Journal of Applied Mathematics and Physics, 2024, 12, 1043-1049
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

The Regularity of Solutions to Mixed Boundary Value Problems of Second-Order Elliptic Equations with Small Angles

Mingyu Wu

School of Mathematics, East China University of Science and Technology, Shanghai, China
Email: Y30211263@mail.ecust.edu.cn

How to cite this paper: Wu, M.Y. (2024) The Regularity of Solutions to Mixed Boundary Value Problems of Second-Order Elliptic Equations with Small Angles. *Journal of Applied Mathematics and Physics*, **12**, 1043-1049.
https://doi.org/10.4236/jamp.2024.124064

Received: March 5, 2024
Accepted: April 8, 2024
Published: April 11, 2024

Copyright © 2024 by author(s) and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).
http://creativecommons.org/licenses/by/4.0/
Open Access

Abstract

This paper considers the regularity of solutions to mixed boundary value problems in small-angle regions for elliptic equations. By constructing a specific barrier function, we proved that under the assumption of sufficient regularity of boundary conditions and coefficients, as long as the angle is sufficiently small, the regularity of the solution to the mixed boundary value problem of the second-order elliptic equation can reach any order.

Keywords

Mixed Boundary Value Problems for Elliptic Equations, Small-Angle Boundary Value Problems, Regularity of Solutions to Elliptic Equations

1. Introduction

This paper studies the regularity of solutions to mixed boundary value problems of two-dimensional linear second-order elliptic equations in an angular region. When the corresponding angle in the angular region tends to 0, we consider the regularity influence of the solution of the equation under the assumption that the boundary conditions, inhomogeneous terms and coefficients have sufficient regularity. The study of mixed boundary value problems for elliptic equations in angular regions has important theoretical significance and practical application value. This type of problem has been systematically discussed in the study of elliptic equations, see [1]-[9]. Recently, in the study of hypersonic flow, we encountered a type of small-angle boundary value problem for elliptic equations. The regularity of its solution is of great significance.

2. Preparatory Work

This article considers the simplest type of mixed boundary value problems. Assume that the origin $O=(0,0)$ is a corner point, and consider the angular region:

$$\Omega_{\varepsilon}:=\{(x,y)\mid 0<y<\varepsilon x, x>0\} \tag{2.1}$$

Its boundary consists of the following two rays:

$$l_0:=\{(x,y)\mid y=0, x\geq 0\}, \tag{2.2}$$

and

$$l_{\varepsilon}:=\{(x,y)\mid y=\varepsilon x, x\geq 0\}, \tag{2.3}$$

where $0<\varepsilon\ll 1$.

Assume that:

$$B:=\{(x,y)\mid x^2+y^2\leq 1\}. \tag{2.4}$$

We consider the regularity of the solution to the following boundary value problem for the elliptic equation at the origin O:

$$u(x,\varepsilon x)=f(x) \text{ on } l_{\varepsilon}\cap B; \tag{2.5}$$

$$bu_x+u_y=0 \text{ on } l_0\cap B; \tag{2.6}$$

$$\Delta u=u_{xx}+u_{yy}=0 \text{ in } \Omega_{\varepsilon}\cap B; \tag{2.7}$$

where b is a fixed constant, $f\in C^{m,\alpha}([0,1])$.

Regarding this problem, we have the results of the Lieberman's weighted Hölder norm corner point elliptic equation problem [6]. *i.e.* $u\in C^{1,\beta}(\Omega_{\varepsilon}\cap B)$, in particular $u\in C^{1,\beta}(O)$, where β is determined by ε and b. [10] proved the case of second-order regularity. Inspired by its results, we will prove the following regularity theorem by constructing a similar barrier function.

Theorem 1. For any natural number $m>2$, there exists a sufficiently small positive number ε_0 such that when $0<\varepsilon<\varepsilon_0$, we have $u\in C^{m,\alpha}(O)$. In fact, we will prove that there exists a polynomial of degree m, denoted as $P_m(x,y)$ such that the following inequality holds:

$$|u-P_m|\leq Cr^{m+\alpha},$$

where C is a constant determined by $\|f\|_{C^{m,\alpha}([0,1])}, b$ and $\|u\|_{L^{\infty}(\Omega_{\varepsilon}\cap\partial B)}$.

3. Proof of the Main Theorem

Consider the higher-order boundary conditions corresponding to (2.5)-(2.7). Taking the k-order derivative of x along l_{ε}, we have:

$$\sum_{i=0}^{k}C_k^i\varepsilon^{k-i}\partial_x^i\partial_y^{k-i}u(x,\varepsilon x)=f^{(k)}(x) \text{ on } l_{\varepsilon}\cap B. \tag{3.1}$$

Taking the $(k-1)$-order derivative of x along l_0, we have:

$$b\partial_x^k u(x,0)+\partial_x^{k-1}\partial_y u(x,0)=0 \text{ on } l_0\cap B. \tag{3.2}$$

Taking the partial derivative $\partial_x^i \partial_y^{k-i-2}$ of the function, where $i = 0, \cdots, k-2$, we have:

$$\partial_x^{i+2} \partial_y^{k-i-2} u(x, y) + \partial_x^i \partial_y^{k-i} u(x, y) = 0 \text{ in } \Omega_\varepsilon \cap B. \tag{3.3}$$

We consider a polynomial of degree m, denoted as P_m, which is defined as:

$$P_m(x, y) := \sum_{k=0}^{m} \sum_{i=0}^{k} P_{i,k-i} x^i y^{k-i}.$$

We have the following theorem:

Theorem 2. There exists a unique polynomial of degree m, denoted as $P_m(x, y)$, which satisfies for all $k = 0, \cdots, m$,

$$\sum_{i=0}^{k} C_k^i \varepsilon^{k-i} \partial_x^i \partial_y^{k-i} P_m(0,0) = f^{(k)}(0); \tag{3.4}$$

$$b \partial_x^k P_m(0,0) + \partial_x^{k-1} \partial_y P_m(0,0) = 0; \tag{3.5}$$

and

$$\partial_x^{i+2} \partial_y^{k-i-2} P_m(0,0) + \partial_x^i \partial_y^{k-i} P_m(0,0) = 0 \tag{3.6}$$

where $i = 0, \cdots, k-2$.

Proof. Obviously, $P_m(x, y)$ is determined by $\partial_x^i \partial_y^{k-i} P_m(0,0), (0 \le i + j \le m)$. We note that for a given $0 \le k \le m$, (3.4)-(3.6) form a linear system of equations of $(k+1)$-order with respect to $\partial_x^i \partial_y^{k-i} P_m(0,0)$. According to Cramer's law, we only need to verify that its coefficient determinant is not zero.

In fact, let $\varepsilon = 0$, the matrix form corresponding to (3.4)-(3.6) is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_x^k P_m(0,0) \\ \partial_x^{k-1} \partial_y P_m(0,0) \\ \partial_x^{k-2} \partial_y^2 P_m(0,0) \\ \partial_x^{k-3} \partial_y^3 P_m(0,0) \\ \vdots \\ \partial_y^k P_m(0,0) \end{pmatrix} = \begin{pmatrix} f^{(k)}(0) \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Obviously, its coefficient determinant value is 1, and the equation has a unique solution. We note that when ε is very small, the coefficient determinant of (3.4)-(3.6) is a small perturbation of the above equation, so there must be a unique solution. The theorem is proved.

Proof of Theorem 1. Thus, for all $k = 0, \cdots, m$, we have:

$$\partial_{l_\varepsilon}^k (u - P_m)(0,0) = 0.$$

And thus, there is a boundary estimate:

$$|u - P_m|(x, \varepsilon x) \le \|f\|_{C^{m,\alpha}} |x|^{m+\alpha} \le 2\|f\|_{C^{m,\alpha}} r^{m+\alpha}. \tag{3.7}$$

At the same time, for all $k = 0, \cdots, m-1$, we have:

$$\partial_{l_0}^k \left((b\partial_x + \partial_y)(u - P_m) \right)(0,0) = 0.$$

And thus, there is a boundary estimate:

$$\left| (b\partial_x + \partial_y)(u - P_m)(x, 0) \right| \le Mr^m \le Mr^{m-1+\alpha}. \tag{3.8}$$

Considering the function $\Delta(u-P_m)$ in Ω_ε, then for all $i=0,\cdots,k-2$ and $k=2,\cdots,m$, we have:

$$\partial_x^i \partial_y^{k-i-2}\left(\Delta(u-P_m)\right)(0,0)=0.$$

Thereby, we obtain an equation estimate:

$$\left|\Delta(u-P_m)(x,y)\right| \le Mr^{m-1} \le Mr^{m-2+\alpha}. \tag{3.9}$$

When considering polar coordinates:

$$\begin{cases} r=\sqrt{x^2+y^2}, \\ \theta=\arctan\dfrac{y}{x}, \end{cases}$$

in this way, we have some differential relationships:

$$\begin{cases} r_x=\dfrac{x}{\sqrt{x^2+y^2}}=\cos\theta, & \theta_x=-\dfrac{y}{x^2+y^2}=-\dfrac{\sin\theta}{r}, \\ r_y=\dfrac{y}{\sqrt{x^2+y^2}}=\sin\theta, & \theta_y=\dfrac{x}{x^2+y^2}=\dfrac{\cos\theta}{r}, \\ r_{xx}=\dfrac{1}{\sqrt{x^2+y^2}}-\dfrac{x^2}{\left(x^2+y^2\right)^{3/2}}=\dfrac{1}{r}-\dfrac{\cos^2\theta}{r}, & \theta_{xx}=\dfrac{2xy}{\left(x^2+y^2\right)^2}=\dfrac{2\sin\theta\cos\theta}{r^2}, \\ r_{yy}=\dfrac{1}{\sqrt{x^2+y^2}}-\dfrac{y^2}{\left(x^2+y^2\right)^{3/2}}=\dfrac{1}{r}-\dfrac{\sin^2\theta}{r}, & \theta_{yy}=-\dfrac{2xy}{\left(x^2+y^2\right)^2}=-\dfrac{2\sin\theta\cos\theta}{r^2}, \\ r_{xy}=-\dfrac{xy}{\left(x^2+y^2\right)^{3/2}}=\dfrac{\sin\theta\cos\theta}{r}, & \theta_{xy}=-\dfrac{1}{x^2+y^2}+\dfrac{2y^2}{\left(x^2+y^2\right)^2}=\dfrac{\cos^2\theta-\sin^2\theta}{r^2}. \end{cases}$$

We assume that the barrier function has the form:

$$U=r^{m+\alpha}\sin\left(\kappa\theta+\varphi_0\right). \tag{3.10}$$

where κ and φ_0 are undetermined constants. In this way, we first have:

$$U\big|_{l_\varepsilon}=U\big|_{\theta=\arctan\varepsilon}=r^{m+\alpha}\sin\left(\kappa\arctan\varepsilon+\varphi_0\right). \tag{3.11}$$

Then, due to

$$\begin{cases} U_x=(m+\alpha)\dfrac{x}{r}r^{m+\alpha-1}\sin\left(\kappa\theta+\varphi_0\right)-\kappa\dfrac{y}{r}r^{m+\alpha-1}\cos\left(\kappa\theta+\varphi_0\right); \\ U_y=(m+\alpha)\dfrac{y}{r}r^{m+\alpha-1}\sin\left(\kappa\theta+\varphi_0\right)+\kappa\dfrac{x}{r}r^{m+\alpha-1}\cos\left(\kappa\theta+\varphi_0\right). \end{cases}$$

We have the expression for the boundary oblique derivative:

$$\begin{aligned} bU_x+U_y\big|_{l_0} &= bU_x+U_y\big|_{\theta=0} \\ &= b(m+\alpha)r^{m+\alpha-1}\sin\varphi_0+\kappa r^{m+\alpha-1}\cos\varphi_0 \\ &= r^{m+\alpha-1}\left(b(m+\alpha)\sin\varphi_0+\kappa\cos\varphi_0\right) \\ &= \kappa\cos\varphi_0 r^{m+\alpha-1}\left(\frac{b(m+\alpha)}{\kappa}\tan\varphi_0+1\right). \end{aligned} \tag{3.12}$$

Finally, applying the Laplace operator Δ, we have:

$$\Delta U = U_{rr} + \frac{U_r}{r} + \frac{U_{\theta\theta}}{r^2} = \left((m+\alpha)^2 - \kappa^2 \right) r^{m+\alpha-2} \sin(\kappa\theta + \varphi_0). \tag{3.13}$$

Now, we set κ and φ_0. On the one hand, let $\kappa > 0$ be sufficiently large and satisfy:

$$(m+\alpha)^2 - \kappa^2 \le -1; \tag{3.14}$$

and

$$\left| \frac{b(m+\alpha)}{\kappa} \right| \le \frac{1}{2}. \tag{3.15}$$

On the other hand, let:

$$\varphi_0 = \frac{3\pi}{4}. \tag{3.16}$$

It can be seen from the above selection that κ and φ_0 here only depend on *m*, *α* and *b*. In this way, for the κ and φ_0 that have been determined above, we can make ε_0 sufficiently small such that when $\varepsilon < \varepsilon_0$, we have:

$$\kappa \arctan \varepsilon + \varphi_0 = \kappa \arctan \varepsilon + \frac{3\pi}{4} \in \left[\frac{3\pi}{4}, \frac{5\pi}{6} \right]. \tag{3.17}$$

Let's summarize the properties following the above settings. First, from (3.11) and (3.17), we have:

$$U\big|_{\theta=\arctan\varepsilon} = r^{m+\alpha} \sin(\kappa \arctan \varepsilon + \varphi_0) \ge \frac{r^{m+\alpha}}{2}. \tag{3.18}$$

Secondly, from (3.14), (3.15) and (3.16), we have:

$$bU_x + U_y\big|_{\theta=0} = \kappa \cos\varphi_0 r^{m+\alpha-1} \left(\frac{b(m+\alpha)}{\kappa} \tan\varphi_0 + 1 \right) \le -\frac{r^{m+\alpha-1}}{4}. \tag{3.19}$$

Finally, from (3.14), (3.16) and (3.17),

$$\Delta U = \left((m+\alpha)^2 - \kappa^2 \right) r^{m+\alpha-2} \sin(\kappa\theta + \varphi_0) \le -\frac{r^{m+\alpha-2}}{2}. \tag{3.20}$$

Here, $\theta \in [0, \arctan\varepsilon]$. Furthermore,

$$U\big|_{r=1} = r^{m+\alpha} \sin(\kappa\theta + \varphi_0)\big|_{r=1} = \sin(\kappa\theta + \varphi_0) \ge \frac{1}{2}. \tag{3.21}$$

Thus, for a positive constant *C*, we consider $CU \pm (u - P_m)$. Combining (3.7)-(3.9) and (3.18)-(3.21), by choosing a sufficiently large *C*, we can get:

$$CU \pm (u - P_m) > 0 \ \text{ on } l_\varepsilon \cap B;$$

$$(b\partial_x + \partial_y)(CU \pm (u - P_m)) < 0 \ \text{ on } l_0 \cap B;$$

$$CU \pm (u - P_m) > 0 \ \text{ on } \Omega_\varepsilon \cap \partial B;$$

$$\Delta(CU \pm (u - P_m)) < 0 \ \text{ in } \Omega_\varepsilon \cap B.$$

According to the extreme value principle, we have:

$$CU \pm (u - P_m) > 0 \ \text{ in } \Omega_\varepsilon \cap B,$$

that is

$$\left|u-P_m\right| \le CU \le Cr^{m+\alpha} \text{ in } \Omega_\varepsilon \cap B.$$

The theorem is proved.

4. Conclusion and Suggestions

This paper studies the regularity of solutions to mixed boundary value problems of second-order linear elliptic equations with small angles in a two-dimensional region. We proved that under the assumption of sufficient regularity of boundary conditions, inhomogeneous terms and coefficients, as long as the angle is sufficiently small, the regularity of the solution to the mixed boundary value problem of the second-order elliptic equation can reach any order. Our proof process mainly used the construction of the barrier function, the theory of Schauder estimation, and the regularity conclusion of the solution to the oblique derivative problems in Lipschitz domains. The results can also be extended to high dimensions and high orders. For details, please refer to Lieberman's work [6] [11], which has a more systematic explanation.

Acknowledgements

Sincerely thank Dian Hu for his professional education and careful guidance.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Grisvard, P. (1985) Elliptic Problems in Nonsmooth Domains. In: *Monographs and Studies in Mathematics*, Vol. 24, Pitman Advanced Publishing Program, Pitman Publishing Inc., Boston, MA, xiv+410p.

[2] Lieberman, G.M. (1985) The Perron Process Applied to Oblique Derivative Problems. *Advances in Mathematics*, **55**, 161-172. https://doi.org/10.1016/0001-8708(85)90019-2

[3] Lieberman, G.M. (1986) Intermediate Schauder Estimates for Oblique Derivative Problems. *Archive for Rational Mechanics and Analysis*, **93**, 129-134. https://doi.org/10.1007/BF00279956

[4] Lieberman, G.M. (1986) Mixed Boundary Value Problems for Elliptic and Parabolic Differential Equations of Second Order. *Journal of Mathematical Analysis and Applications*, **113**, 422-440. https://doi.org/10.1016/0022-247X(86)90314-8

[5] Lieberman, G.M. (1987) Oblique Derivative Problems in Lipschitz Domains. I. Continuous Boundary Data. *Bollettino dell Unione Matematica Italiana B*, **1**, 1185-1210.

[6] Lieberman, G.M. (1988) Oblique Derivative Problems in Lipschitz Domains. II. Discontinuous Boundary Data. *Journal für Die Reine und Angewandte Mathematik*, **389**, 1-21. https://doi.org/10.1515/crll.1988.389.1

[7] Lieberman, G.M. (1989) Optimal Hölder Regularity for Mixed Boundary Value Problems. *Journal of Mathematical Analysis and Applications*, **143**, 572-586. https://doi.org/10.1016/0022-247X(89)90061-9

[8] Lieberman, G.M. (2001) Pointwise Estimates for Oblique Derivative Problems in Nonsmooth Domains. *Journal of Differential Equations*, **173**, 178-211. https://doi.org/10.1006/jdeq.2000.3939

[9] Lieberman, G.M. (2002) Higher Regularity for Nonlinear Oblique Derivative Problems in Lip-Schitz Domains. *Annali Scuola Normale Superiore-Classe Di Scienze*, **1**, 111-151.

[10] Azzam, A. and Kreyszig, E. (1982) On Solutions of Elliptic Equations Satisfying Mixed Boundary Conditions. *SIAM Journal on Mathematical Analysis*, **13**, 254-262. https://doi.org/10.1137/0513018

[11] Lieberman, G.M. (2013) Oblique Derivative Problems for Elliptic Equations. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, xvi+509p. https://doi.org/10.1142/8679