alpha}\sigma_{jk}^{\alpha}). \tag{9}$$

Then, for Ricci curvature

$$R_{jij}^{i} = (\delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji})c + \sum_{\alpha=1}^{p} (\sigma_{ii}^{\alpha}\sigma_{jj}^{\alpha} - \sigma_{ij}^{\alpha}\sigma_{ji}^{\alpha}). \tag{10}$$

As we assumed that Ψ is an immersion which is minimal, (10) yields

$$\mathrm{Ric}(e_i, e_j) = (n-1)c\delta_{ij} - \sum_{\alpha=1}^{n} \sigma_{ik}^{\alpha}\sigma_{jk}^{\alpha}. \tag{11}$$

Let a function $\varphi : M^n \longrightarrow R$ established on a Riemannian manifold, then the Bochner formula (see, e.g., [2]) given as:

$$\frac{1}{2}\Delta|\nabla\varphi|^2 = |\nabla^2\varphi|^2 + \mathrm{Ric}_{M^n}(\nabla\varphi, \nabla\varphi) + g(\nabla\varphi, \nabla(\Delta\varphi)), \tag{12}$$

where Hessian is denoted by $\nabla^2\varphi$ and Ric denotes the Ricci curvature of M^n.

Now, we prove a proposition which authorizes to construct the proof of Theorems 1 and 2, that is the following:

Proposition 4. *Let $\Psi : M^n \longrightarrow \tilde{M}^n(4c)$ be an immersion of a compact Lagrangian submanifold into the complex space form $\tilde{M}^{n+p}(4c)$. Let φ be a first eigenfunction endowed to the Laplacian of M^n and Φ is minimal, then*

$$(nc - \mu_1)\int_M |\nabla\varphi|^2 dV = \int_M \sum_{i=1}^{n} |B(\nabla\varphi, e_i)|^2 dV \\ + c\int_M |\nabla\varphi|^2 dV - \int_M |\nabla^2\varphi|^2 dV. \tag{13}$$

For exceptional, we have

$$\int_M \sum_{i=1}^{n} |B(\nabla\varphi, e_i)|^2 dV = \int_M \left|\nabla^2\varphi + \frac{\mu_1}{n}\varphi I\right|^2 dV \\ + \left\{\frac{(n-1)(nc-\mu_1)}{n}\right\}\int_M |\nabla\varphi|^2 dV, \tag{14}$$

for any orthonormal frame $\{e_1, \cdots, e_n\}$ tangent to M^n.

Proof. If the identity operator on TM is denoted by I, then we have

$$\left|\nabla^2\varphi - t\varphi I\right|^2 = \left|\nabla^2\varphi\right|^2 - 2t\varphi\Delta\varphi + nt^2\varphi^2. \tag{15}$$

Therefore, if $\Delta\varphi + \mu\varphi = 0$, we obtain for any $t \in \mathbb{R}$. The norm of an operator which is given by $|I|^2 = tr(II^*)$. Taking integration in the above equation (15) and from Stokes theorem, we have

$$\int_M \left|\nabla^2\varphi - t\varphi I\right|^2 dV = \int_M \left|\nabla^2\varphi\right|^2 dV + \left(2t + \frac{n}{\mu_1}t^2\right)\int_M |\nabla\varphi|^2 dV. \tag{16}$$

We setting $t = -\mu_1/n$ in (16), we get

$$\int_M \left|\nabla^2\varphi\right|^2 dV = \int_M \left|\nabla^2\varphi + \frac{\mu_1}{n}\varphi I\right|^2 dV + \frac{\mu_1}{n}\int_M |\nabla\varphi|^2 dV. \tag{17}$$

On other hand, (11) yields

$$\mathrm{Ric}(\varphi_i e_i, \varphi_j e_j) = (n-1)c\delta_{ij}\varphi_i\varphi_j - \sum_{\alpha=1}^{p}\sum_{k=1}^{n} \sigma_{ik}^{\alpha}\sigma_{jk}^{\alpha}\varphi_i\varphi_j. \tag{18}$$

Tracing the above equation, we obtain

$$\mathrm{Ric}(\nabla\varphi, \nabla\varphi) = c(n-1)|\nabla\varphi|^2 - \sum_{i=1}^{n} |B(\nabla\varphi, e_i)|^2. \tag{19}$$

Let us assume that $\Delta\varphi = -\mu_1\varphi$. Taking integration in Bochner formula and using Stokes theorem, we get

$$\int_M \left|\nabla^2\varphi\right|^2 dV + \int_M \mathrm{Ric}(\nabla\varphi, \nabla\varphi)dV = \mu_1 \int_M |\nabla\varphi|^2 dV. \tag{20}$$

From (19), we conclude

$$(cn - \mu_1) \int_M |\nabla\varphi|^2 dV = \int_M \sum_{i=1}^n |B(\nabla\varphi, e_i)|^2 dV$$
$$+ c \int_M |\nabla\varphi|^2 dV - \int_M |\nabla^2\varphi|^2 dV. \tag{21}$$

This is the first result of the proposition. On the other hand, using (17) in the last equality, we obtain

$$(cn - \mu_1) \int_M |\nabla\varphi|^2 dV$$
$$= \int_M \sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV + c \int_M |\nabla\varphi|^2 dV \tag{22}$$
$$- \int_M \left|\nabla^2\varphi + \frac{\mu_1}{n}\varphi I\right|^2 dV - \frac{\mu_1}{n} \int_M |\nabla\varphi|^2 dV.$$

After some computation, we get

$$\int_M \sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV = \int_M \left|\nabla^2\varphi + \frac{\mu_1}{n}\varphi I\right|^2 dV$$
$$+ \left\{\frac{(n-1)(nc - \mu_1)}{n}\right\} \int_M |\nabla\varphi|^2 dV. \tag{23}$$

Now, we have reached the proof of the proposition. □

Recall the following lemma which set up to eliminate the proof of Theorem 2.

Lemma 5 [4]. *Let a valid symmetric linear operator* $T : V \longrightarrow V$ *which trace-less defined over a finite dimensional vector space* V. *If it is diagonalized* T, *i.e.,* $Te_i = \mu_i e_i$ *and* $\dim KerT = k$, *they for any* j *we have*

$$\mu_j^2 \leq \frac{(n-k-1)|T|^2}{(n-k)}, \tag{24}$$

for any integer k *and for an orthonormal basis* $\{e_1, \cdots, e_n\}$.

3. Proof of Main Theorems

We are in the position to prove our main results.

3.1. Proof of Theorem 1. Let us consider

$$nc \geq \mu_1. \tag{25}$$

Then, we noticed that left-hand side of (13) of Proposition 4 is different from negative. Therefore, the other side also non-negative, we get

$$\int_M \sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV + c \int_M |\nabla\varphi|^2 dV \geq \int_M |\nabla^2 f|^2 dV. \tag{26}$$

Additionally, the equality holds if and only if the following holds

$$\mu_1 = nc. \tag{27}$$

Moreover, we expressed the first equation of Proposition 4 in a new form

$$\int_M |\nabla^2\varphi|^2 dV - \int_M \sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV$$
$$= \{\mu_1 - c(n-1)\} \int_M |\nabla\varphi|^2 dV. \tag{28}$$

If we consider the following

$$\int_M |\nabla^2\varphi|^2 dV \geq \int_M \sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV. \tag{29}$$

Then, equation (28) implies that

$$\{\mu_1 - c(n-1)\} \geq 0. \tag{30}$$

If we notice that

$$\int_M |\nabla\varphi|^2 dV \geq \frac{nc}{\mu_1} \int_M \sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV. \tag{31}$$

This implies that

$$\int_M |\nabla^2\varphi|^2 dV \geq \int_M \sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV. \tag{32}$$

This completes the proof of Theorem 1.

3.2. Proof of Theorem 2. Let the second fundamental form T which diagonalized via an orthogonal frame $\{e_1, \cdots, e_n\}$, i.e., $Te_i = k_i e_i$, and the angle is denoted by θ_i between $\nabla\varphi$ and e_i. Thus, we find that

$$|h(\nabla\varphi, e_i)|^2 = g(T\nabla\varphi, e_i)^2 = g(\nabla\varphi, Te_i)^2 = k_i^2 \cos^2\theta_i |\nabla\varphi|^2. \tag{33}$$

From the virtue of (13) of Proposition 4, we construct

$$\int_M \left(\sum_{i=1}^n k_i^2 \cos^2\theta_i\right) |\nabla\varphi|^2 dV$$
$$= \int_M |\nabla^2\varphi|^2 dV + \{(n-1)c - \mu_1\} \int_M |\nabla\varphi|^2 dV. \tag{34}$$

Implementation Lemma 5 to the previous equation to establish

$$\left(\frac{n-k-1}{n-k}\right)\int_M S|\nabla\varphi|^2 dV$$
$$\geq \int_M |\nabla^2\varphi|^2 dV + \{(n-1)c - \mu_1\}\int_M |\nabla\varphi|^2 dV. \tag{35}$$

Let us assume the following inequality

$$\int_M |\nabla^2\varphi|^2 dV \geq \frac{\mu_1}{nc}\int_M |\nabla\varphi|^2 dV. \tag{36}$$

Plugging above equation into (35), we arrive at

$$\left(\frac{n-k-1}{n-k}\right)\int_M S|\nabla\varphi|^2 dV$$
$$\geq \left(\frac{n^2c^2 - nc\mu_1 - nc^2 + \mu_1}{nc}\right)\int_M |\nabla\varphi|^2 dV. \tag{37}$$

After some computations, finally, we get

$$\int_M S|\nabla^2\varphi|^2 dV \geq \left\{\frac{(n-k)(nc-1)(nc-\mu_1)}{(n-k-1)nc}\right\}\int_M |\nabla\varphi|^2 dV. \tag{38}$$

This completes the proof of Theorem 2.

3.3. Proof of Theorem 3. As recognize a theorem as a result of Obata in [5], a differentiable function φ on Riemannian manifold M^n is satisfied the following ordinary differential equation

$$\nabla^2\varphi = -\varphi, \tag{39}$$

if and only if M^n is isometric to a unit sphere $\mathbb{S}^n$, where $\nabla^2\varphi$ is two time derivatives of φ and is called Hessian of φ. As we assumed that

$$\nabla\varphi \in \ker hh(\nabla\varphi, e_i) = 0, \ 1 \leq i \leq n. \tag{40}$$

Then, using equation (14), we get

$$\frac{(n-1)(\mu_1 - nc)}{n}\int_M |\nabla\varphi|^2 dV = \int_M \left|\nabla^2\varphi + \frac{\mu_1}{n}\varphi\right|^2 dV. \tag{41}$$

The left-hand side of this previous equation is not negative; we summarize that

$$\mu_1 = nc. \tag{42}$$

Therefore, we have

$$\nabla^2\varphi = -\varphi. \tag{43}$$

Now using Obata theorem [5], we conclude that $\Psi(M^n)$ is isometric to a unit sphere $\mathbb{S}^n$. This completes the proof of first part of Theorem 3.

On the other case, if we consider that

$$\mathrm{Ric}(\nabla\varphi,\nabla\varphi) \geq c(n-1)|\nabla\varphi|^2. \tag{44}$$

Follows the equation (19), we find that

$$\int_M (n-1)c|\nabla\varphi|^2 dV \geq \sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV$$
$$+ (n-1)c\int_M |\nabla\varphi|^2 dV, \tag{45}$$

which implies that

$$\sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV \leq 0. \tag{46}$$

From where we conclude that $h(\nabla\varphi, e_i) = 0$, this means that $\nabla\varphi \in \ker B$. Now, we invoke the first case (i) of Theorem 3 we get required result. The proof of Theorem 3 is completed.

4. Some Applications

It is renowned that the complex Euclidean space $\mathbb{C}^n$, the complex projective n-space $\mathbb{C}P^n(4)$, and complex hyperbolic n-space $\mathbb{C}H^n(-4)$ are special cases of a complex space form $\tilde{M}^n(4c)$ with $c = 0, 1$ and $c = -1$, respectively. Therefore, we define following corollaries for complex projective spaces as consequences of Theorems 1, 2, and 3.

Corollary 6. *Let $\Psi : M^n \longrightarrow \mathbb{C}P^n(4)$ be an immersion of a compact Lagrangian submanifold into complex projective space $\mathbb{C}P^n(4)$. If the Laplacian of M^n endowed to the first eigenvalue μ_1 corresponding eigenfunction φ and Ψ is minimal, then the following inequality holds*

$$\int_M \sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV + \int_M |\nabla\varphi|^2 dV \geq \int_M |\nabla^2\varphi|^2 dV. \tag{47}$$

The equality holds if and only if $\mu_1 = n$. Furthermore, if the following inequality holds

$$\int_M |\nabla^2\varphi|^2 dV \geq \int_M \sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV. \tag{48}$$

Then, we have $\mu_1 \geq (n-1)$. In particular, if the following inequality satisfying

$$\int_M |\nabla\varphi|^2 dV \geq \frac{n}{\mu_1}\int_M \sum_{i=1}^n |h(\nabla\varphi, e_i)|^2 dV. \tag{49}$$

Then, eigenvalue is satisfied $\mu_1 \geq (n-1)$.

Corollary 7. *Let $\Psi : M^n \longrightarrow \mathbb{C}P^n(4)$ be an immersion of a compact Lagrangian submanifold into the complex projective space $\mathbb{C}P^n(4)$. If Ψ is minimal and $\dim \mathrm{Ker} h = k$, then we have*

$$\int_M S|\nabla^2 \varphi|^2 dV \geq \left\{ \frac{(n - \mu_1)(n - 1)(n - k)}{(n - k - 1)n} \right\} \int_M |\nabla \varphi|^2 dV.$$
$$(50)$$

In especial case, if S is constant, then we define

$$S \geq \frac{(n - 1)(n - k)}{n(n - \mu_1)(n - k - 1)}.$$
$$(51)$$

From Theorem 3, we have the following:

Corollary 8. Let $\Psi : M^n \longrightarrow \mathbb{C}P^n(4)$ be an immersion of a compact Lagrangian submanifold into the complex projective space $\mathbb{C}P^n(4)$. Assuming that Ψ is minimal and φ be a first eigenfunction endowed to the Laplacian of M^n. Then, we get following:

(i) If $\nabla \varphi$ on Kerh, then $\Psi(M^n)$ is locally geodesic sphere $\mathbb{S}^n$, or $\Psi(M^n)$ is isometric to standard sphere $\mathbb{S}^n$

(ii) If $Ric_{M^n}(\nabla \varphi, \nabla \varphi) \geq (n - 1)|\nabla \varphi|^2$, then $\Psi(M^n)$ is isometric to sphere $\mathbb{S}^n$

Data Availability

No datasets were generated or analyzed during the current study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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