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V_4- Magic Labelings of Some Shell Related Graphs

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Abstract

For any abelian group A, a graph $G = (V, E)$ is said to be A-magic if there exists a labeling $\ell : E(G) \to A \setminus \{0\}$ such that the induced vertex set labeling $\ell^+ : V(G) \to A$ defined by

$$\ell^+(v) = \sum \{\ell(vu) : vu \in E(G)\}$$

is a constant map. A graph $G = (V, E)$ is said to be a-sum A magic if there there exists an $a \in A$ such that $\ell^+(v) = a$ for all $v \in V$. In particular, if a is the identity element 0, we say that G is zero-sum A magic. In this paper we will consider the Klein-four group $V_4 = \{0, a, b, c\}$ and investigate a class of V_4 magic shell related graphs that belongs to the following categories:

(i) $\mathscr{V}_a$, the class of a-sum V_4 magic graphs,

(ii) $\mathscr{V}_0$, the class of zero-sum V_4 magic graphs,

(iii) $\mathscr{V}_{a,0}$, the class of graphs which are both a-sum and zero -sum V_4 magic.

Keywords: Klein 4-group; V_4- magic graph; Shell graph; multiple shell; umbrella graph.

2010 Mathematics Subject Classification: 05C78, 05C25

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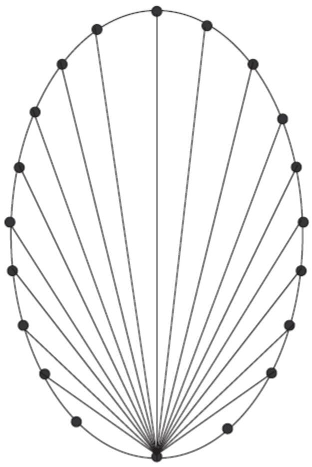

Figure 1: The Shell graph $H(n, n-3)$

1 Introduction

In this paper all graphs are connected, finite, simple, and undirected. For graph theory notations and terminology not directly defined in this paper, we refer readers to [1]. For positive integers n, k, $1 \le k \le n-3$, $H(n,k)$ is used to denote the cycle C_n with k chords sharing a common endpoint called the apex. In general $H(n,k)$ represents a family of graphs. For certain choices of n and k, the family $H(n,k)$ may be singleton. For example, when $k = n-3$, the family $H(n, n-3)$ is singleton, called a shell (see fig.1)[2]. Observe that the shell $H(n, n-3)$ is the same as the fan $F_{n-1} = P_{n-1} + K_1$. For, $2 \le p \le n-r$, let $C_n(p,r)$ denote cycle $C_n : (v_0, v_1, \ldots, v_{n-1}, v_0)$ with consecutive r chords $v_0v_p, v_0v_{p+1}, \ldots, v_0v_{p+r-1}$. Sin-Min Lee and Nien Tsuf [3] defined an umbrella graph $U(m,n)$ to be a graph obtained by joining a path P_n with the apex of a shell $H(m, m-3)$(see fig.2)[3]. An extended umbrella graph $U(m,n,k)$ is a graph obtained by identifying the pendant vertex of the umbrella $U(m,n)$ with the center(apex) of the star $K_{1,k}$ (see fig.2) [3]. A multiple shell $MS(n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r})$ is a graph formed by t_i shells of width n_i each, $1 \le i \le r$, which have a common apex [4]. Thus a multiple shell is a one point union of many shells. Observe that the multiple shell $MS(n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r})$ has $\sum_{i=1}^{n}(n_i - 1)t_i + 1$ vertices. If there are k shells with a common apex, then it is called a k- tuple shell. A multiple shell is said to be balanced if it is of the form $MS(p^t)$ or of the form $MS(p^t, (p+1)^s)$ [4].

For an abelian group A, written additively, any mapping $\ell : E(G) \to A \setminus \{0\}$ is called a labeling, where 0 denote the identity element in A. For any abelian group A, a graph $G = (V; E)$ is said to be A-magic if there exists a labeling $\ell : E(G) \to A \setminus \{0\}$ such that the induced vertex set labeling $\ell^+ : V(G) \to A$ defined by

$$\ell^+(v) = \sum \{\ell(uv) : uv \in E(G)\}$$

is a constant map [5]. If $\ell : E(G) \to A \setminus \{0\}(|A| > 2)$ is a magic labeling of G with sum c, then $-\ell : E(G) \to A \setminus \{0\}$, defined by $(-\ell)(u) = -\ell(u)$ is another A- magic labeling of G with sum $-c$. The labeling $-\ell$ is called the inverse of ℓ. This implies that A- magic labeling of a graph need not be unique. A graph $G = (V, E)$ is called non-magic if for every abelian group A, the graph is not A-magic [5]. The most obvious example of a non-magic graph is $P_n(n > 3)$, the path of order n. As a result, any graph with a path pendant of length at least two would be non-magic. The Klein 4-group, denoted by V_4 is an abelian group of order 4. The Cayley table for V_4 is given below:

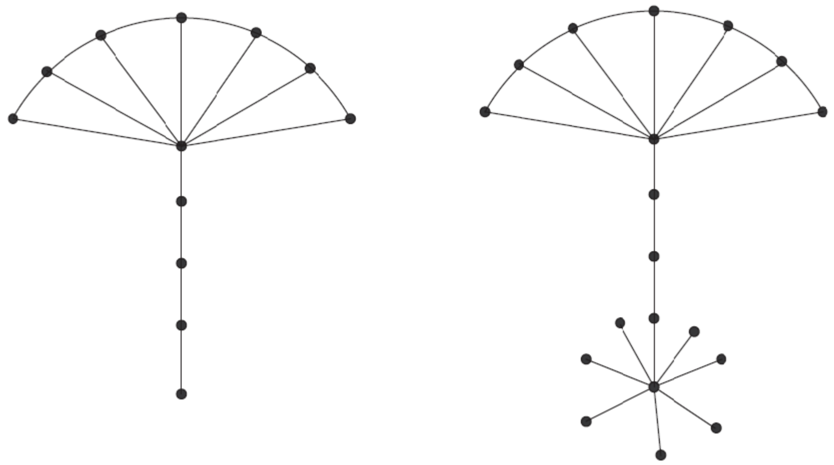

Figure 2: The graphs $U(m, n)$(left) and $U(m, n, k)$ (right)

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Observe that $a + a = b + b = c + c = 0$ and $a + b = c, b + c = a, c + a = b$. Also note that V_4 is not cyclic, since every element has order 2 (except for the identity, of course) and V_4 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The V_4 magic graphs was introduced by S. M. Lee et al. in 2002 [5]. There has been an increasing interest in the study of V_4 magic graphs since the publication of [5]. Let A be a group and let $a \in A$. A graph G is said to be a-sum V_4 magic if there exists a labeling $\ell : E(G) \to V_4 \setminus \{0\}$ such that the induced vertex set labeling $\ell^+ : V(G) \to A$ satisfies $\ell^+(v) = a$ for all $v \in V(G)(a \neq 0)$ [6]. If $\ell^+(v) = 0$, for all $v \in V(G)$, the graph is zero-sum V_4 magic [6]. In [6], the authors classified the class of V_4 magic graphs into the following three categories and identified some wheel related graphs that belongs to these categories. Moreover, investigated necessary and sufficient condition for several wheel related graphs that may fall into the following categories.

(*i*) $\mathscr{V}_a$, the class of a-sum V_4 magic graphs,

(*ii*) $\mathscr{V}_0$, the class of zero-sum V_4 magic graphs,

(*iii*) $\mathscr{V}_{a,0}$, the class of graphs which are both a-sum and zero -sum V_4 magic.

In this paper, we continue the study carried out in [6] and identify some shell related graphs that belongs to the above categories.

2 Main Results

We start with the following lemma.

Lemma 2.1. *If* $\ell : E(H(n, n-3)) \to V_4 \setminus \{0\}$ *is a labeling of the shell* $H(n, n-3)$, *then*

$$\sum_{i=0}^{n-1} \ell^+(u_i) = 0, \tag{2.1}$$

where $v_0, v_1, \ldots, v_{n-1}$ *are the vertices of* C_n *and* v_0 *is the apex.*

Proof. Observe that

$$\begin{aligned}
\ell^+(v_0) &= \sum_{i=1}^{n-1} \ell(v_0 v_i),\\
\ell^+(v_1) &= \ell(v_1 v_0) + \ell(v_1 v_2),\\
\ell^+(v_{n-1}) &= \ell(v_{n-1} v_0) + \ell(v_{n-1} v_{n-2}), \quad \text{and}\\
\ell^+(v_i) &= \ell(v_{i-1} v_i) + \ell(v_i v_{i+1}) + \ell(v_0 v_i), \quad \text{for} \;\; i = 2, 3, \ldots, n-2.
\end{aligned}$$

Adding the above equations, we obtain that

$$\sum_{i=0}^{n-1} \ell^+(v_i) = 0.$$

This completes the proof. ☐

Theorem 2.1. *$H(n, n-3) \in \mathscr{V}_a$ if and only if n is even.*

Proof. Assume that $H(n, n-3) \in \mathscr{V}_a$. Then $\ell^+(u_i) = a$ for $i = 0, 1, \ldots, n-1$. Then by lemma 2.1, we have $na = 0$. This implies that n is even.

Conversely, assume that n is even. We need to show that $H(n, n-3) \in \mathscr{V}_a$. Let the vertices of $H(n, n-3)$ be $v_0, v_1, \ldots, v_{n-1}$. Assume that v_0 be the apex of $H(n, n-3)$. Define $\ell : E(H(n, n-3)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_0 v_i) = \begin{cases} c & \text{for } i = 1, n-1,\\ a & \text{for } i = 2, 3, \ldots, n-2, \end{cases}$$
$$\ell(v_i v_{i+1}) = b \quad \text{for } i = 1, 2, 3, \ldots, n-2.$$

Then we have:

$$\ell^+(v_i) = \begin{cases} c + c + (n-3)a = a, & \text{for} \;\; i = 0,\\ b + c = a, & \text{for} \;\; i = 1, n-1,\\ b + b + a = a, & \text{for} \;\; i = 2, 3, \ldots, n-2. \end{cases}$$

This completes the proof. ☐

Theorem 2.2. *$H(n, n-3) \in \mathscr{V}_0$ if n is even (see [7]).*

Theorem 2.3. *If n is even $H(n, n-3) \in \mathscr{V}_{a,0}$.*

Proof. Proof follows from Theorems 2.1 and 2.2. ☐

Theorem 2.4. *If n is odd, then $H(n, n-3) \notin \mathscr{V}_a$.*

Proof. Assume that n is odd and let $H(n, n-3) \in \mathscr{V}_a$. Then by lemma 2.1, we have $na = 0$. This implies that $a = 0$. This is a contradiction. The result now follows. ☐

Theorem 2.5. *$U(n, m) \notin \mathscr{V}_a$ if $m \geq 2$.*

Proof. Since any graph with a path pendant of length at least two is non-magic, $U(n, m) \notin \mathscr{V}_a$ if $m \geq 2$. ☐

Theorem 2.6. *$U(n, 1) \in \mathscr{V}_a$ if n is odd.*

Proof. Let the vertices of $U(n,1)$ be $\{v_0, v_1, v_2, \ldots, v_{n-1}, u_n\}$, where v_0 is the apex of $H(n, n-3)$ and u_n is the pendant vertex. Define $\ell : E(U(n,1)) \to V_4 \setminus \{0\}$ by

$$\ell(v_0v_i) = \begin{cases} c, & \text{for } i = 1, n-1, n, \\ a, & \text{for } i = 2, 3, \ldots, n-2, \end{cases}$$
$$\ell(v_iv_{i+1}) = b \text{ for } i = 1, 2, 3, \ldots, n-2.$$

Then we have,

$$\ell^+(v_i) = \begin{cases} c + c + (n-3)a + a = a, & \text{for } i = 0, \\ b + c = a, & \text{for } i = 1, n-1, n, \\ b + b + a = a, & \text{for } i = 2, 3, \ldots, n-2. \end{cases}$$

This completes the proof. □

Theorem 2.7. *$U(m, n, k) \notin \mathscr{V}_a$ if $n \geq 2$.*

Proof. Assume that $n \geq 2$ and $U(m, n, k) \in \mathscr{V}_a$. Let v_0 be the apex of $H(m, m-3)$ and u_{n-1} be the apex of $K_{1,k}$. Let $V(H(m, m-3)) = \{v_0, v_1, \ldots, v_{m-1}\}$, $V(P_n) = \{v_0, u_1, u_2, \ldots, u_{n-1}\}$ and $V(K_{1,n}) = \{u_{n-1}, w_1, w_2, \ldots, w_k\}$. Since $\ell^+(v) = a$ for all $v \in V(U(m, n, k))$, we can label all pendant vertices of $U(m, n, k)$ by a. Assume that $\ell(u_{n-2}u_{n-1}) = x$, $x \in V_4 \setminus \{0\}$. Since $\ell^+(u_{n-1}) = a$, $ka + x = a$. This implies that $x = (k-1)a$. Hence, x=0, if k is odd and $x = a$, if k is even. Observe that $x = 0$ is not admissible. Moreover, $x = a$ implies that $\ell(u_{n-3}u_{n-2}) = 0$. This is also not admissible. This completes the proof. □

Theorem 2.8. *If $U(m, 1, k) \in \mathscr{V}_a$ then $m + k$ is odd.*

Proof. Observe that $U(m, 1, k)$ has $m+k+1$ vertices. If $U(m, 1, k) \in \mathscr{V}_a$, then one can easily verify that $(m + k + 1)a = 0$. This implies that $m + k$ is odd. □

Theorem 2.9. *If m is odd and k is even, then $U(m, 1, k) \in \mathscr{V}_a$.*

Proof. Let $V(H(m, m-3)) = \{v_0, v_1, \ldots, v_{m-1}\}$ and $V(K_{1,k}) = \{u_0, u_1, \ldots, u_k\}$. Define $\ell : U(m, 1, k) \to V_4 \setminus \{0\}$ by:

$$\ell(v_0v_i) = \begin{cases} c & \text{for } i = 1, m-1, \\ a, & \text{for } i = 2, 3, \ldots, m-2, \end{cases}$$
$$\ell(v_iv_{i+1}) = b \ \text{ for } \ i = 1, 2, 3, \ldots, m-2,$$
$$\ell(v_0u_0) = a.$$
$$\ell(u_0u_i) = a \ \text{ for } \ i = 1, 2, \ldots, k.$$

Then we have,

$$\ell^+(v_i) = \begin{cases} c + c + (m-3)a + a = a, & \text{for } \ i = 0, \\ b + c = a, & \text{for } \ i = 1, n-1, \\ b + b + a = a, & \text{for } i = 2, 3, \ldots, m-2, \end{cases}$$
$$\ell^+(u_i) = \begin{cases} ka + a = a & \text{for } i = 0, \\ a, & \text{for } i = 1, 2, 3, \ldots, k. \end{cases}$$

This completes the proof. □

Theorem 2.10. *If m is even and n is odd, then $U(m, 1, k) \notin \mathscr{V}_a$.*

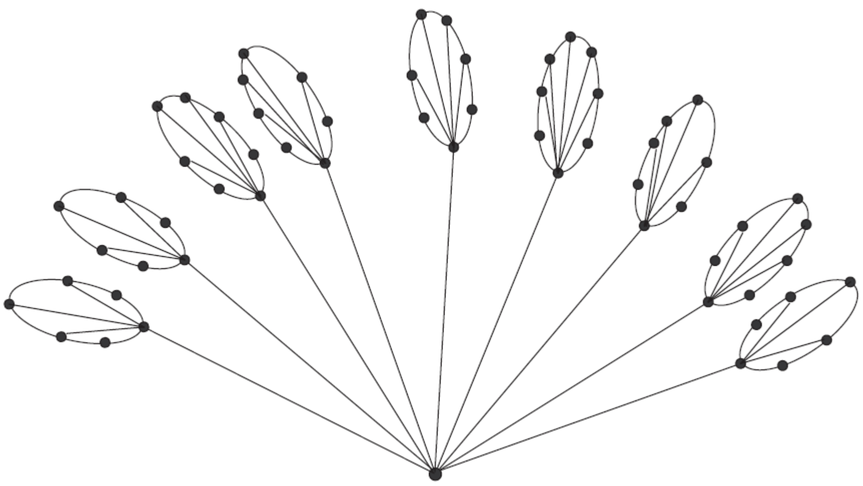

Figure 3: $B(t, n_1, n_2, \dots, n_t)$

Proof. Label all the pendant edges of the star by a and label the edge v_0u_0 by x. If $U(m, 1, k) \in \mathscr{V}_a$, then $ka + x = a$. This implies that $x = 0$. This is a contradiction. The result now follows. □

Let $B(t, n_1, n_2, \dots, n_t)$ be the graph obtained by identifying each pendant vertex v_i of the star $K_{1,t}$ with apex of shells $H(n_i, n_i - 3)$, $i = 1, 2, \dots, t$ (see fig.3). Then we have the following:

Theorem 2.11. *If $B(t, n_1, n_2, \dots, n_t) \in \mathscr{V}_a$, then $n_1 + n_2 + \cdots + n_t$ is odd.*

Proof. Observe that $B(t, n_1, n_2, \dots, n_t)$ has $n_1 + n_2 + \cdots + n_t + 1$ vertices. So, we have $(n_1 + n_2 + \cdots + n_t + 1)a = 0$. This implies that $n_1 + n_2 + \cdots + n_t$ is odd. □

Theorem 2.12. *If n and t are odd then $B(t, n, n, \dots, n) \in \mathscr{V}_a$.*

Proof. Let the vertex set of $K_{1,t}$ be $\{v_0, v_1, v_2, \dots, v_t\}$, where v_0 is the apex. Consider t copies of the shell $H(n, n-3)$. Let $H^i(n, n-3)$ be the i^{th} copy of $H(n, n-3)$. Let the vertex set of $H^i(n, n-3)$ be $\{v_i, v_1^i, v_2^i, \dots, v_{n-1}^i\}$, where v_i is the apex. Define a labeling $\ell : E(B(t, n, n, \dots, n)) \to V_4 \setminus \{0\}$ by

$$\begin{aligned}
&\ell(v_0 v_i) = a, \text{ for } i = 1, 2, \dots, t,\\
&\text{for } i = 1, 2, \dots, t:\\
&\begin{cases}
\ell(v_i v_1^i) = c,\\
\ell(v_i v_{n-1}^i) = c,\\
\ell(v_j^i v_{j+1}^i) = b, \text{ for } j = 1, 2, \dots, n-2,\\
\ell(v_i v_j^i) = a, \text{ for } j = 2, 3, \dots, n-2.
\end{cases}\\
&\text{end for}
\end{aligned}$$

Obviously ℓ is an a-sum magic labeling of $B(t, n, n, \dots, n)$. □

Let $H(2n, n-2)$ be the graph obtained by taking the cycle $C_{2n} : (v_0, v_1, \dots, v_{2n-1}, v_0)$ and its chords $v_0v_3, v_0v_5, \dots, v_0v_{2n-3}$. Observe that $H(2n, n-2)$ has $n-2$ chords. We have the following Theorem:

Theorem 2.13. $H(2n, n-2) \in \mathscr{V}_a$.

Proof. We consider two cases.

Case 1 Assume that n is even. Let $n = 2t$. Observe that in this case, the graph $H(2n, n-2)$ has $4t$ vertices. Let the vertex set of $H(2n, n-2)$ be $\{v_0, v_1, \ldots, v_{4t-1}\}$, where v_0 is the apex. For convenience, we denote the vertex v_0 by v_{4t}. Define $\ell : E(H(2n, n-2)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_{i-1}v_i) = \begin{cases} c & \text{for } i = 1, 4t-1, \\ b & \text{for } i = 2, 4t, \end{cases}$$

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} c & \text{for } i = 3, 7, 11, \ldots, 4t-5, \\ b & \text{for } i = 5, 9, 13, \ldots, 4t-3, \end{cases}$$

$$\ell(v_0v_i) = a \text{ for } i = 3, 5, 7, \ldots, 4t-3.$$

Obviously,

$$\ell^+(v_i) = \begin{cases} b + c + (2t-2)a = a & \text{for } i = 0, \\ b + c = a, & \text{for } i = 1, 2, 4, 6, \ldots, 2t+4, 4t-4, 4t-2, 4t-1, \\ c + c + a = a, & \text{for } i = 3, 7, \ldots, 4t-5, \\ b + b + a = a, & \text{for } i = 5, 9, \ldots, 4t-3. \end{cases}$$

Case 2: Assume that n is odd. Let $n = 2t+1$. In this case, the graph has $4t+2$ vertices. Let the vertex set of $H(n, n-2)$ be $\{v_0, v_1, v_2, \ldots, v_{4t+2}\}$. For convenience, we denote the vertex v_0 by v_{4t+2}. Define $\ell : E(H(2n, n-2)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_{i-1}v_i) = \begin{cases} c & \text{for } i = 1, 4t+2, \\ b & \text{for } i = 2, 4t+1, \end{cases}$$

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} c & \text{for } i = 3, 7, 11, \ldots, 4t-5, 4t-1, \\ b & \text{for } i = 5, 9, 13, \ldots, 4t-3, \end{cases}$$

$$\ell(v_0v_i) = a, \text{ for } i = 3, 5, 9, \ldots, 4t-1.$$

Obviously,

$$\ell^+(v_i) = \begin{cases} c + c + (2t-1)a = a, & \text{for } i = 0, \\ b + c = a, & \text{for } i = 1, 2, 4, 6, \ldots, 4t, 4t+1, \\ c + c + a = a, & \text{for } i = 3, 7, \ldots, 4t-1, \\ b + b + a = a, & \text{for } i = 5, 9, \ldots, 4t-3. \end{cases}$$

This completes the proof.

An a-sum V_4 magic labelings of $H(16, 6)$ and $H(18, 7)$ is shown in figure 4. □

Theorem 2.14. $H(2n, n-2) \in \mathscr{V}_0$.

Proof. We consider two cases.

Case 1: Suppose n is even. Let $n = 2t$. Let the vertex set of $H(2n, n-2)$ be $\{v_0, v_1, v_2, \ldots, v_{4t-1}\}$. Define $\ell : E(H(2n, n-1)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} c & \text{for } i = 2, 6, 10, 14, \ldots, 4t-6, 4t-2, 4t, \\ b & \text{for } i = 4, 8, 12, \ldots, 4t-4, \end{cases}$$

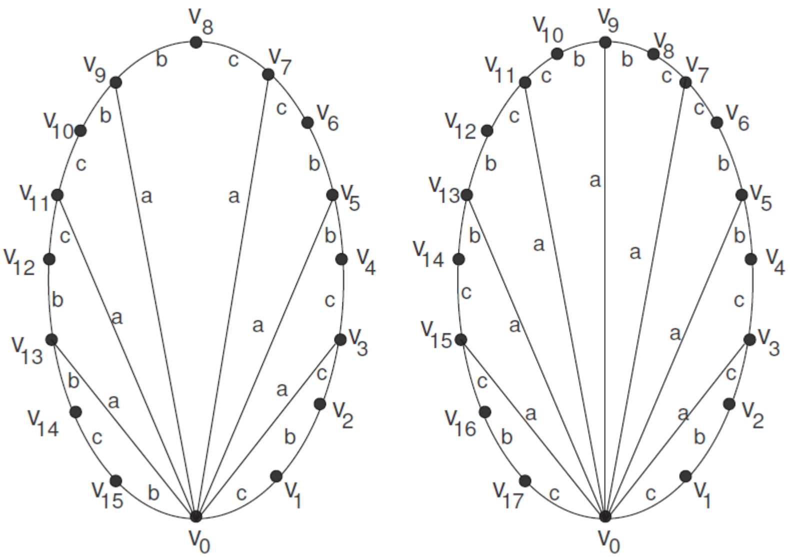

Figure 4: An a-sum V_4 magic labelings of $H(16,6)$ and $H(18,7)$

$$\ell(v_0v_i) = a \text{ for } i = 3, 5, 7, \ldots, 4t-3.$$

where $v_{4t} = v_0$. Obviously,

$$\ell^+(v_i) = \begin{cases} c + c + (2t-2)a = 0 & \text{for } i = 0, \\ c + c = 0, & \text{for } i = 1, 2, 6, \ldots, 4t-2, 4t-1, \\ a + b + c = 0, & \text{for } i = 3, 5, 7, \ldots, 4t-3, \\ b + b = 0, & \text{for } i = 4, 8, 12, \ldots, 4t-4. \end{cases}$$

Case 2: Assume that n is odd. Let $n = 2t+1$. In this case, the graph has $4t+2$ vertices. Define $\ell : E(H(2n, n-1)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} c & \text{for } i = 1, 2, 6, 10, 14, \ldots, 4t-2, \\ b & \text{for } i = 4, 8, 12, \ldots, 4t-4, 4t, 4t+1, \end{cases}$$

$$\ell(v_0v_i) = a \text{ for } i = 3, 5, 7, \ldots, 4t-1,$$

where $v_{4t+2} = v_0$. Obviously,

$$\ell^+(v_i) = \begin{cases} b + c + (2t-1)a = 0, & \text{for } i = 0, \\ c + c = 0, & \text{for } i = 1, 2, 6, \ldots, 4t-2, \\ a + b + c = 0, & \text{for } i = 3, 5, \ldots, 4t-1, \\ b + b = 0, & \text{for } i = 4, 8, \ldots, 4t, 4t+1. \end{cases}$$

This completes the proof. □

A 0-sum V_4 magic labelings of $H(16,6)$ and $H(18,7)$ is shown in figure 5.

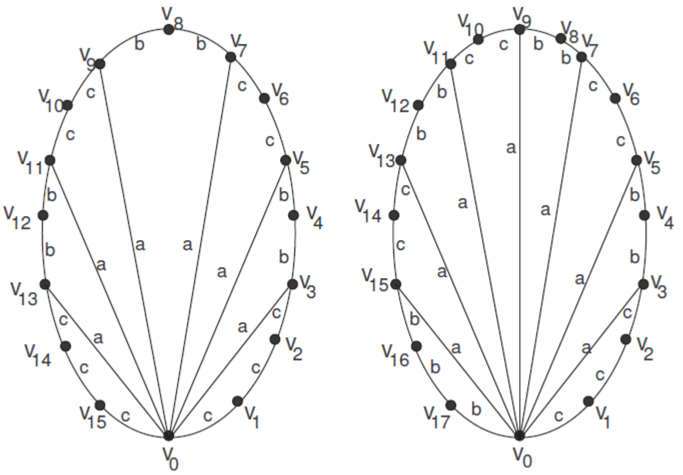

Figure 5: A 0-sum V_4 magic labelings of $H(16, 6)$ and $H(18, 7)$

Theorem 2.15. $H(2n, n-2) \in \mathscr{V}_{a,0}$.

Proof. Proof follows from Theorems 2.13 and 2.14. □

Let $H(2n, n-1)$ be the graph obtained by taking the cycle $C_{2n} : (v_0, v_1, \dots, v_{2n-1}, v_0)$ and its alternate chords $v_0v_2, v_0v_4, \dots, v_0v_{2n-2}$. Observe that $H(2n, n-1)$ has $n-1$ chords. We have the following Theorem:

Theorem 2.16. $H(2n, n-1) \in \mathscr{V}_a$.

Proof. **Case 1:** Assume that n is even. Let $n = 2t$. Observe that in this case, the graph $H(2n, n-1)$ has $4t$ vertices. Define $\ell : E(H(2n, n-1)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} c & \text{for } i = 4, 8, 12, \dots, 4t-4, 4t, \\ b & \text{for } i = 2, 6, 10, \dots, 4t-2, \end{cases}$$
$$\ell(v_0v_i) = a \text{ for } i = 2, 4, 6, \dots, 4t-2,$$

Obviously,

$$\ell^+(v_i) = \begin{cases} c + c + (2t-1)a = a, & \text{for } i = 0, \\ b + c = a, & \text{for } i = 1, 3, 5, 7, \dots, 4t-3, 4t-1, \\ a + b + b = a, & \text{for } i = 2, 6, \dots, 4t-2, \\ c + c + a = a, & \text{for } i = 4, 8, \dots, 4t-4. \end{cases}$$

Case 2: Assume that n is odd. Let $n = 2t+1$. In this case, the graph has $4t+2$ vertices. Define $\ell : E(H(2n, n-1)) \to V_4 \setminus \{0\}$ by:

$$\ell(v_{i-1}v_i) = \begin{cases} c & \text{for } i = 1, \\ b & \text{for } i = 4t+2, \end{cases}$$

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} c & \text{for } i = 4, 8, 12, \dots, 4t-4, 4t, \\ b & \text{for } i = 2, 6, 10, \dots, 4t-2, \end{cases}$$
$$\ell(v_0v_i) = a \quad \text{for } i = 2, 4, 6, \dots, 4t,$$

where $v_{4t} = v_0$.
Obviously,

$$\ell^+(v_i) = \begin{cases} b + c + (2t)a = a, & \text{for } i = 0, \\ b + c = a, & \text{for } i = 1, 3, 5, 7, \dots, 4t-1, 4t+1, \\ c + c + a = a, & \text{for } i = 4, 8, \dots, 4t, \\ b + b + a = a, & \text{for } i = 2, 6, \dots, 4t-2. \end{cases}$$

This completes the proof. □

Theorem 2.17. $H(2n, n-1) \in \mathscr{V}_0$ *(see [7])*

Theorem 2.18. $H(2n, n-1) \in \mathscr{V}_{a,0}$

Proof. Proof follows from Theorems 2.16 and 2.17. □

Let $H(4n+1, 2n)$ be the graph obtained by taking the cycle $C_{4n+1} := (v_0, v_1, \dots, v_{4n}, v_0)$, the consecutive middle chords v_0v_{2n} and v_0v_{2n+1} and all alternate chords symmetrically placed between the apex, that is, the chords:$v_0v_2, v_0v_4, \dots, v_0v_{2n-2}$; $v_0v_{2n+3}, v_0v_{2n+5}, \dots, v_0v_{4n-1}$.

Theorem 2.19. $H(4n+1, 2n) \notin \mathscr{V}_a$

Proof. Since the order of the graph $H(4n+1, 2n)$ is odd, $H(4n+1, 2n) \notin \mathscr{V}_a$. □

Theorem 2.20. $H(4n+1, 2n) \in \mathscr{V}_0$*(see [7]).*

Let $U(4n+1, 2n, 1)$ be the graph obtained by identifying the apex of $H(4n+1, 2n)$ with a vertex of K_2. Then we have the following:

Theorem 2.21. $U(4n+1, 2n, 1) \in \mathscr{V}_a$.

Proof. We consider two cases:

Case 1: Suppose $n = 2t$. Then $U(4n+1, 2n, 1)$ has $8t+2$ vertices. Let the vertex set of $H(4n+1, 2n)$ be $\{v_0, v_1, v_2, \dots, v_{8t}\}$ and let u be the pendant vertex. Define $\ell : E(U(4n+1, 2n, 1)) \to V_4 \setminus \{0\}$ by

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} c & \text{for } i = 2, 6, 10, \dots, 4t-2, 4t+3, 4t+7, \dots, 8t-1, \\ b & \text{for } i = 4, 8, 10, \dots, 4t-4, 4t, 4t+1, 4t+4, 4t+8, \dots, 8t+1 \end{cases}$$
$$\ell(v_0v_i) = a \text{ for } i = 2, 4, \dots, 4t, 4t+1, 4t+3, \dots, 8t-1,$$
$$\ell(v_0u) = a.$$

where $v_{8t+1} = v_0$ and $v_{8t+2} = v_1$. We have, $\ell^+(u) = a$ and

$$\ell^+(v_i) = \begin{cases} a + b + b + 4ta = a & \text{for } i = 0, \\ b + c = a & \text{for } i = 1, 3, 5, \dots, 4t-1, 4t+2, 4t+4, \dots, 8t, \\ c + c + a = a & \text{for } i = 2, 6, 10, \dots, 4t-2, 4t+3, 4t+7, \dots, 8t-1, \\ b + b + a = a & \text{for } i = 4, 8, 10, \dots, 4t-4, 4t, 4t+1, 4t+4, 4t+8, \dots, 8t-3 \end{cases}$$

Case 2: Suppose $n = 2t + 1$. In this case $U(4n + 1, 2n, 1)$ has $8t + 6$ vertices. Let the vertex set of $H(4n + 1, 2n)$ be $\{v_0, v_1, \ldots, v_{8t+4}\}$ and let the pendant vertex be u. Define $\ell : E(U(4n + 1, 2n, 1)) \to V_4 \setminus \{0\}$ by

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} b & \text{for } i = 2, 6, 10, \ldots, 4t + 2, 4t + 3, 4t + 7, \ldots, 8t + 3, \\ c & \text{for } i = 4, 8, 10, \ldots, 4t - 4, 4t, 4t + 5, 4t + 9, \ldots, 8t + 5 \end{cases}$$

where $v_{8t+5} = v_0$ and $v_{8t+6} = v_1$ and

$$\ell(v_0v_i) = a \text{ for } i = 2, 4, \ldots, 4t + 2, 4t + 3, 4t + 5, \ldots, 8t + 3.$$

We have, $\ell^+(u) = a$ and

$$\ell^+(v_i) = \begin{cases} a + c + c + (4t + 2)a = a & \text{for } i = 0, \\ b + c = a & \text{for } i = 1, 3, \ldots, 4t + 1, 4t + 6, 4t + 8, \ldots, 8t + 4, \\ c + c + a = a & \text{for } i = 4, 8, 10, \ldots, 4t - 4, 4t, 4t + 5, 4t + 9, \ldots, 8t + 1, \\ b + b + a = a & \text{for } i = 2, 6, 10, \ldots, 4t + 2, 4t + 3, 4t + 7, \ldots, 8t + 3. \end{cases}$$

This completes the proof. □

Let $H(4n + 3, 2n + 2)$ denotes cycle $C_{4n+3} := (v_0, v_1, \ldots, v_{4n+2}, v_0)$, the four consecutive middle chords $v_0v_{2n}, v_0v_{2n+1}, v_0v_{2n+2}, v_0v_{2n+3}$ and all alternate chords symmetrically placed between the apex, that is, the chords:$v_0v_2, v_0v_4, \ldots, v_0v_{2n-2}; v_0v_{2n+5}, v_0v_{2n+7}, \ldots, v_0v_{4n+1}$. Then we have the following:

Theorem 2.22. *$H(4n + 3, 2n + 2) \notin \mathscr{V}_a$.*

Proof. Obvious. □

Theorem 2.23. *$H(4n + 3, 2n + 2) \in \mathscr{V}_0$ (see [7]).*

Let $U(4n + 3, 2n + 2, 1)$ be the graph obtained by identifying the apex of $H(4n + 1, 2n + 2)$ with a vertex of K_2. Then we have the following:

Theorem 2.24. *$U(4n + 3, 2n + 2, 1) \in \mathscr{V}_a$.*

Proof. We consider two cases

Case 1: Suppose $n = 2t$. Then $U(4n + 3, 2n + 2, 1)$ has $8t + 4$ vertices. Let the vertex set of $U(4n + 3, 2n + 2)$ be $\{v_0, v_1, \ldots, v_{8t+2}\}$. Let u be the pendant vertex of $U(4n + 3, 2n + 2, 1)$. Define $\ell : E(U(4n + 3, 2n + 2, 1)) \to V_4 \setminus \{0\}$ by

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} c & \text{for } i = 2, 6, 10, \ldots, 4t - 2, 4t + 5, 4t + 9, \ldots, 8t + 1, \\ b & \text{for } i = 4, 8, \ldots, 4t, 4t + 1, 4t + 2, 4t + 3, 4t + 7, 4t + 11, \ldots, 8t + 3. \end{cases}$$

where $v_{8t+3} = v_0$ and $v_{8t+4} = v_1$ and

$$\ell(v_0v_i) = a \text{ for } i = 2, 4, 6, \ldots, 4t - 2, 4t, 4t + 1, 4t + 2, 4t + 3, 4t + 5, \ldots 8t + 1,$$
$$\ell(v_0u) = a.$$

We have, $\ell^+(u) = a$ and

$$\ell^+(u_i) = \begin{cases} a + (4t + 2)a + b + b = a \text{ for} i = 0, \\ b + c = a \text{ for } i = 1, 3, 5, \ldots, 4t - 1, 4t + 4, 4t + 6, 4t + 8, \ldots 8t + 2, \\ c + c + a = a \text{ for } i = 2, 6, 10, \ldots, 4t - 2, 4t + 5, 4t + 9, \ldots, 8t + 1, \\ b + b + a = a \text{ for } i = 4, 8, \ldots, 4t, 4t + 1, 4t + 2, 4t + 3, 4t + 7, 4t + 11, \ldots, 8t + 3. \end{cases}$$

Case 2: Suppose $n = 2t + 1$. In this case, the graph has $8t + 7$ vertices. Define $\ell : E(U(4n + 3, 2n + 2, 1)) \to V_4 \setminus \{0\}$ by

$$\ell(v_{i-1}v_i) = \ell(v_iv_{i+1}) = \begin{cases} b \text{ for } i = 2, 6, 10, \ldots, 4t+2, 4t+3, 4t+4, 4t+5, 4t+9, \ldots, 8t+5, \\ c \text{ for } i = 4, 8, \ldots, 4t, 4t+7, 4t+11, \ldots, 8t+7. \end{cases}$$

where $v_{8t+7} = v_0$ and $v_{8t+8} = v_1$ and

$$\ell(v_0v_i) = a \text{ for } i = 2, 4, 6, \ldots, 4t+2, 4t+3, 4t+4, 4t+5, 4t+7, \ldots, 8t+5,$$
$$\ell(v_0u) = a.$$

We have, $\ell^+(u) = a$,

$$\ell(v_i) = \begin{cases} c + c + (4t+2)a + a = a \text{ for } i = 0, \\ b + b + a = a \text{ for } i = 2, 6, 10, \ldots, 4t+2, 4t+3, 4t+4, 4t+5, 4t+9, \ldots, 8t+5, \\ c + c + a = a \text{ for } i = 4, 8, \ldots, 4t, 4t+7, 4t+11, \ldots, 8t+3, \\ b + c = a \text{ for } i = 1, 3, 5, \ldots, 4t-3, 4t+6, 4t+8, \ldots, 8t+6. \end{cases}$$

This completes the proof. □

Theorem 2.25. *$C_n(2, r) \in \mathscr{V}_a$ if n is even and $2 \le r \le n - 3$.*

Proof. Let the vertex set of $C_n(2, r)$ be $\{u_0, u_1, u_2, \ldots, u_{n-1}\}$, where u_0 is the apex. Here we consider two cases:

Case 1: Suppose r is odd. Now, we give the labeling to the edges of G as follows:

$$\ell(uu_1) = b,$$

$$\begin{cases} \text{for } i = 1, 2, \ldots, r+1: \\ \qquad \ell(u_iu_{i+1}) = c. \\ \text{end for} \end{cases}$$

$$\begin{cases} \text{for } i = r+2, r+4, \ldots, n-1: \\ \qquad \ell(u_iu_{i+1}) = b. \\ \text{end for} \end{cases}$$

$$\begin{cases} \text{for } i = r+3, r+5, \ldots, n-2: \\ \qquad \ell(u_iu_{i+1}) = c. \\ \text{end for} \end{cases}$$

$$\begin{cases} \text{for } i = 2, 3, \ldots, r+1: \\ \qquad \ell(uu_i) = a. \\ \text{end for} \end{cases}$$

Observe that,

$$\ell^+(u_i) = \begin{cases} b + b + ra = a, & \text{for } i = 0, \\ b + c = a, & \text{for } i = 1, \\ c + c + a = a, & \text{for } i = 2, 3, \ldots, r+1, \\ b + c = a, & \text{for } i = r+2, r+3, \ldots, n-1. \end{cases}$$

Case 2: Suppose r is even. In this case, labeling is similar to case 1.

This completes the proof. □

Theorem 2.26. *If $2 \leq p \leq n-r$ and n is even, then $C_n(p,r) \in \mathscr{V}_a$.*

Proof. Proof is similar to theorem 2.25. □

Lemma 2.2. *Let $\mathcal{G}_r(n, n-3)$ denote graph $P_{n-1} + \overline{K_r}$ (see fig.6). Let the vertex sets of P_{n-1} and K_r be $\{v_1, v_2, \dots, v_{n-1}\}$ and $\{u_1, u_2, \dots, u_r\}$, respectively. If $\ell : E(\mathcal{G}_r(n, n-3) \to V_4 \setminus \{0\}$ is a labeling of $\mathcal{G}_r(n, n-3)$, then*

$$\sum_{i=1}^{n-1} \ell^+(v_i) + \sum_{i=1}^{r} \ell^+(u_i) = 0. \tag{2.2}$$

Proof. Proof is similar to Lemma 2.1. □

Theorem 2.27. *$\mathcal{G}_r(n, n-3) \in \mathscr{V}_a$ if and only if $n+r$ is odd.*

Proof. Assume that $\mathcal{G}_r(n, n-3) \in \mathscr{V}_a$. Then by lemma 2.2, we have $(n-1)a + ra = 0$. This implies that $n+r$ is odd.
Conversely, assume that $n+r$ is odd. We consider two cases.

Case 1: Suppose n is odd and r is even. Let $n = 2s+1$. Now, we give the labeling to the edges of $\mathcal{G}_r(n, n-3)$ as follows:

$$\begin{aligned}
\ell(u_i v_1) &= b, \text{ for } i = 1, 2, \dots, r \\
\ell(u_i v_{2s}) &= c, \text{ for } i = 1, 2, \dots, r \\
\ell(u_1 v_i) &= a \quad \text{for } i = 2, 3, \dots, 2s-1, \\
\ell(v_i v_{i+1}) &= a \quad \text{for } i = 1, 3, \dots, 2s-1, \\
\ell(v_i v_{i+1}) &= c \quad \text{for } i = 2, 4, \dots, 2s-2,
\end{aligned}$$

$$\begin{cases} \text{for } i = 2, 3, \dots, r: \\ \ell(u_i v_j) = b \quad \text{for } j = 2, 3, \dots, 2s-1 \\ \text{end for} \end{cases}$$

We have

$$\ell^+(v_i) = \begin{cases} rb + a = a & \text{for } i = 1, \\ rc + a = a & \text{for } i = 2s, \\ a + c + a + (r-1)b = a, & \text{for } i = 2, 3, \dots, 2s-1 \end{cases}$$

$$\ell^+(u_i) = \begin{cases} b + c + (2s-2)a = a & \text{for } i = 1 \\ b + c + (2s-2)b = a, & \text{for } i = 2, 3, \dots, r. \end{cases}$$

Case 2: Suppose n is even and r is odd. Let $n = 2s$. Now, we give the labeling to the edges of $\mathcal{G}_r(n, n-3)$ as follows:

$$\ell(v_i v_{i+1}) = c \text{ for } i = 1, 2, \dots, 2s-2,$$

for $i = 1, 2, \dots, r$:

$$\begin{cases} \ell(u_i v_1) = b, \\ \ell(u_i v_{2s-1}) = b, \\ \ell(u_i v_j) = a \text{ for } j = 2, 3, \dots, 2s-1. \end{cases}$$

end for

We have

$$\ell^+(v_i) = \begin{cases} rb + c = a & \text{for } i = 1, 2s-1 \\ c + c + ra = a & \text{for } i = 2, 3, \ldots, 2s-2 \end{cases}$$

$$\ell^+(u_i) = b + b + (2s-3)a = a \text{ for } \ i = 1, 2, \ldots, r.$$

This completes the proof. □

Theorem 2.28. *$\mathcal{G}_r(n, n-3) \notin \mathscr{V}_a$ if $n + r$ is even.*

Proof. Suppose $\mathcal{G}_r(n, n-3) \in \mathscr{V}_a$. Then by lemma 2.2, we have $(n + r - 1)a = 0$. Since $n + r$ is even, we have $a = 0$. This is a contradiction. The result now follows. □

Theorem 2.29. *$\mathcal{G}_r(n, n-3) \in \mathscr{V}_0$ if n+r is even.*

Proof. Assume that $n + r$ is even. We consider two cases.

Case 1: Suppose n and r are both odd. Let $n = 2s + 1$. Now, we give the labeling to the edges of $\mathcal{G}_r(n, n-3)$ as follows:

$$\ell(v_i v_{i+1}) = \begin{cases} b & \text{for } i = 1, 3, \ldots, 2s-1, \\ c & \text{for } i = 2, 4, \ldots, 2s-2, \end{cases}$$

$$\ell(u_1 v_i) = b \text{ for } \ i = 1, 2s$$

$$\ell(u_1 v_i) = a \text{ for } \ i = 2, 3, \ldots, 2s-1$$

$$\begin{cases} \text{for } \ i = 2, 3, \ldots, r: \\ \ell(u_i v_j) = a \text{ for } \ j = 1, 2, 3, \ldots, 2s \\ \text{end for} \end{cases}$$

We have,

$$\ell^+(v_i) = \begin{cases} b + b + (r-1)a = 0 & \text{for } i = 1, \\ b + c + ra = 0 & \text{for } \ i = 2, 3, \ldots, 2s-1, \\ b + b + (r-1)a = 0 & \text{for } \ i = 2s. \end{cases}$$

$$\ell^+(u_i) = \begin{cases} b + b + (2s-2)a = 0 & \text{for } i = 1, \\ 2sa = 0 & \text{for } \ i = 2, 3, \ldots, r. \end{cases}$$

Case 2: Suppose both n and r are both even.

$$\ell(u_1 v_{2s-1}) = c,$$

$$\ell(u_1 v_s) = a,$$

$$\ell(v_i v_{i+1}) = a, \quad \text{for } i = 1, 2, \ldots, 2s-2,$$

$$\ell(u_1 v_i) = b, \quad \text{for } i = 1, 2, 3, \ldots, s-1, s+1, s+2, \ldots 2s, 2s-2$$

$$\begin{cases} \text{for } \ i = 2, 3, \ldots, r: \\ \ell(u_i v_1) = c \\ \ell(u_i v_{2s-1}) = b \\ \ell(u_i v_s) = a \\ \ell(u_i v_j) = b, \quad \text{for } i = 2, 3, \ldots, s-1, s+1, s+2, \ldots, 2s-2 \\ \text{end for} \end{cases}$$

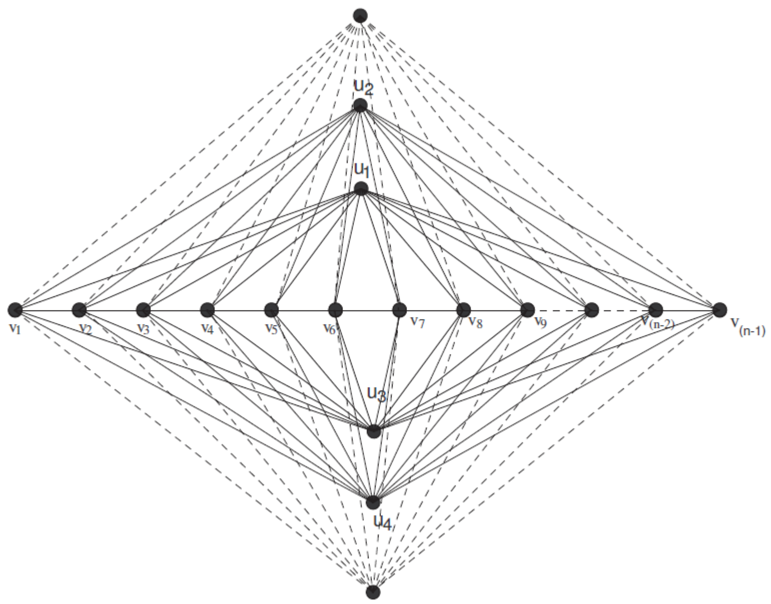

Figure 6: The graph $\mathcal{G}_r(n, n-3)$

We have,

$$\ell^+(v_i) = \begin{cases} a + b + (r-1)c = 0 & \text{for } \; i = 1 \\ a + a + b + (r-1)b = 0 & \text{for } \; i = 2, 3, \ldots, s-1, s+1, s+2, \ldots, 2s-2 \\ a + a + a + (r-1)a = 0 & \text{for } \; i = s \\ c + a + (r-1)b = 0 & \text{for } \; i = 2s-1 \end{cases}$$

$$\ell^+(u_i) = (2s-3)b + a + c = 0 \;\; \text{for } \; i = 1, 2, \ldots, r,$$

This completes the proof. □

A zero magic labeling of $\mathcal{G}_2(8, 5)$ is shown in figure 7.

Lemma 2.3. *Let $G(n, n-3, k)$ denote the graph obtained by taking the union of k copies of $H(n, n-3)$ having the edges v_0v_1's identified (see figure 8). Let $\{v_{1,0}, v_{1,1}, v_{i,2}, v_{i,3}, \ldots, v_{i,n-1}\}$ be the vertex set of the i^{th} copy of $H(n, n-3)$. Then*

$$\sum_{i=0}^{n-1} \ell^+(v_{1,i}) + \sum_{i=2}^{k}\sum_{j=2}^{n-1} \ell^+(v_{i,j}) = 0, \tag{2.3}$$

Theorem 2.30. *$G(n, n-3, k) \in \mathscr{V}_a$ if and only if nk is even.*

Proof. Assume that $G(n, n-3, k) \in \mathscr{V}_a$. Then by lemma 2.3, we have $na + (k-1)(n-2)a = 0$. This implies that nk is even.
Conversely, assume that nk is even. We consider the following cases:

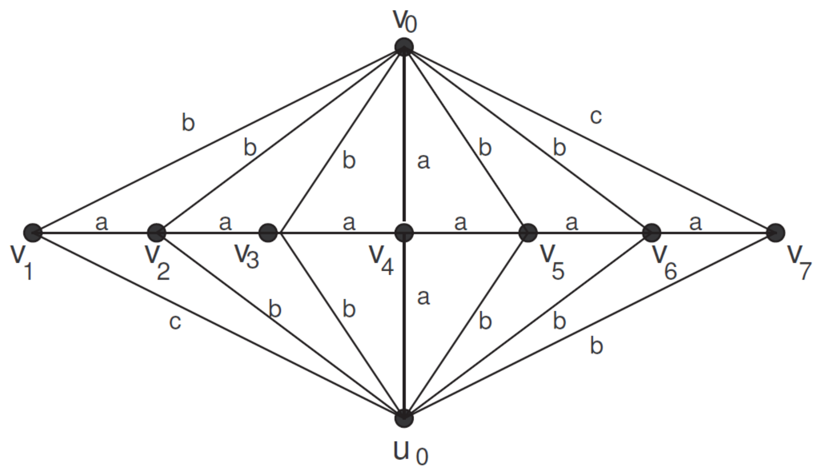

Figure 7: A zero magic labeling of $\mathcal{G}_2(8,5)$

Case 1: Assume that both n and k are even. In this case, we label the edges of $G(n, n-3, k)$ as follows:

$$\ell(v_{1,0}v_{1,1}) = a,$$

$$\begin{cases} \text{for} \quad i = 1, 2, \ldots, k \\ \qquad \ell(v_{i,j}v_{i,j+1}) = c, \quad \text{for} \quad j = 1, 2, 3, \ldots, n-2, \\ \text{end for} \end{cases}$$

$$\begin{cases} \text{for} \quad i = 1, 2, \ldots, k \\ \qquad \ell(v_{1,0}v_{i,j+1}) = a, \quad \text{for} \quad j = 1, 2, 3, \ldots, n-3, \\ \text{end for} \end{cases}$$

$$\ell(v_{1,0}v_{i,n-1}) = b \quad \text{for} \quad i = 1, 2, 3, \ldots, k.$$

So, we have

$$\ell^+(v_{i,j}) = \begin{cases} kb + k(n-3)a + a = a & \text{for} \quad i = 1, j = 0, \\ a + kc = a, & \text{for} \quad i = 1, j = 1, \\ b + c = a, & \text{for} \quad i = 1, 2, 3, \ldots, k, j = n-1, \\ c + c + a = a, & \text{for} \quad i = 1, 2, 3, \ldots, k; j = 2, 3, \ldots, n-2. \end{cases}$$

Case 2: Assume that n is even and k is odd. In this case, the labeling is exactly, similar to case 1 with only difference is that $\ell(v_{1,0}v_{1,1}) = a$ is to be replaced by $\ell(v_{1,0}v_{1,1}) = b$.

Case 3: Assume that n is odd and k is even. In this case, the labeling is obvious.

This completes the proof □

Theorem 2.31. *$G(n, n-3, k) \notin \mathscr{V}_a$ if n and k are both odd.*

Proof. Assume that $G(n, n-3, k) \in \mathscr{V}_a$. Since n and k are both odd, nk is odd. Then by lemma 2.3, we have $nka = 0$. This implies that $a = 0$. This is a contradiction. The result now follows. □

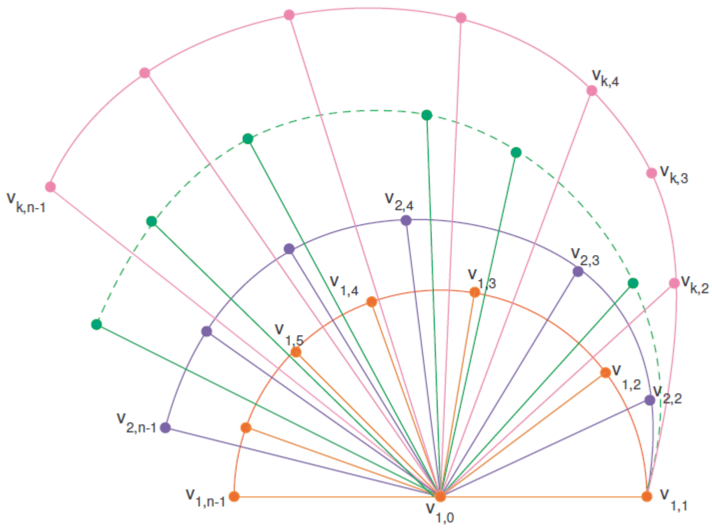

Figure 8: The graph $G(n, n-3, k)$

Theorem 2.32. *$G(n, n-3, k) \in \mathscr{V}_0$ if n and k are odd.*

Proof. Now, we give the labeling to the edges of G as follows:

$$\ell(v_{1,0}v_{1,1}) = b,$$
$$\ell(v_{i,n-1}v_{1,0}) = b, \quad \text{for } i = 1, 2, \ldots, k,$$
$$\text{for} \quad i = 1, 2, \ldots, k:$$
$$\begin{cases} \ell(v_{i,j}v_{i,j+1}) = b, \; j = 1, 3, \ldots, n-2, \\ \ell(v_{i,j}v_{i,j+1}) = c, j = 2, 4, \ldots, n-3, \\ \ell(v_{1,0}v_{i,j}) = a, j = 2, 3, \ldots, n-2. \end{cases}$$
$$\text{end for}$$

Obviously,

$$\ell^+(v_{i,j}) = \begin{cases} k(n-3)a + 2b = 0, & \text{for } i = 0, j = 1, \\ b + b = 0, & \text{for } i = 1, 2, 3, \ldots, k; j = n-1 \\ b + kb = 0, & \text{for } i = 1, j = 1, \\ b + c + a = 0, & \text{for } i = 1, 2, \ldots, k; j = 2, 3, \ldots, n-2. \end{cases}$$

This completes the proof. □

Theorem 2.33. *$G(n, n-3, k) \in \mathscr{V}_0$ if n is even and k odd.*

Proof. Now, we give the labeling to the edges of G as follows:

$$\ell(v_{1,0}v_{1,1}) = c,$$

$$\begin{aligned}
&\ell(v_{i,n-1}v_{1,0}) = b, \quad i = 1,2,3,\ldots,k,\\
&\text{for} \quad i = 1,2,\ldots,k:\\
&\left\{\begin{array}{l}
\ell(v_{i,j}v_{i,j+1}) = c,\ j = 1,3,\ldots,n-3,\\
\ell(v_{i,j}v_{i,j+1}) = b,\ j = 2,4,\ldots,n-2,\\
\ell(v_{1,0}v_{i,j}) = a,\ j = 2,3,\ldots,n-2.
\end{array}\right.\\
&\text{end for}
\end{aligned}$$

Obviously,

$$\ell^+(v_{i,j}) = \begin{cases} k(n-3)a + kb + c = a+b+c = 0, & \text{for } i=1, j=0,\\ b+b=0, & \text{for } i = 1,2,3,\ldots,k; j = n-1,\\ c+kc = 0, & \text{for } i=1, j=n-1,\\ b+c+a=0, & \text{for } i=1,2,\ldots,k; j=2,3,\ldots,n-2. \end{cases}$$

This completes the proof. □

Theorem 2.34. *$G(n, n-3, k) \in \mathscr{V}_0$ if n and k are even.*

Proof. Now, we give the labeling to the edges of G as follows:

$$\begin{aligned}
&\ell(v_{1,0}v_{1,1}) = a,\\
&\ell(v_{1,n-1}v_{1,0}) = b,\\
&\ell(v_{1,j}v_{1,j+1}) = c, \quad ,\text{for} \quad j = 1,3,\ldots,n-3,\\
&\ell(v_{1,j}v_{1,j+1}) = b, \quad ,\text{for} \quad j = 2,4,\ldots,n-2,\\
&\text{for} \quad i = 2,3,\ldots,k:\\
&\begin{cases}\ell(v_{i,j}v_{i,j+1}) = b,\ j = 1,3,\ldots,n-3,\\ \ell(v_{i,j}v_{i,j+1}) = c,\ j = 2,4,\ldots,n-2,\end{cases}\\
&\text{end for}\\
&\ell(v_{1,0}v_{i,n-1}) = c, \quad ,\text{for} \quad i = 2,3,\ldots,k,\\
&\text{for} \quad i = 1,2,3,\ldots,k:\\
&\qquad \ell(v_{1,0}v_{i,j}) = a,\ j = 2,3,\ldots,n-2,\\
&\text{end for}
\end{aligned}$$

Then we have,

$$\ell^+(v_{i,j}) = \begin{cases} b + a + (n-3)ka + (k-1)c = a+b+c = 0, & \text{for } i=1, j=0,\\ a + c + (k-1)b = a+b+c = 0, & \text{for } i=1, j=1,\\ c+c=0, & \text{for } i=2,3,\ldots,k; j=n-1\\ b+b=0, & \text{for } i=1, j=n-1,\\ b+c+a=0 & \text{for } i=1,2,\ldots,k; j=2,3,\ldots,n-2. \end{cases}$$

□

Theorem 2.35. *$G(n, n-3, k) \in \mathscr{V}_0$ if n is odd and k is even.*

Proof. Now, we give the labeling to the edges of G as follows:

$$\ell(v_{1,n-1}v_{1,0}) = c$$

$\ell(v_{1,0}v_{1,1}) = a$
for $i = 1, 2, \ldots, k$:
$\quad \ell(v_{1,0}v_{i,j}) = a,\quad$ for $j = 1, 2, \ldots, n-2,$
end for
for $\quad i = 2, 3, \ldots, k$:
$\quad \ell(v_{1,0}v_{i,n-1}) = b$
end for
$\quad \ell(v_{1,i}v_{1,i+1}) = c,\quad$ for $\quad i = 1, 3, \ldots, n-2,$
$\quad \ell(v_{1,i}v_{1,i+1}) = b,\quad$ for $\quad i = 2, 4, \ldots, n-3,$
$\quad \ell(v_{1,1}v_{i,2}) = b,\quad$ for $\quad i = 2, \ldots, k,$
for $\quad i = 2, 3, \ldots, k$:
$\quad \ell(v_{i,j}v_{i,j+1}) = c,\ j = 2, 4, \ldots, n-3,$
$\quad \ell(v_{i,j}v_{i,j+1}) = b\ j = 3, 5, \ldots, n-2,$
end for

Obviously,

$$\ell^+(v_{i,j}) = \begin{cases} c + (k-1)b + (n-3)ka + a = 0, & \text{for } i = 1, j = 0, \\ a + c + (k-1)b = 0, & \text{for } i = 1, j = 1, \\ b + b = 0, & \text{for } i = 1, 2, \ldots, k, j = n-1, \\ a + b + c = 0, & \text{for } i = 1, 2, \ldots, k; j = 2, \ldots, n-2. \end{cases}$$

This completes the proof. □

Lemma 2.4. *Let G denotes the multiple shell $MS(n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r})$. Let $\{u, v_{k,j}^{t_i}\}$, $j = 1, 2, \ldots, n_i$, $1 \le i \le r$ be the vertices of G with apex u. If $\ell : E(G) \to V_4 \setminus \{0\}$ is a labeling of G, then*

$$\sum_{i=1}^{r}\sum_{k=1}^{t_i}\sum_{j=1}^{n_i-1} \ell^+(v_{k,j}^{t_i}) + \ell^+(u) = 0.$$

Proof. Proof is exactly similar to lemma 2.1. □

Theorem 2.36. *If $MS(n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}) \in \mathscr{V}_a$, then $\sum_{i=1}^{r}[(n_i-1)t_i]$ is odd.*

Proof. Proof follows from lemma 2.4. □

Conjecture 2.37. *If $\sum_{i=1}^{r}[(n_i-1)t_i]$ is odd, then $MS\{n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}\} \in \mathscr{V}_a$.*

We prove that the conjecture is true for $r = 1$.

Corollary 2.38. *$MS(n^t) \in \mathscr{V}_a$ if $(n-1)t$ is odd.*

Proof. Assume that $(n-1)t$ is odd. This implies that n is even and t is odd. Observe that $MS(n^t)$ is the one point union of t shells $H^i(n, n-3)$, $i = 1, 2, \ldots, t$. Let the vertex set of $H^i(n, n-3)$ be $\{u_{0,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,n-1}\}$, where $u_{0,0}$ is the apex of all shells. Now, we give the labeling to the edges of G as follows:

$\ell(u_{0,0}u_{i,1}) = b,\quad$ for $\quad i = 1, 2, \ldots, t,$
$\ell(u_{0,0}u_{i,n-1}) = b,\quad$ for $\quad i = 1, 2, \ldots, t,$

for $\quad i = 1, 2, 3, \ldots, t$:

$$\begin{cases} \ell(u_{i,j}u_{i,j+1}) = c, \ j = 1, 2, \ldots, n-2, \\ \ell(uu_{i,j}) = a, \ j = 2, 3, \ldots, n-2, \end{cases}$$

end for

Then we have,

$$\ell^+(u_{i,j}) = \begin{cases} (n-3)ta + tb + tb = a + b + b = a, & \text{for } i = 0, j = 0, \\ b + c = a, & \text{for } i = 1, 2, 3, \ldots, t; j = 1, n-1, \\ c + c + a = a, & \text{for } i = 1, 2, 3, \ldots, t; 2, 3, \ldots, n-2. \end{cases}$$

This completes the proof. ☐

Next we prove that the conjecture is true for $r = 2$, $n_1 = n$, $n_2 = n + 1$ and $t_1 = t_2 = 1$.

Corollary 2.39. $MS(n, n+1) \in \mathscr{V}_a$.

Proof. Observe that $MS(n, n+1)$ is the one point union of $H(n, n-3)$ and $H(n+1, n-2)$. Let

$$V(H(n, n-3)) = \{u_0, u_1, u_2, \ldots, u_{n-1}\}$$
$$V(H(n, n-3)) = \{v_0, v_1, v_2, \ldots, v_n\}$$

Assume that $u_o = v_0$ be the apex of both the shells $H(n, n-3)$ and $H(n+1, n-2)$. Now, we give the labeling to the edges of $MS(n, n+1)$ as follows:

$$\begin{aligned} \ell(u_0u_1) &= b, \\ \ell(u_0u_{n-1}) &= b, \\ \ell(u_iu_{i+1}) &= c, \quad \text{for} \quad i = 1, 2, 3, \ldots, n-2, \\ \ell(u_0u_i) &= a, \quad \text{for} \quad i = 2, 3, \ldots, n-2, \\ \ell(v_0v_1) &= b, \\ \ell(v_0v_n) &= b \\ \ell(v_iv_{i+1}) &= c, \quad \text{for} \quad j = 1, 2, 3, \ldots, n-1, \\ \ell(v_0v_i) &= a, \quad \text{for} \quad i = 2, 3, \ldots, n-1. \end{aligned}$$

We have,

$$\ell^+(u_i) = \begin{cases} (n-3)a + (n-2)a + 4b = (2n-5)a = a, & \text{for } i = 0, \\ b + c = a, & \text{for } i = 1, n-1 \\ a + c + c = a, & \text{for } 2, 3, \ldots, n-2, \end{cases}$$

$$\ell^+(v_i) = \begin{cases} b + c = a, & \text{for } i = 1, n \\ a + c + c = a, & \text{for } 2, 3, \ldots, n-1, \end{cases}$$

This completes the proof. ☐

Corollary 2.40. $MS(n^t, (n+1)^t) \in \mathscr{V}_a$ *for all odd* t.

Proof. Proof is similar to theorem 2.39. ☐

Corollary 2.41. $MS(n, m) \in \mathscr{V}_a$*, if and only if* $m + n$ *is odd.*

Proof. Assume that $MS(n,m) \in \mathscr{V}_a$. Observe that $MS(n,m)$ has $(m+n-1)$ vertices. Then by Theorem 2.36, $m+n$ is odd.

Conversely, assume that $m+n$ is odd. We need to show that $MS(n,m) \in \mathscr{V}_a$. We consider two cases.

Case 1: Suppose n is even and m is odd. Assume that v_0 is the apex of both the shells and, let

$$V(H(n,n-3)) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$$
$$V(H(m,m-3)) = \{v_0, u_1, u_2, \ldots, u_{m-1}\}.$$

Now, we give the labeling to the edges of $H(n,m,n-3,m-3)$ as follows:

$$\begin{aligned} \ell(v_0v_i) &= b \ \text{ for } i = 1, n-1,\\ \ell(v_iv_{i+1}) &= c \ \text{ for } i = 1,2,\ldots,n-2,\\ \ell(v_0v_i) &= a \ \text{ for } i = 2,\ldots,n-2,\\ \ell(v_0u_i) &= b \ \text{ for } i = 1, m-1,\\ \ell(v_iv_{i+1}) &= c \ \text{ for } i = 1,2,\ldots,m-2,\\ \ell(v_0u_i) &= a \ \text{ for } i = 2,\ldots,m-2. \end{aligned}$$

We have

$$\ell^+(v_i) = \begin{cases} b+b+(n-3)a+b+b+(m-3)a = a & \text{for } i = 0,\\ b+c = a & \text{for } i = 1, n-1,\\ c+c+a = a & \text{for } i = 2,3,4,\ldots,n-2. \end{cases}$$

$$\ell^+(u_i) = \begin{cases} b+b+a = a & \text{for } i = 1, m-1,\\ c+c+a = a & \text{for } i = 2,3,4,\ldots,m-2. \end{cases}$$

Case 2: Suppose n is odd and m is even. In this case the labeling is exactly similar to case 1. This completes the proof.

Conjecture 2.42. *$MS(n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}) \in \mathscr{V}_0$ for all n_i and t_i.*

We prove some special cases of conjecture 2.42.

Corollary 2.43. *$MS(n^t) \in \mathscr{V}_0$ if n is even and t is odd.*

Now, we give the labeling to the edges of $MS(n^t)$ as follows:

$$\begin{aligned} &\ell(u_{0,0}u_{i,1}) = b, \quad \text{for} \quad i = 1,2,\ldots,t,\\ &\ell(u_{0,0}u_{i,n-1}) = c, \quad \text{for} \quad i = 1,2,\ldots,t,\\ &\text{for} \quad i = 1,2,3,\ldots,t: \\ &\left\{ \begin{array}{l} \ell(u_{i,j}u_{i,j+1}) = b, \quad \text{for} \quad j = 1,3,\ldots,n-3,\\ \ell(u_{i,j}u_{i,j+1}) = c, \quad \text{for} \quad j = 2,4,\ldots,n-2,\\ \ell(uu_{i,j}) = a, \quad \text{for} \quad j = 2,3,\ldots,n-2, \end{array} \right.\\ &\text{end for} \end{aligned}$$

Then we have,

$$\ell^+(u_{i,j}) = \begin{cases} (n-3)ta + tb + tc = a+b+c = 0, & \text{for } i = 0, j = 0,\\ b+b = 0, & \text{for } i = 1,2,3,\ldots,t; j = 1,\\ c+c = 0, & \text{for } i = 1,2,3,\ldots,t; j = n-1,\\ b+c+a = 0, & \text{for } i = 1,2,3,\ldots,t; j = 2,3,\ldots,n-3. \end{cases}$$

This completes the proof.

Corollary 2.44. *$MS(n^t) \in \mathscr{V}_0$ if n is even and t is even.*

Proof. Proof is exactly similar to theorem 2.43. □

Corollary 2.45. *$MS(n^t) \in \mathscr{V}_0$ if n is odd and t is odd.*

Proof. Now, we give the labeling to the edges of $MS(n^t)$ as follows:

$$\begin{aligned}
&\ell(u_{0,0}u_{i,1}) = b, \quad \text{for} \quad i = 1, 2, \ldots, t,\\
&\ell(u_{0,0}u_{i,n-1}) = b, \quad \text{for} \quad i = 1, 2, \ldots, t,\\
&\text{for} \quad i = 1, 2, 3, \ldots, t:\\
&\left\{\begin{array}{l}
\ell(u_{i,j}u_{i,j+1}) = b, \quad \text{for} \quad j = 1, 3, \ldots, n-2,\\
\ell(u_{i,j}u_{i,j+1}) = c, \quad \text{for} \quad j = 2, 4, \ldots, n-3,\\
\ell(uu_{i,j}) = a, \quad \text{for} \quad j = 2, 3, \ldots, n-3,
\end{array}\right.\\
&\text{end for}
\end{aligned}$$

Then we have,

$$\ell(u_{i,j}) = \begin{cases} (n-3)ta + tb + tb = 0, & \text{for } i = 0, j = 0\\ b + b = 0, & \text{for } i = 1, 2, 3, \ldots, t, j = 1, n-1\\ b + c + a = 0, & \text{for } i = 1, 2, 3, \ldots, t, j = 2, 3, \ldots, n-2. \end{cases}$$

This completes the proof. □

Corollary 2.46. *$MS(n^t) \in \mathscr{V}_0$ if n is odd and t is even.*

Proof. Proof is exactly similar to theorem 2.45 □

Corollary 2.47. *If $m + n$ is even, then $MS(n, m) \in \mathscr{V}_0$.*

We consider two cases

Case 1: Suppose both n and m are even. Now, we give the labeling to the edges of $MS(n, m)$ as follows:

$$\ell(v_0v_i) = \begin{cases} b & \text{for } i = 1,\\ c & \text{for } i = n-1, \end{cases}$$

$$\ell(v_iv_{i+1}) = \begin{cases} c & \text{for } i = 2, 4, 6, \ldots, n-2,\\ b & \text{for } i = 3, 5, 8, \ldots, n-3, \end{cases}$$

$$\ell(v_0v_i) = a \ \text{ for } i = 2, \ldots, n-2,$$

$$\ell(v_0u_i) = \begin{cases} b & \text{for } i = 1,\\ c & \text{for } i = m-1, \end{cases}$$

$$\ell(u_iu_{i+1}) = \begin{cases} c & \text{for } i = 2, 4, 6, \ldots, m-2,\\ b & \text{for } i = 3, 5, 8, \ldots, m-3, \end{cases}$$

$$\ell(u_0u_i) = a \ \text{ for } i = 2, \ldots, m-2,$$

We have,

$$\ell^+(v_i) = \begin{cases} b+c+(n-3)a+b+c+(m-3)a = 0 & \text{for } i=0, \\ b+b=0 & \text{for } i=1,n-1, \\ b+c+a=0 & \text{for } i=2,3,4,\dots,n-2. \end{cases}$$

$$\ell^+(u_i) = \begin{cases} b+b=0 & \text{for } i=1,m-1, \\ a+b+c=0 & \text{for } i=2,3,4,\dots,m-2. \end{cases}$$

Case 2: Suppose both m and n are odd. In this case, the labeling is similar to case 1.

This completes the proof. □

Consider the multiple shell $MS\{n_1^{t_1}, n_2^{t_2}, \dots, n_r^{t_r}\}$ with vertex set $\{u, v_{k,j}^{t_i}\}$, $j = 1, 2, \dots, n_i$, $1 \le i \le r$. Let $K_{1,m}$ denotes the star graph with vertex set $\{v, v_1, v_2, \dots, v_m\}$. Here v denotes the apex of $K_{1,n}$. Let $MS\{n_1^{t_1}, n_2^{t_2}, \dots, n_r^{t_r}\} \diamond K_{1,m}$ denotes the graph obtained by identifying the vertices u and v. Then we have the following lemma:

Lemma 2.5. *Let G denotes the graph $MS\{n_1^{t_1}, n_2^{t_2}, \dots, n_r^{t_r}\} \diamond K_{1,m}$. Let $\{u, v_{k,j}^{t_i}\}$, $j = 1, 2, \dots, n_i$, $1 \le i \le r$ be the vertices of $MS\{n_1^{t_1}, n_2^{t_2}, \dots, n_r^{t_r}\}$ with apex u and let $\{v, v_1, v_2, \dots, v_m\}$ be the vertices of $K_{1,m}$ with apex v. If $\ell : E(G) \to V_4 \setminus \{0\}$ is a labeling of G, then*

$$\sum_{i=1}^{r} \sum_{k=1}^{t_i} \sum_{j=1}^{n_i-1} \ell^+(v_{k,j}^{t_i}) + \sum_{i=1}^{m} \ell^+(v_i) + \ell^+(u) = 0.$$

Theorem 2.48. *If $MS\{n_1^{t_1}, n_2^{t_2}, \dots, n_r^{t_r}\} \diamond K_{1,m} \in \mathscr{V}_a$, then $\sum_{i=1}^{r}[(n_i-1)t_i] + m$ is odd.*

Proof. Proof follows from lemma 2.5. □

Conjecture 2.49. *If $\sum_{i=1}^{r}[(n_i-1)t_i] + m$ is odd, then $MS\{n_1^{t_1}, n_2^{t_2}, \dots, n_r^{t_r}\} \diamond K_{1,m} \in \mathscr{V}_a$.*

We prove some special cases of the conjecture 2.49.

Corollary 2.50. *$MS(n^t) \diamond K_{1,m} \in \mathscr{V}_a$ if and only if $(n-1)t + m$ is odd.*

Proof. Assume that $MS(n^t) \odot K_{1,m} \in \mathscr{V}_a$. Then by lemma 2.5, we have $[(n-1)t + m + 1]a = 0$. This implies that $(n-1)t + m$ is odd.

Conversely, assume that $(n-1)t + m$ is odd. Then we have the following cases:

Case 1: Suppose n is even, t is odd and m is even. Let $u_{0,0}$ be the apex of both $MS(n^t)$ and $K_{1,m}$. Let $\{u_{0,0}, u_{i,1}, u_{i,2}, \dots, u_{i,n-1}\}$ be the vertex set of the i^{th} copy of $H^i(n, n-3)$ and let $\{u_{0,0}, v_{1,1}, v_{1,2}, \dots, v_{1,m}\}$ be the vertex set of $K_{1,m}$. Now, we give the labeling to the edges of G as follows:

$\ell(u_{0,0}u_{i,1}) = b, \quad \text{for} \quad i = 1, 2, \dots, t,$

$\ell(u_{0,0}u_{i,n-1}) = b, \quad \text{for} \quad i = 1, 2, \dots, t,$

for $\quad i = 1, 2, 3, \dots, t:$

$$\begin{cases} \ell(u_{i,j}u_{i,j+1}) = c, & \text{for} \quad j = 1, 2, \dots, n-2, \\ \ell(uu_{i,j}) = a, & \text{for} \quad j = 2, 3, \dots, n-2, \end{cases}$$

end for

$\ell(u_{0,0}v_{1,i}) = a, \quad \text{for} \quad j = 1, 2, 3, \dots, m.$

Then we have,

$$\ell^+(u_{i,j}) = \begin{cases} (n-3)ta + tb + tb + ma = a + b + b = a, & \text{for } i = 0, j = 0 \\ b + c = a, & \text{for } i = 1, 2, \ldots, t, j = 1, n-1 \\ c + c + a = a, & \text{for } i = 1, 2, \ldots, t, j = 1, 2, 3, \ldots, n-2 \end{cases}$$

$$\ell^+(v_{i,j}) = a \quad \text{for } i = 1; j = 1, 2, 3, \ldots, m$$

Case 2: n is even, t is even and m is odd. In this case, the labeling is obvious.

Case 3: n is odd, t is even and m is odd. In this case, the labeling is obvious.

Case 4; m, n and t are odd. In this case, the labeling is obvious.

This completes the proof. □

Corollary 2.51. *$MS(n, n+1) \odot K_{1,m} \in \mathscr{V}_a$ if m is even.*

Proof. First, label all the edges of $K_{1,m}$ by a. Next, label all edges of $MS(n, n+1)$ as described in Corollary 2.39. Then one can easily verify that this labeling is an a-sum V_4 magic labeling of $MS(n, n+1) \odot K_{1,m}$. □

Corollary 2.52. *$MS(n^t, (n+1)^t) \odot K_{1,m} \in \mathscr{V}_a$ if m is even and t is odd.*

Proof. Labeling is similar to Corollary 2.51. □

3 Conclusion

Let $V_4 = \{0, a, b, c\}$ be the Klein 4-group. In this paper, we identified a class of V_4-magic shell related graphs in the following categories:

(i) $\mathscr{V}_a$, the class of a-sum V_4 magic graphs,

(ii) $\mathscr{V}_0$, the class of zero-sum V_4 magic graphs,

(iii) $\mathscr{V}_{a,0}$, the class of graphs which are both a-sum and zero -sum V_4 magic.

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Competing Interests

The authors declare that no competing interests exist.

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