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*Research Article*

# New Exact Solutions, Dynamical and Chaotic Behaviors for the Fourth-Order Nonlinear Generalized Boussinesq Water Wave Equation

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Based on the extended homogeneous balance method, the auto-Bäcklund transformation transformation is constructed and some new explicit and exact solutions are given for the fourth-order nonlinear generalized Boussinesq water wave equation. Then, the fourth-order nonlinear generalized Boussinesq water wave equation is transformed into the planer dynamical system under traveling wave transformation. We also investigate the dynamical behaviors and chaotic behaviors of the considered equation. Finally, the numerical simulations show that the change of the physical parameters will affect the dynamic behaviors of the system.

## 1. Introduction

Many nonlinear phenomena can be described by nonlinear wave equations, where Boussinesq equation is one of the important mathematical models, which is widely used in shallow water and the percolation in horizontal porous media [1]. Boussinesq equation has many different forms [2–8]. Clarkson and Kruskal gave a new method for determining similarity reductions of Boussinesq equation in Ref. [2]. Nonclassical symmetry reduction can be used to reduce the Boussinesq equation in Ref. [3]. Based on the asymptotic expansion for small phase speed, an asymptotic semianalytical solution was found in Ref. [4]. Higher-order Boussinesq equations (fourth-order and sixth-order Boussinesq equations ) for two-way propagation of shallow water waves were studied in Ref. [5]. The generalized Boussinesq equation was reduced using Lie group and some exact solutions were obtained in Ref. [6]. The Hirota bilinear method is used to construct two soliton solutions for the $(2+1)$-dimensional Boussinesq equation, the $(3+1)$-dimensional Boussinesq equation, and the sixth-order Boussinesq equation. But the three variants of the Boussinesq equation are nonintegrable and do not admit $N$-soliton solutions in Ref. [7]. The generalized $(2+1)$-dimensional Boussinesq equation is investigated by the bifurcation method of dynamical systems, and some exact solutions were obtained in Ref. [8].

In this work, we consider the fourth-order nonlinear generalized Boussinesq water wave equation:

$$u_{tt} - au_{xx} - 2bu_x^2 - 2buu_{xx} + cu_{xxxx} = 0, \tag{1}$$

where $u = u(x, t)$ is a real function and $a$, $b$, and $c$ are real nonzero arbitrary constants [9–15].

As far as the authors know, Lie symmetry method is used to analyze Equation (1) and some soliton wave solutions are obtained in Ref. [9]. Based on Hirota's bilinear method, Qin got some rational solutions and interaction solutions of Equation (1) in Ref. [10]. The rogue wave and semirational solutions of Equation (1) are studied by Hirota's bilinear method [11]. The general rogue wave solutions of arbitrary orders for Boussinesq equation were found by bilinear Kadomtsev-Petviashvili (KP) reduction method in Ref. [12]. Some travelling wave solutions of Equation (1) were given by extended tanh method and rational method [13, 14]; the solitary wave and shock wave solutions of Equation (1) were obtained by method of undetermined coefficients [15].

Many powerful methods for solving exact solutions of partial differential equations have been developed such as Darboux transform [16, 17] and Lie group method [18–20]. The extended homogeneous balance method (EHBM) is an effective method solving nonlinear partial differential equations [21–23]. So, we apply EHBM to construct new solutions of Equation (1) in this paper.

The analysis of bifurcation and chaos behavior is a very interesting nonlinear phenomenon, which has been applied in many fields, such as engineering, telecommunication, and ecology [24–27]. By analyzing the dynamic behavior of differential equation, we can study whether the periodic external perturbation will lead to the chaotic behavior of differential equation. So, we will study the dynamical behaviors and chaotic behaviors of Equation (1) in this paper.

The rest of this paper is arranged as follows. In Section 2, based on the extended homogeneous balance method, the auto-Bäcklund transformation is constructed and some new explicit and exact solutions of Equation (1) are obtained. In Section 3, based on the theory of plane dynamic system, dynamical behaviors and chaotic behaviors of Equation (1) are studied. Some conclusions are provided at the end of the paper.

## 2. Abundant New Explicit and Exact Solutions to Equation (1)

In terms of the idea of extended homogeneous balance method [21, 22], we assume that the solution of Equation (1) has the following form:

$$u(x,t)=\frac{\partial^{(m+n)}f(\phi)}{\partial x^m\partial t^n}+u_0=f^{(m+n)}\phi_x^m\phi_t^n+\cdots, \tag{2}$$

where $u_0=u_0(x,t)$ is the arbitrary known seed solution and $m$, $n$, and function $f(\phi)$ are to be determined later.

From (2), we obtain

$$\begin{aligned} u_{xxxx}&=f^{(m+n+4)}\phi_x^{m+4}\phi_t^n+\cdots,\\ u_x^2&=f^{(m+n+1)}f^{(m+n+1)}\phi_x^{2m+2}\phi_t^{2n}+\cdots,\\ uu_{xx}&=f^{(m+n)}f^{(m+n+2)}\phi_x^{2m+2}\phi_t^{2n}+\cdots. \end{aligned} \tag{3}$$

According to the homogeneous principle method [21, 22], balancing the highest-order derivative term $u_{xxxx}$ and the highest-order nonlinear term, $u_x^2, uu_{xx}$ can be obtained

$$2m+2=2m+2=m+4,\ n=2n=2n, \tag{4}$$

which gives

$$m=2,\ n=0. \tag{5}$$

Thus, (2) can be rewritten as follows

$$u(x,t)=\frac{\partial^2 f(\phi)}{\partial x^2}+u_0=f''\phi_x^2+f'\phi_{xx}+u_0. \tag{6}$$

From (6), it is easy to deduce that

$$\begin{aligned} u_{tt}=&f^{(4)}\phi_t^2\phi_x^2+f^{(3)}\phi_{tt}\phi_x^2+4f^{(3)}\phi_t\phi_x\phi_{xt}\\ &+2f''\phi_{xt}^2+2f''\phi_x\phi_{xtt}+f^{(3)}\phi_t^2\phi_{xx}\\ &+f''\phi_{xx}\phi_{tt}+f''\phi_t\phi_{xxt}+f'\phi_{xxtt}+u_{0tt}, \end{aligned} \tag{7}$$

$$\begin{aligned} u_{xx}=&f^{(4)}\phi_x^4+6f^{(3)}\phi_x^2\phi_{xx}+3f''\phi_{xx}^2\\ &+4f''\phi_x\phi_{xxx}+f'\phi_{xxxx}+u_{0xx}, \end{aligned} \tag{8}$$

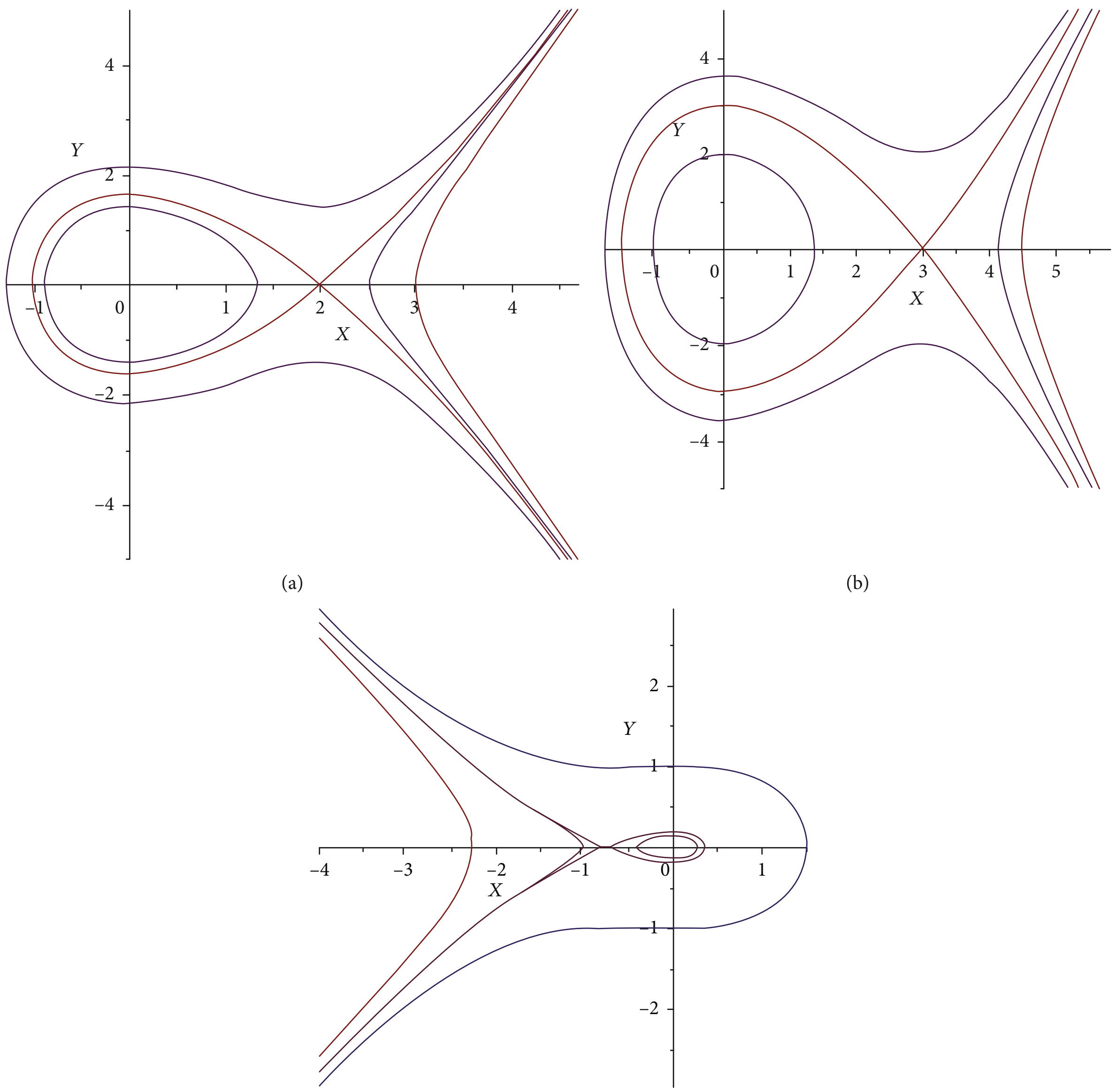
$$\begin{aligned} u_x^2=&\left(f^{(3)}\right)^2\phi_x^6+9\left(f''\right)^2\phi_x^2\phi_{xx}^2+\left(f'\right)^2\phi_{xxx}^2\\ &+u_{0x}^2+6f^{(3)}f''\phi_x^4\phi_{xx}+2f^{(3)}f'\phi_x^3\phi_{xxx}\\ &+2u_{0x}f^{(3)}\phi_x^3+6f''f'\phi_x\phi_{xx}\phi_{xxx}\\ &+6u_{0x}f''\phi_x\phi_{xx}+2u_{0x}f'\phi_{xxx}, \end{aligned} \tag{9}$$

$$\begin{aligned} uu_{xx}=&f''f^{(4)}\phi_x^6+6f''f^{(3)}\phi_x^4\phi_{xx}+3\left(f''\right)^2\phi_x^2\phi_{xx}^2\\ &+4\left(f''\right)^2\phi_x^3\phi_{xxx}+f'f''\phi_x^2\phi_{xxxx}+f''\phi_x^2u_{0xx}\\ &+f'f^{(4)}\phi_x^4\phi_{xx}+6f'f^{(3)}\phi_x^2\phi_{xx}^2+3f'f''\phi_{xx}^3\\ &+4f'f''\phi_x\phi_{xx}\phi_{xxx}+\left(f'\right)^2\phi_{xx}\phi_{xxxx}\\ &+f'\phi_{xx}u_{0xx}+f^{(4)}u_0\phi_x^4+6f^{(3)}u_0\phi_x^2\phi_{xx}\\ &+3f''u_0\phi_{xx}^2+4f''u_0\phi_x\phi_{xxx}+f'u_0\phi_{xxxx}+u_0u_{0xx}, \end{aligned} \tag{10}$$

$$\begin{aligned} u_{xxxx}=&f^{(6)}\phi_x^6+15f^{(5)}\phi_x^4\phi_{xx}+45f^{(4)}\phi_x^2\phi_{xx}^2\\ &+20f^{(4)}\phi_x^3\phi_{xxx}+15f^{(3)}\phi_{xx}^3\\ &+60f^{(3)}\phi_x\phi_{xx}\phi_{xxx}+15f^{(3)}\phi_x^2\phi_{xxxx}\\ &+10f''\phi_{xx}^2+15f''\phi_{xx}\phi_{xxxx}\\ &+6f''\phi_x\phi_{xxxx}+f'\phi_{xxxxxx}+u_{0xxxx}. \end{aligned} \tag{11}$$

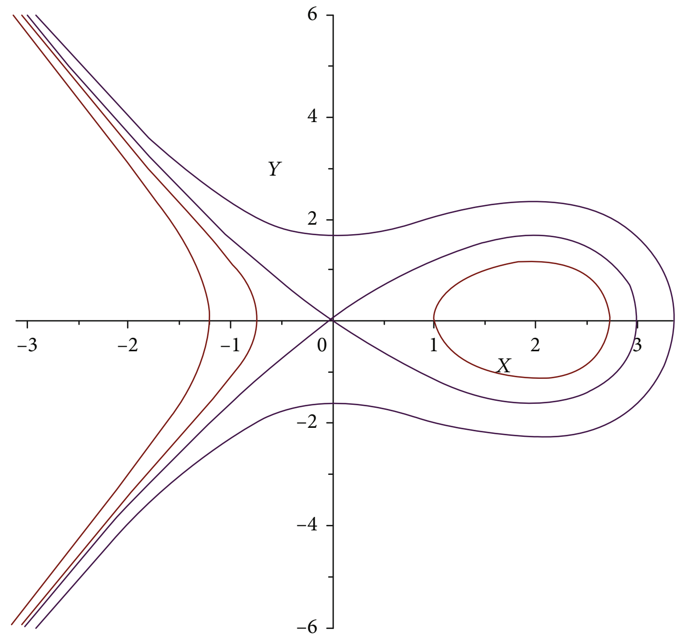
Table 1: Information of equilibrium points.

| No. | Values of parameters | det $M(P_1)$ | det $M(P_2)$ | $P_1$ | $P_2$ | Phase portraits |
|---|---|---|---|---|---|---|
| 1 | $c > 0, a < 0$ | $J(P_1) > 0$ | $J(P_2) < 0$ | center point | saddle point | Figure 1(a) |
| | $c > 0, a > 0, \|k_2\| > \sqrt{a}\|k_1\|$ | $J(P_1) > 0$ | $J(P_2) < 0$ | center point | saddle point | Figure 1(b) |
| | $c < 0, a > 0, \|k_2\| < \sqrt{a}\|k_1\|$ | $J(P_1) > 0$ | $J(P_2) < 0$ | center point | saddle point | Figure 1(c) |
| | $c < 0, a < 0$ | $J(P_1) < 0$ | $J(P_2) > 0$ | saddle point | center point | Figure 2(a) |
| | $c > 0, a > 0, \|k_2\| < \sqrt{a}\|k_1\|$ | $J(P_1) < 0$ | $J(P_2) > 0$ | saddle point | center point | Figure 2(b) |
| | $c < 0, a > 0, \|k_2\| > \sqrt{a}\|k_1\|$ | $J(P_1) < 0$ | $J(P_2) > 0$ | saddle point | center point | Figure 2(c) |

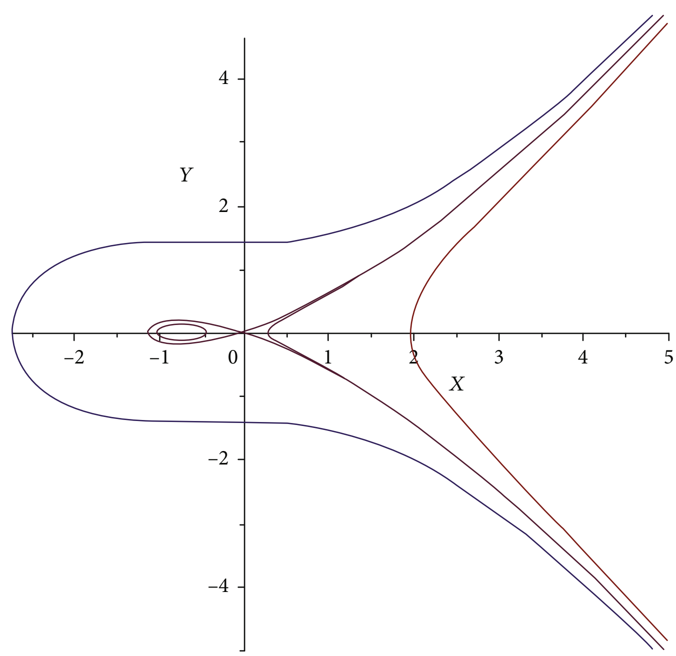



(a) (b) (c)

Figure 1: Phase portraits of system (50).

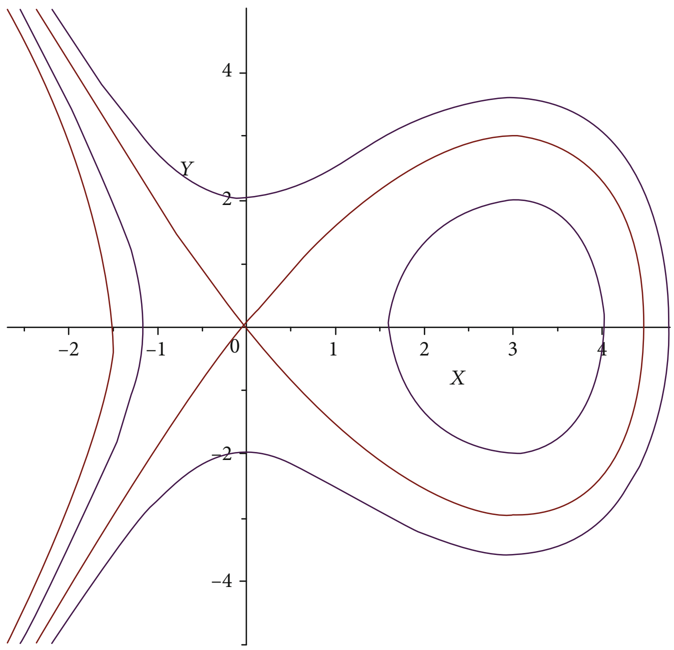



(a)




(b)

Figure 2: Continued.




(c)

Figure 2: Phase portraits of system (50).

Now, substituting (7)–(11) into Equation (1) and simplifying, it yields

$$\begin{aligned}
&\phi_x^6\left[-2b\left(f^{(3)}\right)^2 - 2bf^{(4)}f'' + cf^{(6)}\right]\\
&\quad + \phi_x^4\left[-af^{(4)} - \left(24bf''f^{(3)} + 2bf'f^{(4)} - 15cf^{(5)}\right)\phi_{xx}\right.\\
&\quad \left. - 2bu_0 f^{(4)}\right] + \phi_x^3\left[\left(-4bf'f^{(3)} - 8b\left(f'\right)^2\right.\right.\\
&\quad \left.\left. + 20cf^{(4)}\right)\phi_{xxx} - 4bf^{(3)}u_{0x}\right] + \phi_x^2\left[f^{(4)}\phi_t^2 + f^{(3)}\phi_{tt}\right.\\
&\quad - (6a + 12bu_0)f^{(3)}\phi_{xx} + \left(-24b\left(f''\right)^2\right.\\
&\quad \left. + 2bf'f^{(3)} + 45cf^{(4)}\right)\phi_{xx}^2\\
&\quad \left. + \left(-2bf'f'' + 15cf^{(3)}\right)\phi_{xxxx} - 2bf''u_{0xx}\right]\\
&\quad + \phi_x\left[4f^{(3)}\phi_t\phi_{xt} + 2f''\phi_{xtt}\right.\\
&\quad + \left(-4af'' - 20bf'f''\phi_{xx} + 60cf^{(3)}\phi_{xx} - 8bu_0f''\right)\phi_{xxx}\\
&\quad \left. + 6cf''\phi_{xxxx} - 12bu_{0x}f''\phi_{xx}\right],
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\phi_{xx}^3\left(-6bf'f'' + 15cf^{(3)}\right) - (3a + 6bu_0)f''\phi_{xx}^2\\
&\quad + \phi_{xx}\left[f^{(3)}\phi_t^2 - 2b\left(f'\right)^2\phi_{xxxx} + 15cf''\phi_{xxxx}\right.\\
&\quad \left. + f''\phi_{tt} - 2bf'u_{0xx}\right] + \phi_{xxx}^2\left[-2b\left(f'\right)^2 + 10cf''\right]\\
&\quad - 4bu_{0x}f'\phi_{xxx} - (a + 2bu_0)f'\phi_{xxxx} + 2f''\phi_{xt}^2\\
&\quad + f'\phi_{ttxx} + 2f''\phi_t\phi_{xxt} + cf'\phi_{xxxxxx} + u_{0tt} - au_{0xx}\\
&\quad + cu_{0xxxx} - 2bu_{0x}^2 - 2bu_0u_{0xx} = 0.
\end{aligned} \tag{13}$$

Setting the coefficient of the term $\phi_x^6$ in Equation (13) to zero, we obtain a nonlinear ordinary differential equation for function $f(\phi)$:

$$-2b\left(f^{(3)}\right)^2 - 2bf^{(4)}f'' + cf^{(6)} = 0. \tag{14}$$

Integrating Equation (14) once and neglecting the constant of integration gives

$$-2bf^{(3)}f'' + cf^{(5)} = 0. \tag{15}$$

Again integrating Equation (15) and letting the constant of integration be zero, we obtain

$$-b\left(f''\right)^2 + cf^{(4)} = 0, \tag{16}$$

which has particular solution

$$f(\phi) = r\ln\phi, \tag{17}$$

where $r = -6c/b$.

According to (17), we get the following results:

$$\left(f'\right)^2 = -rf'', f''f' = -\frac{r}{2}f^{(3)}, f^{(3)}f' = -\frac{r}{3}f^{(4)}, \tag{18}$$

$$f^{(4)}f' = -\frac{r}{4}f^{(5)}, \left(f''\right)^2 = -\frac{r}{6}f^{(4)}, f''f^{(3)} = -\frac{r}{12}f^{(5)}. \tag{19}$$

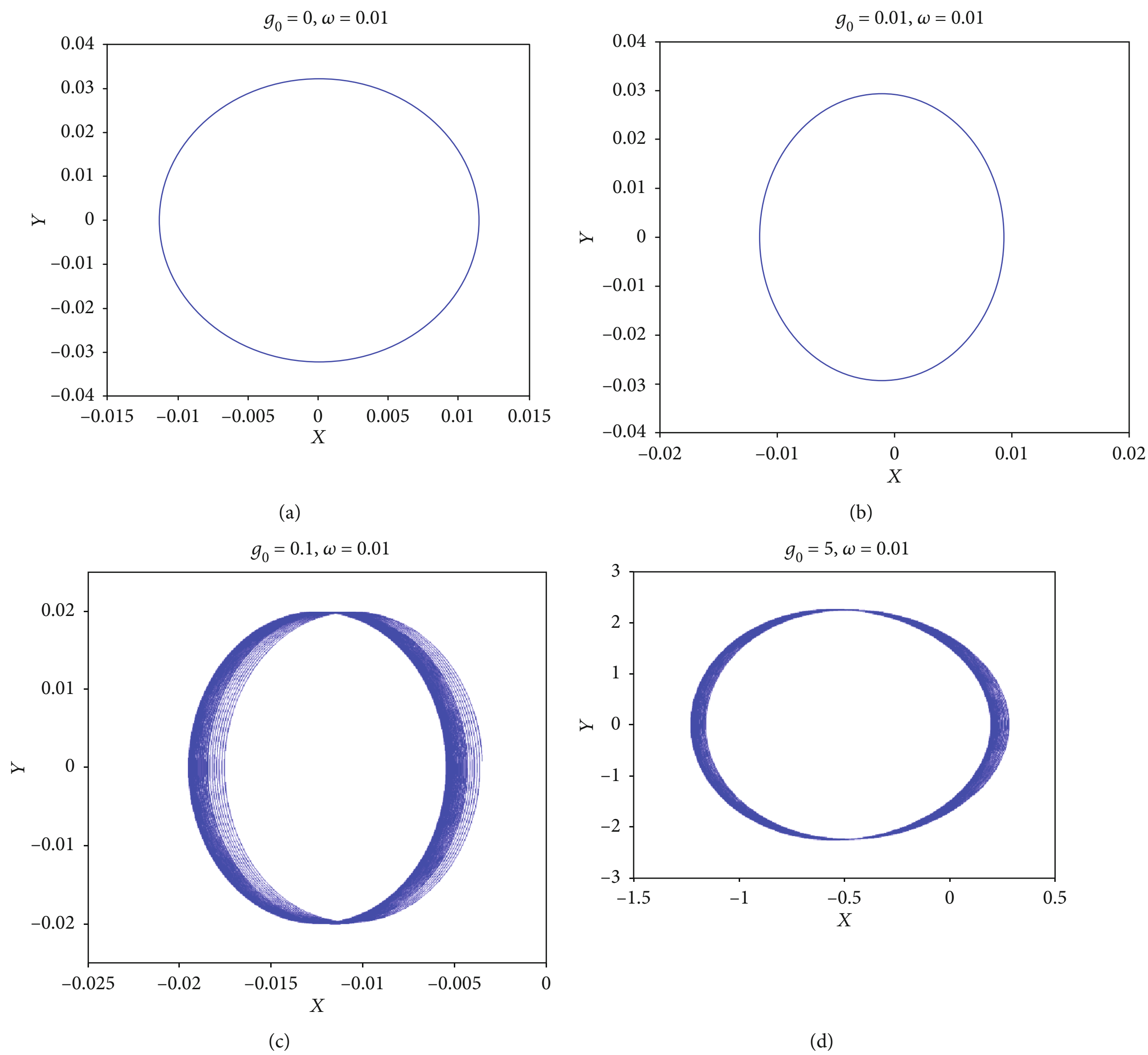



Figure 3: Phase portraits of system (54) with $\omega = 0.01$ and different values of $g_0$. (a) $g_0 = 0$, (b) $g_0 = 0.01$, (c) $g_0 = 0.1$, and (d) $g_0 = 5$.

By (6) and (17), we obtain the auto-Bäcklund transformation of Equation (1) as follows:

$$u(x,t) = -\frac{r\phi_x^2}{\phi^2} + \frac{r\phi_{xx}}{\phi} + u_0. \tag{20}$$

Letting the seed solution

$$u_0(x,t) = 0, \tag{21}$$

and using results (19), then Equation (13) can be simplified as

$$\begin{aligned} & f^{(4)}\left[-a\phi_x^2 + 4c\phi_x\phi_{xxx} + \phi_t^2 - 3c\phi_{xx}^2\right]\phi_x^2 \\ & + f^{(3)}\left[(\phi_{tt} - 6a\phi_{xx} + 9c\phi_{xxxx})\phi_x^2 \right. \\ & \left. + 4\phi_x\phi_t\phi_{xt} - 3c\phi_{xx}^3 + \phi_{xx}\phi_t^2\right] \\ & + f''\left[\phi_{xt}^2 + 2\phi_x\phi_{xtt} + \phi_{tt}\phi_{xx} + 2\phi_t\phi_{xxt} \right. \\ & - 3a\phi_{xx}^2 - 4a\phi_x\phi_{xxx} + 6c\phi_x\phi_{xxxxx} \\ & \left. - 2c\phi_{xxx}^2 + 3c\phi_{xx}\phi_{xxxx}\right] \\ & + f'\left[\phi_{xxtt} - a\phi_{xxxx} + c\phi_{xxxxxx}\right] = 0. \end{aligned} \tag{22}$$

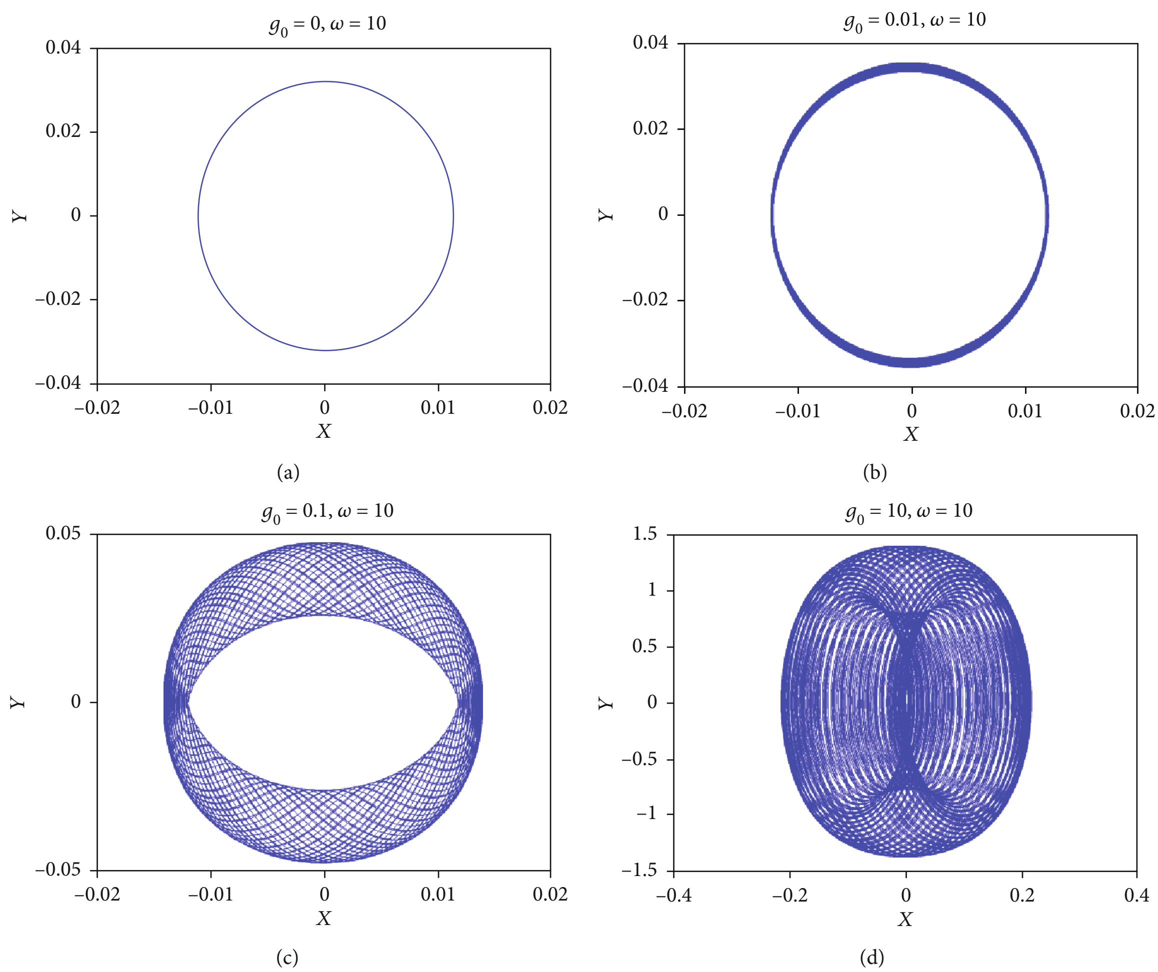



FIGURE 4: Phase portraits of system (54) with $\omega = 10$ and different values of $g_0$. (a) $g_0 = 0$, (b) $g_0 = 0.01$, (c) $g_0 = 0.1$, and (d) $g_0 = 10$.

Setting the coefficients of $f^{(4)}$, $f^{(3)}$, $f''$, and $f'$ in Equation (22) to zero yields a set of partial differential equations for $\phi(x,t)$ as follows:

$$\phi_x^2\left(-a\phi_x^2 + 4c\phi_x\phi_{xxx} + \phi_t^2 - 3c\phi_{xx}^2\right) = 0, \tag{23}$$

$$(\phi_{tt} - 6a\phi_{xx} + 9c\phi_{xxxx})\phi_x^2 + 4\phi_x\phi_t\phi_{xt} - 3c\phi_{xx}^3 + \phi_{xx}\phi_t^2 = 0, \tag{24}$$

$$\begin{aligned}&\phi_{xt}^2 + 2\phi_x\phi_{xtt} + \phi_{tt}\phi_{xx} + 2\phi_t\phi_{xxt} - 3a\phi_{xx}^2\\&\quad - 4a\phi_x\phi_{xxx} + 6c\phi_x\phi_{xxxxx} - 2c\phi_{xxx}^2 + 3c\phi_{xx}\phi_{xxxx} = 0,\end{aligned} \tag{25}$$

$$(\phi_{tt} - a\phi_{xx} + c\phi_{xxxx})_{xx} = 0. \tag{26}$$

Here, Equations (24) and (25) can be rewritten as

$$\begin{aligned}&\phi_x^2[\phi_{tt} - a\phi_{xx} + c\phi_{xxxx}] + \phi_{xx}\left[-a\phi_x^2 + 4c\phi_x\phi_{xxx} + \phi_t^2 - 3c\phi_{xx}^2\right]\\&\quad + 2\phi_x\left[-a\phi_x^2 + 4c\phi_x\phi_{xxx} + \phi_t^2 - 3c\phi_{xx}^2\right]_x = 0,\end{aligned} \tag{27}$$

$$\begin{aligned}&\phi_{tt}[\phi_{tt} - a\phi_{xx} + c\phi_{xxxx}] + 2\phi_x[\phi_{tt} - a\phi_{xx} + c\phi_{xxxx}]_x\\&\quad + \left[-a\phi_x^2 + 4c\phi_x\phi_{xxx} + \phi_t^2 - 3c\phi_{xx}^2\right]_{xx} = 0.\end{aligned} \tag{28}$$

By analysis of Equations (27) and (28), we find that Equations (23)–(26) are satisfied automatical under the following conditions:

$$\phi_{tt} - a\phi_{xx} + c\phi_{xxxx} = 0, \tag{29}$$

$$-a\phi_x^2 + 4c\phi_x\phi_{xxx} + \phi_t^2 - 3c\phi_{xx}^2 = 0. \tag{30}$$

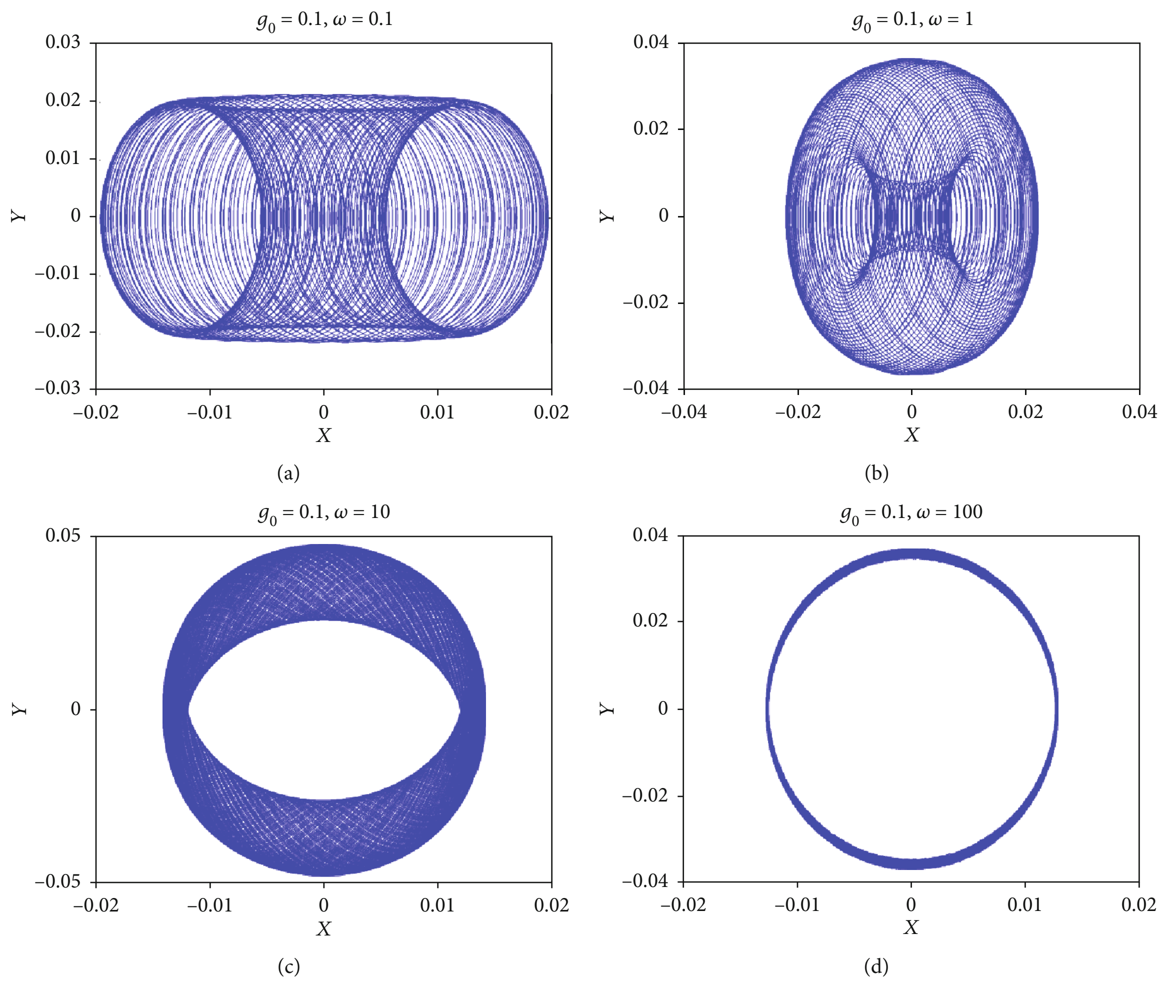



FIGURE 5: Phase portraits of system (54) with $g_0 = 0.1$ and different values of $\omega$. (a) $\omega = 0.1$. (b) $\omega = 1$. (c) $\omega = 10$. (d) $\omega = 100$.

To obtain some exact solutions of Equations (29) and (30), we assume the solutions of the form:

$$\phi(x, t) = A + B \sin(\xi) \exp(\eta), \tag{31}$$

where $\xi = Kx + \Omega t + \xi_0, \eta = kx + \omega t + \eta_0$ ; $A, B, K, \Omega, \xi_0, k, \omega$, and $\eta_0$ are constants which are to be determined.

Substituting (31) into Equations (29) and (30) and simplifying, it leads to a system of nonlinear algebraic equations with respect to $K$, $\Omega$, $k$,and $\omega$ as follows

$$\begin{gathered}
-\Omega^2 + \omega^2 + aK^2 - ak^2 + cK^4 - 6cK^2k^2 + ck^4 = 0, \\
2\Omega\omega - 2aKk - 4ckK^3 + 4cKk^3 = 0, \\
-2aKk + 2\Omega\omega + (12c - 4)kK^3 - (12c - 4)Kk^3 = 0, \\
-ak^2 + \omega^2 + ck^4 - 6cK^2k^2 + aK^2 - \Omega^2 + cK^4 = 0, \\
\Omega^2 - aK^2 - 4cK^4 = 0.
\end{gathered} \tag{32}$$

By solving the above equations, the following sets of solutions are obtained

*Case 1.* $K = 0, \Omega = 0, k = k, \omega = \omega$, where $k \neq 0, c \neq 1/3, \omega^2 + c k^4 - ak^2 = 0, ac > 0$.

*Case 2.* $k = 0, \Omega = 0, K = \sqrt{-a/4c}, \omega = \pm\sqrt{3a^2/16c}$, where $c > 0, a < 0, c \neq 1/3$.

*Case 3.* $k = 0, \Omega = 0, K = -\sqrt{-a/4c}, \omega = \pm\sqrt{3a^2/16c}$, where $c > 0, a < 0, c \neq 1/3$.

*Case 4.* $\Omega = \Omega, k = \sqrt[4]{-\Omega^2/4c}, K = -\sqrt[4]{-\Omega^2/4c}, \omega = 2\Omega$, where $c < 0, c \neq 1/3$.

*Case 5.* $\Omega = \Omega, k = -\sqrt[4]{-\Omega^2/4c}, K = \sqrt[4]{-\Omega^2/4c}, \omega = 2\Omega$, where $c < 0, c \neq 1/3$.

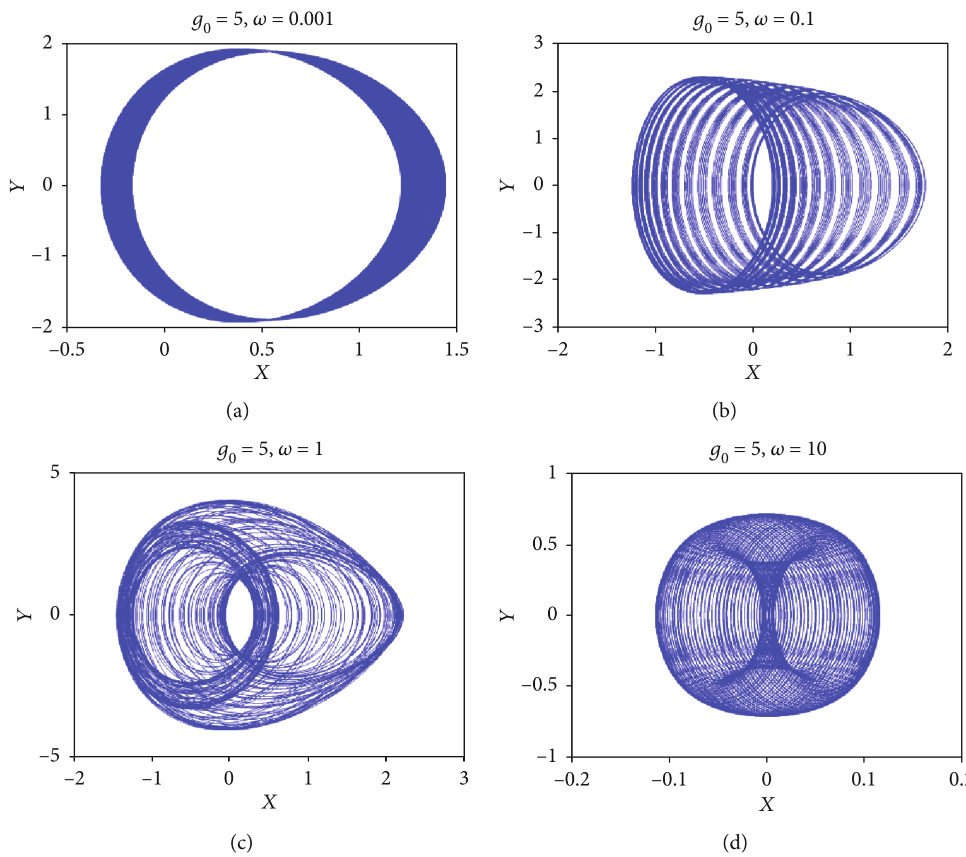



FIGURE 6: Phase portraits of system (54) with $g_0 = 5$ and different values of $\omega$. (a) $\omega = 0.001$. (b) $\omega = 0.1$. (c) $\omega = 1$. (d) $\omega = 10$.

*Case 6.* $\Omega = \Omega, k = \sqrt[4]{3\Omega^2/4c}, K = \sqrt[4]{3\Omega^2/4c}, \omega = -2\Omega,$ where $c > 0, c \neq 1/3$.

*Case 7.* $\Omega = \Omega, k = -\sqrt[4]{3\Omega^2/4c}, K = -\sqrt[4]{3\Omega^2/4c}, \omega = -2\Omega,$ where $c > 0, c \neq 1/3$.

According to (20), (21), and (31), the solution of the fourth-order nonlinear generalized Boussinesq water wave equation is as follows:

$$\begin{aligned} u(x,t) &= -\frac{r\phi_x^2}{\phi^2} + \frac{r\phi_{xx}}{\phi} \\ &= \frac{6cB^2 \exp(2\eta)[K\cos(\xi) + k\sin(\xi)]^2}{b[A + B\sin(\xi)\exp(\eta)]^2} \\ &\quad - \frac{6cB\exp(\eta)\left[\left(k^2 - K^2\right)\sin(\xi) + 2Kk\cos(\xi)\right]}{b[A + B\sin(\xi)\exp(\eta)]}, \end{aligned} \tag{33}$$

where $\xi = Kx + \Omega t + \xi_0, \eta = kx + \omega t + \eta_0$.

For Case 1, the exact explicit solutions to the fourth-order nonlinear generalized Boussinesq water wave equation are obtained as follows:

$$\begin{aligned} u(x,t) = {} & \frac{6cB^2k^2\sin^2\xi_0 \exp 2(kx + \omega t + \eta_0)}{b[A + B\sin\xi_0 \exp 2(kx + \omega t + \eta_0)]^2} \\ & - \frac{6cBk^2 \sin\xi_0 \exp(kx + \omega t + \eta_0)}{b[A + B\sin\xi_0 \exp 2(kx + \omega t + \eta_0)]}. \end{aligned} \tag{34}$$

If $A = 1, B\sin\xi_0 = 1$, Equation (34) is reduced to the solitary wave solutions as follows:

$$u(x,t) = -\frac{3ck^2}{2b} sech^2\left[\frac{1}{2}(kx + \omega t + \eta_0)\right]. \tag{35}$$

For Case 2, the exact explicit solutions to the fourth-order nonlinear generalized Boussinesq water wave equation are obtained as follows:

$$u(x,t)=\frac{-3aB^2\exp\left(\pm\left(\sqrt{3a^2/16c}\right)t+\eta_0\right)\left[\exp\left(\pm\left(\sqrt{3a^2/16c}\right)t+\eta_0\right)+A\sin\left(\left(\sqrt{-a/4c}\right)x+\xi_0\right)\right]}{2cb\left[A+B\sin\left(\left(\sqrt{-a/4c}\right)x+\xi_0\right)\exp\left(\pm\left(\sqrt{3a^2/16c}\right)t+\eta_0\right)\right]^2}. \tag{36}$$

If $A=0$, Equation (36) is reduced to

$$u(x,t)=\frac{-3a}{2b}csc^2\left(\sqrt{\frac{-a}{4c}}x+\xi_0\right), \tag{37}$$

which are periodic wave solutions.

For Case 3, the exact explicit solutions to the fourth-order nonlinear generalized Boussinesq water wave equation are obtained as follows:

$$u(x,t)=\frac{-3aB^2\exp\left(\pm\left(\sqrt{3a^2/16c}\right)t+\eta_0\right)\left[\exp\left(\pm\left(\sqrt{3a^2/16c}\right)t+\eta_0\right)+A\sin\left(-\left(\sqrt{-a/4c}\right)x+\xi_0\right)\right]}{2cb\left[A+B\sin\left(-\sqrt{-a/4c}x+\xi_0\right)\exp\left(\pm\sqrt{3a^2/16c}t+\eta_0\right)\right]^2}. \tag{38}$$

If $A=0$, Equation (38) is reduced to Equation (37).

For Case 4, the exact explicit solutions to the fourth-order nonlinear generalized Boussinesq water wave equation are obtained as follows:

$$\begin{aligned}u(x,t)={}&\frac{-6cB\left(\sqrt[4]{-\Omega^2/4c}\right)\exp\left(\left(\sqrt[4]{-\Omega^2/4c}\right)x+2\Omega t+\eta_0\right)\left[\cos\left(-\left(\sqrt[4]{-\Omega^2/4c}\right)x+\Omega t+\xi_0\right)-\sin\left(-\left(\sqrt[4]{-\Omega^2/4c}\right)x+\Omega t+\xi_0\right)\right]}{b\left[A+B\sin\left(-\left(\sqrt[4]{-\Omega^2/4c}\right)x+\Omega t+\xi_0\right)\exp\left(\left(\sqrt[4]{-\Omega^2/4c}\right)x+2\Omega t+\eta_0\right)\right]^2}\\&+\frac{12cB\left(\sqrt{-\Omega^2/4c}\right)\exp\left(\left(\sqrt[4]{-\Omega^2/4c}\right)x+2\Omega t+\eta_0\right)\cos\left(-\left(\sqrt[4]{-\Omega^2/4c}\right)x+\Omega t+\xi_0\right)}{b\left[A+B\sin\left(-\left(\sqrt[4]{-\Omega^2/4c}\right)x+\Omega t+\xi_0\right)\exp\left(\left(\sqrt[4]{-\Omega^2/4c}\right)x+2\Omega t+\eta_0\right)\right]}.\end{aligned} \tag{39}$$

For Case 5, the exact explicit solutions to the fourth-order nonlinear generalized Boussinesq water wave equation are obtained as follows:

$$\begin{aligned}u(x,t)={}&\frac{6cB\sqrt[4]{-\Omega^2/4c}\exp\left(-\sqrt[4]{-\Omega^2/4c}x+2\Omega t+\eta_0\right)\left[\cos\left(\sqrt[4]{-\Omega^2/4c}x+\Omega t+\xi_0\right)-\sin\left(\sqrt[4]{-\Omega^2/4c}x+\Omega t+\xi_0\right)\right]}{b\left[A+B\sin\left(\sqrt[4]{-\Omega^2/4c}x+\Omega t+\xi_0\right)\exp\left(-\sqrt[4]{-\Omega^2/4c}x+2\Omega t+\eta_0\right)\right]^2}\\&+\frac{12cB\sqrt{-\Omega^2/4c}\exp\left(-\sqrt[4]{-\Omega^2/4c}x+2\Omega t+\eta_0\right)\cos\left(\sqrt[4]{-\Omega^2/4c}x+\Omega t+\xi_0\right)}{b\left[A+B\sin\left(\sqrt[4]{-\Omega^2/4c}x+\Omega t+\xi_0\right)\exp\left(-\sqrt[4]{-\Omega^2/4c}x+2\Omega t+\eta_0\right)\right]}.\end{aligned} \tag{40}$$

For Case 6, the exact explicit solutions to the fourth-order nonlinear generalized Boussinesq water wave equation are obtained as follows:

$$u(x,t)=\frac{6cB\sqrt[4]{3\Omega^2/4c}\exp\left(\sqrt[4]{3\Omega^2/4c}x-2\Omega t+\eta_0\right)\left[\cos\left(\sqrt[4]{3\Omega^2/4c}x+\Omega t+\xi_0\right)-\sin\left(\sqrt[4]{3\Omega^2/4c}x+\Omega t+\xi_0\right)\right]}{b\left[A+B\sin\left(\sqrt[4]{-\Omega^2/4c}x+\Omega t+\xi_0\right)\exp\left(\sqrt[4]{3\Omega^2/4c}x-2\Omega t+\eta_0\right)\right]^2} - \frac{12cB\sqrt{3\Omega^2/4c}\exp\left(\sqrt[4]{3\Omega^2/4c}x-2\Omega t+\eta_0\right)\cos\left(\sqrt[4]{3\Omega^2/4c}x+\Omega t+\xi_0\right)}{b\left[A+B\sin\left(\sqrt[4]{3\Omega^2/4c}x+\Omega t+\xi_0\right)\exp\left(\sqrt[4]{3\Omega^2/4c}x-2\Omega t+\eta_0\right)\right]}. \tag{41}$$

For Case 7, the exact explicit solutions to the fourth-order nonlinear generalized Boussinesq water wave equation are obtained as follows:

$$u(x,t)=-\frac{6cB\sqrt[4]{3\Omega^2/4c}\exp\left(-\sqrt[4]{3\Omega^2/4c}x-2\Omega t+\eta_0\right)\left[\cos\left(-\sqrt[4]{3\Omega^2/4c}x+\Omega t+\xi_0\right)-\sin\left(-\sqrt[4]{3\Omega^2/4c}x+\Omega t+\xi_0\right)\right]}{b\left[A+B\sin\left(-\sqrt[4]{-\Omega^2/4c}x+\Omega t+\xi_0\right)\exp\left(-\sqrt[4]{3\Omega^2/4c}x-2\Omega t+\eta_0\right)\right]^2} - \frac{12cB\sqrt{3\Omega^2/4c}\exp\left(-\sqrt[4]{3\Omega^2/4c}x-2\Omega t+\eta_0\right)\cos\left(-\sqrt[4]{3\Omega^2/4c}x+\Omega t+\xi_0\right)}{b\left[A+B\sin\left(-\sqrt[4]{3\Omega^2/4c}x+\Omega t+\xi_0\right)\exp\left(-\sqrt[4]{3\Omega^2/4c}x-2\Omega t+\eta_0\right)\right]}. \tag{42}$$

*Remark 1.* Equations (29) and (30) may also have other forms of solutions as follows:

$$\begin{aligned}\phi(x,t)&=A+B\cos(\xi)\exp(\eta),\\ \phi(x,t)&=A+B\sinh(\xi)\exp(\eta),\\ \phi(x,t)&=A+B\cosh(\xi)\exp(\eta),\end{aligned} \tag{43}$$

where $\xi=Kx+\Omega t+\xi_0, \eta=kx+\omega t+\eta_0$; $A, B, K, \Omega, \xi_0, k, \omega,$ and $\eta_0$ are constants which are to be determined.

## 3. Dynamical Behaviors and Chaotic Behaviors of Equation (1)

In this section, based on the theory of the plane dynamic system, the bifurcation analysis of the unperturbed dynamic system which is obtained by traveling wave transformation (special combinations of Lie point symmetries) is carried out. The periodic perturbed term is added to the obtained unperturbed dynamic system, and the chaotic behaviors of perturbed system is analyzed under different values of the physical parameters.

*3.1. Bifurcation and Phase Portraits of Equation (1).* In Ref. [9], Lie point symmetries admitted by Equation (1) were given as follows:

$$\begin{aligned}V_1&=\frac{\partial}{\partial t}, V_2=\frac{\partial}{\partial x},\\ V_3&=2bt\frac{\partial}{\partial t}+bx\frac{\partial}{\partial x}-(a+2bu)\frac{\partial}{\partial u},\end{aligned} \tag{44}$$

which form a three-dimensional Lie algebra.

We consider the Lie point symmetry as follows:

$$k_1V_1+k_2V_2=k_1\frac{\partial}{\partial t}+k_2\frac{\partial}{\partial x}, \tag{45}$$

where $k_1, k_2$ are constants.

The corresponding characteristic equation is

$$\frac{dt}{k_1}=\frac{dx}{k_2}=\frac{du}{0}. \tag{46}$$

Solving the above characteristic equation, we obtain the group invariant solution:

$$u(x,t)=X(\xi), \tag{47}$$

where $\xi = k_1 x - k_2 t$ is the similarity invariant. This is a traveling wave transformation, which we will use to analyze dynamic behaviors of the equation.

Under the traveling wave transformation, Equation (1) is reduced to

$$\left(k_2^2 - ak_1^2\right)X'' - 2bk_1^2\left[\left(X'\right)^2 + XX''\right] + ck_1^4 X^{(4)} = 0, \quad (48)$$

where the "′" denotes the differentiation with respect to the variable $\xi$.

Integrating Equation (48) twice with respect to $\xi$ and vanishing the constants of integration yields

$$X'' = \alpha X^2 + \beta X. \quad (49)$$

where $\alpha = b/ck_1^2$, $\beta = (ak_1^2 - k_2^2)/ck_1^4$.

Letting $X' = Y$, Equation (49) is equivalent to the two-dimensional plane autonomous system:

$$\begin{cases} \dfrac{dX}{d\xi} = Y, \\ \dfrac{dY}{d\xi} = \alpha X^2 + \beta X. \end{cases} \quad (50)$$

Obviously, system (50) is an integrable Hamiltonian system. Applying the first integral, we get Hamiltonian functions as follows:

$$\frac{1}{2}Y^2 - \frac{\alpha}{3}X^3 - \frac{\beta}{2}X^2 = h, \quad (51)$$

where $h$ is the constant of integration.

By solving the following system:

$$\begin{cases} Y = 0, \\ \alpha X^2 + \beta X = 0, \end{cases} \quad (52)$$

equilibrium points of system (52) can be obtained as $P_1(0, 0)$ and $P_2(-\beta/\alpha, 0)$.

Let $M(P_i)$ denote the coefficient matrix of the linearized system of system (50) at equilibrium points $P_i(X_i, 0)(i = 1, 2)$ as follows:

$$M(P_i) = \begin{pmatrix} 0 & 1 \\ 2\alpha X_i + \beta & 0 \end{pmatrix}, \quad (53)$$

where the trace of $M(P_i)$ is zero and the determinant of $M$ is $J(P_i) = \det M(P_i) = -(2\alpha X_i + \beta)$. Then, we obtain $\det M(P_1) = -\beta$ and $\det M(P_2) = \beta$.

By the theory of planar dynamical systems [24–30], we draw the following conclusion as in Table 1. The different phase portraits of system (50) are shown in Figures 1 and 2.

*3.2. Chaotic Behaviors of Equation (1).* In this section, the dynamics of system (50) perturbed by periodic term is to be investigated via numerical simulations. For this purpose, suppose that periodic term is $g_0 \cos(\omega\xi)$. Letting $Z = \omega\xi$, system (50) is modified as following three-dimensional system:

$$\begin{cases} \dfrac{dX}{d\xi} = Y, \\ \dfrac{dY}{d\xi} = \alpha X^2 + B\beta X + g_0 \cos(Z), \\ \dfrac{dZ}{d\xi} = \omega. \end{cases} \quad (54)$$

In the simulations, choosing parameters $\alpha = 2, \beta = -8$, the effects of amplitude and frequency on the dynamics of system (54) are to be discussed.

Firstly, the effect of amplitude is considered. For this end, frequency $\omega$ is fixed while $g_0$ is chosen as different values. The corresponding phase portraits are depicted in Figures 3 and 4. By analyzing Figures 3 and 4, we can get the following results.

*Case 1.* When $g_0 = 0$. System (54) is undisturbed by periodic signal, and it is provided with periodic solution (see Figures 3(a) and 4(a)).

*Case 2.* When $g_0 \neq 0$. Figure 3 means that the system (54) is disturbed by periodic signal, and system (54) presents limit cycles for low frequency (see Figures 3(b)–3(d)). Figure 4 shows that, when system (54) is disturbed by periodic signal with higher frequency, it can appear as limit cycles (see Figures 4(b) and 4(c)) and chaotic phenomenon (see Figure 4(d))with the change of amplitude.

Consequently, Figures 3 and 4 indicate that the amplitude has much effect on the dynamics of system (54).

Secondly, $g_0$ taken as a constant, and $\omega$ is chosen as different values. The corresponding phase portraits are calculated and shown in Figures 5 and 6. In Figure 5, it is obvious to see that, for small amplitude $g_0 = 0.1$, with the change of frequency, system (54) has various phenomenon, such as chaos (see Figures 5(a) and 5(b)) and limit cycle (Figures 5(c) and 5(d)). Figure 6 suggests that, for small amplitude $g_0 = 5$, system (54) mainly shows chaos.

## 4. Conclusions

The extended homogeneous balance method is an effective method solving nonlinear partial differential equations. We applied it to obtain the auto-Bäcklund transformation transformation and some new exact explicit solutions for the fourth-order nonlinear generalized Boussinesq water wave equation. Using the theory of plane dynamic system, the dynamical behavior analysis of the perturbed dynamic system and the chaotic behaviors of the perturbed system are analyzed. The change of the physical parameters will affect the dynamic behavior of the dynamic system.

## Data Availability

There is no database involved in the manuscript.

## Conflicts of Interest

The authors declare no potential conflict of interest.

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