Journal of Scientific Research & Reports
8(4): 1-9, 2015; Article no.JSRR.18661
ISSN: 2320-0227

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

n-Normal and n-Quasi-normal Composite Multiplication Operator on L^2-Spaces

S. Senthil[1], P. Thangaraju[2*] and D. C. Kumar[1]

[1]Department of Mathematics, Vickram College of Engineering, Enathi, Tamilnadu, India.
[2]Department of Mathematics, Saraswathi Narayanan College, Madurai, Tamilnadu, India.

Authors' contributions

This work was carried out in collaboration with all the authors. Author SS designed the study, wrote the theorems, corollary and wrote the first draft of the manuscript. Authors PT and DCK managed the literature searches; analysed the study performed, the operator theory and developed the concepts on it. All the above said authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JSRR/2015/18661
Editor(s):
(1) Chun-Gang Zhu, School of Mathematical Sciences, Dalian University of Technology, China.
Reviewers:
(1) Anonymous, Mexico.
(2) Anonymous, Central University of Venezuela, Venezuela.
(3) Anonymous, The John Paul II Catholic of Lublin, Poland.
(4) Anonymous, Hanyang University, Republic of Korea.
Complete Peer review History: http://sciencedomain.org/review-history/10231

Original Research Article

Received 4th May 2015
Accepted 23rd June 2015
Published 17th July 2015

ABSTRACT

An operator A is called n-normal operator if $A^n A^* = A^* A^n$ and n-quasi-normal operator if $A^n A^* A = A^* A A^n$. In this paper we characterize the essential isometry, essential co-isometry n-normal and n-quasi-normal composite multiplication operator on L^2 -Spaces under certain conditions.

Keywords: Essential isometry; essential co-isometry; n-normal; n-quasi-normal; conditional expectation; composition operator; multiplication operator and composite multiplication operator.

Mathematics Subject Classification: 2010: 47B33, 47B37, 47B15, 47B07

**Corresponding author: Email: senthilsnc83@gmail.com;*

1. INTRODUCTION

Let X be a non-empty set, C be the field of complex numbers and $V(X)$ be a vector space of complex valued functions on X under the point wise operations of addition and scalar multiplication. Let T be a mapping of X into X such that $f \circ T$ is in $V(X)$ whenever f is in $V(X)$. Definition of the composition transformation C_T on $V(X)$ as $C_T f = f \circ T$ for every f in $V(X)$. If $V(X)$ has a Banach Space structure and C_T is bounded, then C_T is called the composition operator on $V(X)$ induced by T. Let $u: X \to C$ be a function such that M_u, defined as $M_u f = u \cdot f$ for every f in $V(X)$ be bounded linear operator on $V(X)$. Then the product $C_T M_u$ which becomes a bounded operator on $V(X)$ is called a composite multiplication operator.

Let $B(H)$ be the Banach algebra of all bounded operators on a Hilbert Space H. If (X, Σ, μ) is a σ-finite measure space and $T: X \to X$ is a measurable transformation such that $C_T \in B(L^2(\mu))$, then in [1] R. K. Singh and D. C. Kumar have proved that the measure μT^{-1}, defined as $\mu T^{-1}(E) = \mu(T^{-1}(E))$ for every E in Σ, is absolutely continuous with respect to the measure μ. Let f_0 denote the Radon-Nikodym derivative of μT^{-1} with respect to μ and if $C_T \in B(L^2(\mu))$, then in [1] R. K. Singh has proved that $C_T{}^* C_T = M_{f_0}$. A composite multiplication operator is a linear transformation acting on a set of complex valued Σ measurable functions f of the form

$$M_{u,T}(f) = C_T M_u(f) = u f \circ T = u \circ T \; f \circ T$$

Where u is a complex valued, Σ measurable function. In case $u = 1$ almost everywhere, $M_{u,T}$ becomes a composition operator, denoted by C_T.

Let X be a non-empty set and let Σ be a σ-algebra on X. Let μ and μT^{-1} be measures on Σ and $f_0: X \to [0, \infty]$ be a measurable function. Then the following are equivalent:

(i) μT^{-1} is absolutely continuous with respect to μ and f_0 is Radon-Nikodym derivative of μT^{-1} with respect to μ.

(ii) For every measurable function $f: X \to [0, \infty]$, the equality

$$\int_X f \, d\mu T^{-1} = \int_X f_0 f \, d\mu$$

holds.

In the study considered is the using conditional expectation of weighted composition operator on L^2-spaces. For each $f \in L^p(X, \Sigma, \mu)$, $1 \le p \le \infty$, there exists an unique $T^{-1}(\Sigma)$-measurable function $E(f)$ such that

$$\int_A g f \, d\mu = \int_A g E(f) \, d\mu$$

for every $T^{-1}(\Sigma)$-measurable function g, for which the left integral exists. The function $E(f)$ is called the conditional expectation of f with respect to the subalgebra $T^{-1}(\Sigma)$. As an operator of L^p, E is the projection onto the closure of range of T and E is the identity on $L^p(\mu)$, $p \ge 1$ if and only if $T^{-1}(\Sigma) = \Sigma$. Detailed discussion of E is found in [2,3].

1.1 Normal

A is said to be normal if $(AA^*)(x) = (A^*A)(x)$ for all $x \in H$.

1.2 n-Normal

A is said to be n-normal if $A^n A^* = A^* A^n$ for $n \in N$.

1.3 Quasi –normal

A is said to be quasi - normal if

$A(A^*A) = (A^*A)A$.

1.4 n-Quasi –normal

A is said to be n-quasi - normal if $A^n(A^*A)=(A^*A)A^n$ for $n \in N$.

2. RELATED WORK IN THE FIELD

During the last thirty years several authors have defined $W_{u,T}=M_u C_T = u\,(f\circ T)$ and have studied the properties of various classes of weighted composition operators on L^2 Spaces. The study of weighted composition operator was initiated, $M_{u,T}(f)=C_T\,M_u(f)=u\,f\circ T=u\circ T\;\; f\circ T$ by R.K. Singh and D. C. Kumar [1]. The concept of normality of bounded linear operators on a Hilbert Space has been generalized by different authors.

Recently, S. A. Alzuraiqi and A. B. Patel [4], define an operator A is called n-normal if $A^n A^* = A^* A^n$ for $n \in N$ and also proved that A is n-normal if and only if A^n is normal, that is $A^n A^* = A^* A^n$ for $n \in N$. O. Ahmed and M.S. Ahmed [5], defined an operator A which is called n-quasi-normal operator if $A^n A^* A = A^* A A^n$ for $n \in N$. Anuradha Gupta and Neha Bhatia [6] have proved, a weighted composition operator $W_{u,T}=M_u C_T = u\,(f\circ T)$ is n-normal and n-quasi-normal composition and weighted composition operators on $L^2(\mu)$. R. K. Singh, Kuldip Raj and Sunil K. Sharma [7,8], many results have been found, in the characterization of isometry, partial isometry, adjoint weighted composition operator.

3. ISOMETRIC AND UNITARY OF COMPOSITE MULTIPLICATION OPERATOR

Throughout the paper, by an operator we mean a bounded linear operator on a Hilbert space. If H denotes an infinite dimensional complex separable Hilbert Space, denote the algebra of all operators on H by $B(H)$. An operator A on H is said to be an isometry if $A^*A=I$, co isometry if $AA^*=I$, partial isometry if it is an isometry on the orthogonal complement of the kernel of A and unitary if $A^*A=AA^*=I$. The characterization of isometric and co-isometric composition operators on $L^2(\mu)$ have been studied by R.K. Singh [9]. We characterize isometric, co-isometric and partial isometric composite multiplication operator on $L^2(\mu)$.

Theorem 3.1

The composite multiplication operator $M_{u,T}$ on $L^2(\mu)$ is an isometry if and only if u^2 is the inverse of f_0.

Proof:

Suppose $M_{u,T}$ is an isometry, then

$$M^*_{u,T}\;M_{u,T} = (C_T M_u)^*(C_T M_u) = I$$

This implies that

$$M_{u^2 f_0} = I$$

Hence $u^2 f_0 = 1$ almost everywhere and this shows that u^2 is the inverse of f_0.

The converse can be proved by reversing the above argument.

Corollary 3.2

The composition operator C_T on $L^2(\mu)$ is an isometry if and only if $f_0 = 1$ almost everywhere.

Corollary 3.3

The composition operator C_T on l^2 is an isometry if and only if $\mu T^{-1}(n)=1$ almost everywhere.

Theorem 3.4

The composite multiplication operator $M_{u,T}$ on $L^2(\mu)$ is a co- isometry if and only if $u^2\circ T$ is the inverse of $f_0\circ T\cdot E(f)$ and $M_{u,T}$ is surjective.

Proof:

Suppose $M_{u,T}$ is a co-isometry. Then $M_{u,T}$ is surjective and

$$M_{u,T}\ M_{u,T}^{*} = (C_T M_u)\ (C_T M_u)^{*} = I$$

This implies that

$$M_{u^2 \circ T \cdot f_0 \circ T \cdot E(f)} = I$$

Hence $u^2 \circ T \cdot f_0 \circ T \cdot E(f) = 1$ almost everywhere and this shows that $u^2 \circ T$ is the inverse of $f_0 \circ T \cdot E(f)$.

The converse of this theorem is straight forward.

Corollary 3.5

The composition operator C_T on $L^2(\mu)$ is a co-isometry if and only if C_T is surjective and $f_0 \circ T \cdot E(f) = 1$ almost everywhere.

Theorem 3.6

The composite multiplication operator $M_{u,T}$ on $L^2(\mu)$ is unitary if and only if C_T has dense range and $u^2 f_0 = 1$ almost everywhere.

Theorem 3.7

The composite multiplication operator $M_{u,T}$ on $L^2(\mu)$ is a partial isometry if and only if $u^2 f_0$ is a characteristic function.

Proof:

Suppose $M_{u,T}$ is a partial-isometry then the problem of Halmos [10] is

$$M_{u,T} = (C_T M_u)(C_T M_u)^{*}(C_T M_u)$$

This implies that

$$(C_T M_u)^{*}\ (C_T M_u) = (C_T M_u)^{*}\ (C_T M_u)\ (C_T M_u)^{*}\ (C_T M_u)$$

$$M_{u^2 f_0} = M_{(u^2 f_0)^2}$$

Hence $u^2 f_0$ is a characteristic function.

Conversely, suppose $u^2 f_0$ is a characteristic function, then

$$\begin{aligned} \operatorname{Ker} M_{u,T} &= \operatorname{Ker} C_T M_u = \operatorname{Ker} (C_T M_u)^{*}(C_T M_u) \\ &= M_{u^2 f_0} \\ &= L^2 [\, Z_{u^2 f_0}] \end{aligned}$$

$$\operatorname{Ker} M^1{}_{u,T} = L^2(X_1, \Sigma_1, \mu),$$

When

$X_1 = X - Z_{u^2 f_0}$ and $\Sigma_1 = E \cap X_1 ; E \in \Sigma$

This shows that $M^{*}{}_{u,T} M_{u,T} f = f$, for every f in $\operatorname{Ker} M^1{}_{u,T}$

Hence $M_{u,T}$ is a partial isometry.

Corollary 3.8

If $u \in L^{\infty}(\mu)$ be such that $u \neq 0$ almost everywhere then $M_{u,T}$ is a partial isometry if and only if $u^2 \circ T \cdot f_0 \circ T = 1$ almost everywhere.

Proof:

Suppose $M_{u,T}$ is a partial-isometry, then

$$\begin{aligned} M_{u,T} &= (C_T M_u)(C_T M_u)^{*}(C_T M_u) \\ C_T M_u &= C_T M_u\, M_{u^2 f_0} \end{aligned}$$

This implies that

$$M_{u \circ T}\, C_T = M_{u \circ T \cdot u^2 \circ T \cdot f_0 \circ T}\ C_T .$$

Hence by theorem $u^2 \circ T \cdot f_0 \circ T = 1$ almost everywhere.

Then the converse is straight forward.

3.1 Example:

Let $X = [0, \infty)$ and μ be a Lebesgue measure on it .Define $T: X \to X$ as $T(x) = x + a,\ a \in X$ and Define $u: X \to C$ as

$$u(x) = \begin{cases} 2 & \text{if } x \in [0, a) \\ 1 & \text{if } x \in [a, \infty) \end{cases}$$

Then $M_{u,T} \in B(L^2(\mu))$ and $M_{u,T}$ is a partial isometry.

3.1 Definition

An operator A on H is said to be essential isometry if $A^*A - I$ is compact and essential co - isometry if $AA^* - I$ is compact.

The following theorems give the characterizations of essential isometric and essential co-isometric composite multiplication operators on $L^2(\mu)$ of a non-atomic measure space.

Theorem 3.9

The composite multiplication operator $M_{u,T}$ on $L^2(\mu)$ is an essential isometry if and only if $M_{u,T}$ is a partial isometry.

Proof:

Suppose $M_{u,T}$ is an essential isometry, then $M^*{}_{u,T}\, M_{u,T} - I = (C_T M_u)^* (C_T M_u) - I$ is compact.

This implies that
$M_{u^2 f_0 - 1}$ is compact and hence $u^2 f_0 - 1 = 0$ almost everywhere.

This proves that $u^2 f_0 = 1$ almost everywhere.

Thus $M_{u,T}$ is an isometry.

The converse is obvious.

Theorem 3.10

The composite multiplication operator $M_{u,T}$ on $L^2(\mu)$ is an essential co-isometry if and only if $M_{u,T}$ is a co- isometry.

Proof:

Suppose $M_{u,T}$ is an essential co- isometry, then $M_{u,T}\, M^*{}_{u,T} - I = (C_T M_u)(C_T M_u)^* - I$ is compact.

This implies that

$$(C_T M_u)(C_T M_u)^* - I = \begin{cases} M_{u^2 \circ T \cdot f_0 \circ T \cdot E(f) - 1} & \text{on } \overline{\operatorname{ran} M_{u,T}} \\ -I & \text{on } \operatorname{Ker}(M_{u,T})^* \end{cases}$$

Since $\overline{\operatorname{ran} M_{u,T}}$ and $\operatorname{Ker}(M_{u,T})^*$ are invariant under $M_{u^2 \circ T \cdot f_0 \circ T \cdot E(f) - 1}$ and I respectively,

$M_{u^2 \circ T \cdot f_0 \circ T \cdot E(f) - 1}$ and I are compact.

Since μ is non-atomic $\operatorname{Ker}(M_{u,T})^* = 0$ and so $M_{u,T}$ has dense range,
$u^2 \circ T \cdot f_0 \circ T \cdot E(f) = 1$ almost everywhere.

This proves that $M_{u,T} M^*{}_{u,T} = I$ and thus $M_{u,T}$ is a co-isometry.

The converse is straight forward.

Theorem 3.11

The composite multiplication operator $M_{u,T}$ on $L^2(\mu)$ is an essential unitary if and only if it is unitary.

Proof:

The operator $M_{u,T}$ is an essential unitary if and only if it is essential isometry and essential co-isometry.

Hence the results follow from the above theorem.

4. NORMAL AND QUASI-NORMAL COMPOSITE MULTIPLICATION OPERATOR

Now we give a characterization of normal and quasi-normal composite multiplication operator.

Theorem 4.1

Let the composite multiplication operator $M_{u,T}$ on $L^2(\mu)$, then for $u \geq 0$, $M_{u,T}$ is normal if and only if $u^2 \circ T \cdot f_0 \circ T \cdot E(f) = u^2 f_0\ f$.

Proof:

Suppose $M_{u,T}$ is normal, then

Since $M_{u,T}\, f = C_T M_u\, f = (u\, f) \circ T = u \circ T \cdot f \circ T$,

The adjoint $M^*_{u,T}$ of $M_{u,T}$ is given by $M^*_{u,T}\, f = u\, f_0 \cdot E(f) \circ T^{-1}$

$$\text{(i) } M^*_{u,T} M_{u,T}\, f = M^*_{u,T}(u\, f \circ T) = u\, f_0 \cdot E(u\, f \circ T) \circ T^{-1} = u^2 f_0\, f$$

$$\text{(ii) } M_{u,T} M^*_{u,T}\, f = M_{u,T}(u\, f_0 \cdot E(f) \circ T^{-1}) = (u^2 f_0 \cdot E(f) \circ T^{-1}) \circ T = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$$

$$\Leftrightarrow u^2 \circ T \cdot f_0 \circ T \cdot E(f) = u^2 f_0\, f.$$

Corollary 4.2

The composite multiplication operator $M_{u,T}$ on $L^2(\mu)$. The following statements are equivalent:

(i) $M_{u,T}$ is normal operator.

(ii) $M^*_{u,T}$ is normal operator.

(iii) $u^2 \circ T \cdot f_0 \circ T \cdot E(f) = u^2 f_0\, f$.

Corollary 4.3

The composition operator C_T on $L^2(\mu)$ is normal if and only if $f_0 \circ T \cdot E(f) = f_0\, f$.

Proof:

The proof is obtained from theorem 4.1 by putting $u = 1$.

Theorem 4.4

Let the composite multiplication operator $M_{u,T}$ on $L^2(\mu)$. Then $M_{u,T}$ is quasi-normal if and only if $u^3 \circ T \cdot f_0 \circ T \cdot f \circ T = u^2 \cdot u \circ T \cdot f_0 \cdot f \circ T$.

Proof:

Suppose $M_{u,T}$ is quasi-normal, then

$$M_{u,T}\,(M^*_{u,T} M_{u,T}) f = M_{u,T}\,(u^2 f_0\, f) = u^3 \circ T \cdot f_0 \circ T \cdot f \circ T$$

$$(M^*_{u,T} M_{u,T})\, M_{u,T}\, f = (M^*_{u,T} M_{u,T})\,(uf \circ T) = u^2 \cdot u \circ T \cdot f_0 \cdot f \circ T$$

Hence $u^3 \circ T \cdot f_0 \circ T \cdot f \circ T = u^2 \cdot u \circ T \cdot f_0 \cdot f \circ T$.

Corollary 4.5

The composition operator C_T on $L^2(\mu)$ is quasi-normal if and only if $f_0 \circ T \cdot f \circ T = f_0 \cdot f \circ T$.

Proof:

The proof is obtained from theorem 4.4 by putting $u = 1$.

Theorem 4.6

Let the composite multiplication operator $M_{u,T}$ on $L^2(\mu)$. Then the adjoint $M^*_{u,T}$ of $M_{u,T}$ is quasi-normal if and only if $u\, f_0 \cdot E(u^2 f_0 f) \circ T^{-1} = u^3 f_0^2 \cdot E(f) \circ T^{-1}$.

Proof:

Suppose $M^*_{u,T}$ is quasi-normal. Then

$$M^*_{u,T}\,(M^*_{u,T} M_{u,T}) f = u\, f_0 \cdot E(u^2 f_0 f) \circ T^{-1}$$

$$(M^*_{u,T} M_{u,T})\, M^*_{u,T}\, f = u^3 f_0^2 \cdot E(f) \circ T^{-1}$$

Hence $u\, f_0 \cdot E(u^2 f_0 f) \circ T^{-1} = u^3 f_0^2 \cdot E(f) \circ T^{-1}$.

4.1 Example

Let $X = [0, 1]$ and the Lebesgue measure μ on the Lebesgue measurable subsets. The transformation $T: X \to X$ is given by $T(x) = 2x(1-x)$ and $u(x) = 2x$ for every $x \in X$. Then direct computation shows that $f_0(x) = \frac{1}{2\sqrt{1-2x}} \chi_{\left[0, \frac{1}{2}\right]}(x)$ and for each $f \in L^2(\mu)$

$$E(f)(x) = \frac{1}{2}[f(x) + f(1-x)]\chi_{\left[0, \frac{1}{2}\right]}(x)$$

$$u^2 \circ T \cdot f_0 \circ T \cdot E(f)$$
$$= \frac{2\,(2x - 2x^2)^2}{\sqrt{1 - 4x(1-x)}} \cdot \frac{1}{2}[f(x) + f(1-x)]\chi_{\left[0, \frac{1}{2}\right]}(x)$$

$$u^2 f_0 f = \frac{4x^2}{2\sqrt{1-2x}} f(x)\chi_{\left[0, \frac{1}{2}\right]}(x)$$

Then $M_{u,T} \in B(L^2(\mu))$ and $M_{u,T}$ is not normal.

5. n-NORMAL AND n-QUASI-NORMAL COMPOSITE MULTIPLICATION OPERATOR

Alzuraiqi and Patel [4] have proved that, the operators A on H is n- normal if and only if A^n is normal that is $A^n A^{*^n} = A^{*^n} A^n$ for $n \in N$. O. Ahmed and M.S. Ahmed [5] have proved that, the operator A is called n -quasi-normal operator if $A^n A^* A = A^* A A^n$ for $n \in N$. In an analogous manner, we derive characterize the n - normal and n -quasi-normal composite multiplication operator on L^2-Spaces.

Theorem 5.1

Let the composite multiplication operators $M_{u,T}$ on $L^2(\mu)$. Then $M_{u,T}$ is n-normal if and only if

$$u_n \cdot u \circ T^n \cdot f_0 \circ T^n \cdot E[u f_0] \circ T^{n-1} \cdot E(f)$$
$$= u f_0 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(u_n) \circ T^{-n} \; f$$

Proof:

Suppose $M_{u,T}$ is n-normal, then

$$M^n{}_{u,T}\, M^{*^n}{}_{u,T}\; f$$
$$= M^n{}_{u,T}\; [\, u f_0 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(f) \circ T^{-n}\,]$$
$$= u_n \cdot u \circ T^n \cdot f_0 \circ T^n \cdot E[u f_0] \circ T^{n-1} \cdot E(f)$$
$$M^{*^n}{}_{u,T}\; M^n{}_{u,T}\; f = M^{*^n}{}_{u,T}\; [\, u_n \cdot f \circ T^n\,]$$
$$= u f_0 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(u_n) \circ T^{-n} \; f$$

Where $u_n = u \circ T \cdot u \circ T^2 \ldots\ldots u \circ T^n$

$$E(u f_0) \circ T^{-(n-1)}$$
$$= E(u f_0) \circ T^{-1} \cdot E(u f_0) \circ T^{-2} \ldots. E(u f_0) \circ T^{-(n-1)}$$

$$E(u f_0) \circ T^{n-1}$$
$$= E(u f_0) \circ T^1 \cdot E(u f_0) \circ T^2 \ldots. E(u f_0) \circ T^{n-1}$$

Hence

$$u_n \cdot u \circ T^n \cdot f_0 \circ T^n \cdot E[u f_0] \circ T^{n-1} \cdot E(f)$$
$$= u f_0 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(u_n) \circ T^{-n} \; f \,.$$

Corollary 5.2

The composite multiplication operator $M_{u,T}$ on $L^2(\mu)$. The following statements are equivalent:

(i) $M_{u,T}$ is n-normal operator.

(ii) $M^*{}_{u,T}$ is n-normal operator.

(iii) $u_n \cdot u \circ T^n \cdot f_0 \circ T^n \cdot E[u f_0] \circ T^{n-1} \cdot E(f)$
$= u f_0 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(u_n) \circ T^{-n} \; f$

Corollary 5.3

The composition operator $C_T{}^n$ on $L^2(\mu)$ is n - normal if and only if

$$f_0 \circ T^n \cdot E(f_0) \circ T^{n-1} \cdot E(f) = f_0 \cdot E(f_0) \circ T^{-(n-1)} \; f$$

Proof:

The proof is obtained from theorem 5.1 by putting $u = 1$.

Theorem 5.4

Let the composite multiplication operator $M_{u,T}$ on $L^2(\mu)$. Then $M_{u,T}$ is n-quasi-normal if and only if $u_n \cdot u^2 \circ T^n \cdot f_0 \circ T^n \cdot f \circ T^n = u_n \cdot u^2 f_0 \cdot f \circ T^n$.

Proof:

Suppose $M_{u,T}$ is n-quasi-normal. Then

$$M^n{}_{u,T}\, M^*{}_{u,T}\, M_{u,T}\; f = M^n{}_{u,T}\,(u^2 f_0 f)$$
$$= u_n \cdot u^2 \circ T^n \cdot f_0 \circ T^n \cdot f \circ T^n$$

$$M^*{}_{u,T}\, M_{u,T}\; M^n{}_{u,T}\; f = M^*{}_{u,T}\; M_{u,T}\,(u_n \cdot f \circ T^n)$$
$$= u_n \cdot u^2 f_0 \cdot f \circ T^n$$

Where, $u_n = u \circ T \cdot u \circ T^2 ....u \circ T^n$
Hence

$$u_n \cdot u^2 \circ T^n \cdot f_0 \circ T^n \cdot f \circ T^n = u_n \cdot u^2 f_0 \cdot f \circ T^n .$$

Corollary 5.5

The composition operator $C_T{}^n$ on $L^2(\mu)$ is n-quasi-normal if and only if

$$f_0 \circ T^n \cdot f \circ T^n = f_0 \cdot f \circ T^n$$

Proof:

The proof is obtained from theorem 5.4 by putting $u = 1$.

Theorem 5.6

Let the composite multiplication operator $M_{u,T}$ on $L^2(\mu)$. Then the adjoint $M^*{}_{u,T}$ of $M_{u,T}$ is n-quasi-normal if and only if

$$\begin{aligned} & u f_0 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(u^2 f_0 f) \circ T^{-n} \\ & = u^3 f_0{}^2 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(f) \circ T^{-n} \end{aligned} .$$

Proof:

Suppose $M^*{}_{u,T}$ is n-quasi-normal, then

$$\begin{aligned} & M^{*^n}{}_{u,T} \, M^*{}_{u,T} \, M_{u,T} \, f = M^{*^n}{}_{u,T} \; (u^2 f_0 f) \\ & = u f_0 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(u^2 f_0 f) \circ T^{-n} \end{aligned}$$

$$\begin{aligned} & M^*{}_{u,T} \, M_{u,T} \;\; M^{*^n}{}_{u,T} \;\; f \\ & = M^*{}_{u,T} \, M_{u,T} (u f_0 \cdot E(u f_0) \circ T^{-(n-1)} E(f) \circ T^{-n}) \\ & = u^3 f_0{}^2 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(f) \circ T^{-n} \end{aligned}$$

Where

$$\begin{aligned} & E(u f_0) \circ T^{-(n-1)} \\ & = E(u f_0) \circ T^{-1} \cdot E(u f_0) \circ T^{-2} .... E(u f_0) \circ T^{-(n-1)} \end{aligned}$$

$$\begin{aligned} & E(u f_0) \circ T^{n-1} \\ & = E(u f_0) \circ T^{1} \cdot E(u f_0) \circ T^{2} .... E(u f_0) \circ T^{n-1} \end{aligned}$$

Hence

$$\begin{aligned} & u f_0 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(u^2 f_0 f) \circ T^{-n} \\ & = u^3 f_0{}^2 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(f) \circ T^{-n} \end{aligned}$$

Corollary 5.7

The composition operator $C_T{}^{*^n}$ on $L^2(\mu)$ is n-quasi-normal if and only if

$$\begin{aligned} & f_0 \cdot E(f_0) \circ T^{-(n-1)} \cdot E(f_0 f) \circ T^{-n} \\ & = f_0{}^2 \cdot E(f_0) \circ T^{-(n-1)} \cdot E(f) \circ T^{-n} \end{aligned}$$

Proof:

The proof is obtained from theorem 5.6 by putting $u = 1$.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

1. Singh RK, Kumar DC. Weighted composition operators, Ph.D. Thesis, Univ. of Jammu; 1985.
2. Campbell J, Jamison J. On some classes of weighted composition operators, Glasgow Math. J. 1990;32:82-94.
3. Embry Wardrop M, Lambert A. Measurable transformations and Centred composition Operators. Proc. Royal Irish Acad. 2009;2(1):23-25.
4. Alzuraiqi SA, Patel AB. On n-normal operators, General Math. Notes. 2010;1:61-73.
5. Ahmed O, Ahmed MS. On the class of n-power quasi-normal operators on Hilbert space. Bull. Math. Anal. Appl. 2011;2:213-228.
6. Anuradha Gupta, Neha Bhatia. n-normal and n-quasi-normal composition and weighted composition operators on L^2, Mat. Vesnik. 2014;66(4):364-370.
7. Singh RK. A survey of weighted substitution operators and generalizations of banach –stone theorem. International Journal of Mathematics and Mathematical

Sciences, Hindawi Publishing Corporation. 2005;937–948.
8. Raj K, Sharma SK. Weighted substitution operators between -spaces of vector-valued functions. Fasc. Math; 2011.
9. Singh RK. Composition operators, Thesis, Univ. of New Hampshire; 1972.
10. Halmos PR. A Hilbert space problem book, Van Nostrand, Princeton, N.J; 1967.

Peer-review history:
The peer review history for this paper can be accessed here:
http://sciencedomain.org/review-history/10231