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On Generalized Reverse 3-primes Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, we introduce and investigate the generalized reverse 3-primes sequences and we deal with, in detail, three special cases which we call them reverse 3-primes, reverse Lucas 3-primes and reverse modified 3-primes sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

Keywords: Reverse 3-primes numbers; reverse Lucas 3-primes numbers; 3-primes numbers; Lucas 3-primes numbers; Tribonacci numbers.

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1 INTRODUCTION

The sequence of Fibonacci numbers $\{F_n\}$ and the sequence of Lucas numbers $\{L_n\}$ are defined by

$$F_n = F_{n-1} + F_{n-2}, \;\; n \geq 2, \quad F_0 = 0, \; F_1 = 1,$$

and

$$L_n = L_{n-1} + L_{n-2}, \;\; n \geq 2, \quad L_0 = 2, \; L_1 = 1$$

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respectively. There are several nice and interesting generalizations of Fibonacci and Lucas sequences.

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \ \ n \geq 3 \tag{1.1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers.

This sequence has been studied by many authors, see for example [1,2,3,4,5,6,7,8,9,10,11,12,13].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

As $\{W_n\}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{1.2}$$

whose roots are

$$\begin{aligned} \alpha &= \alpha(r,s,t) = \frac{r}{3} + A + B \\ \beta &= \beta(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B \\ \gamma &= \gamma(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B \end{aligned}$$

where

$$\begin{aligned} A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \ B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3} \\ \Delta &= \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \ \ \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3) \end{aligned}$$

Note that we have the following identities

$$\begin{aligned} \alpha + \beta + \gamma &= r, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -s, \\ \alpha\beta\gamma &= t. \end{aligned}$$

If $\Delta(r, s, t) > 0$, then the Equ. (1.2) has one real (α) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that generalized Tribonacci numbers can be expressed, for all integers n, using Binet's formula

$$W_n = \frac{b_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{b_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{b_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \tag{1.3}$$

where

$$b_1 = W_2 - (\beta + \gamma)W_1 + \beta\gamma W_0, \ b_2 = W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0, \ b_3 = W_2 - (\alpha + \beta)W_1 + \alpha\beta W_0. \tag{1.4}$$

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n, for a proof of this result see [14]. This result of Howard and Saidak [14] is even true in the case of higher-order recurrence relations.

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 1.1. *Suppose that* $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n x^n$ *is the ordinary generating function of the generalized Tribonacci sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum\limits_{n=0}^{\infty} W_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{1 - rx - sx^2 - tx^3}. \tag{1.5}$$

Proof. Using the definition of generalized Tribonacci numbers, and substracting $rx\sum_{n=0}^{\infty} W_n x^n$, $sx^2\sum_{n=0}^{\infty} W_n x^n$ and $tx^3\sum_{n=0}^{\infty} W_n x^n$ from $\sum_{n=0}^{\infty} W_n x^n$ we obtain

$$\begin{aligned}
(1 - rx - sx^2 - tx^3)\sum_{n=0}^{\infty} W_n x^n &= \sum_{n=0}^{\infty} W_n x^n - rx\sum_{n=0}^{\infty} W_n x^n - sx^2\sum_{n=0}^{\infty} W_n x^n - tx^3\sum_{n=0}^{\infty} W_n x^n \\
&= \sum_{n=0}^{\infty} W_n x^n - r\sum_{n=0}^{\infty} W_n x^{n+1} - s\sum_{n=0}^{\infty} W_n x^{n+2} - t\sum_{n=0}^{\infty} W_n x^{n+3} \\
&= \sum_{n=0}^{\infty} W_n x^n - r\sum_{n=1}^{\infty} W_{n-1} x^n - s\sum_{n=2}^{\infty} W_{n-2} x^n - t\sum_{n=3}^{\infty} W_{n-3} x^n \\
&= (W_0 + W_1 x + W_2 x^2) - r(W_0 x + W_1 x^2) - sW_0 x^2 \\
&\quad + \sum_{n=3}^{\infty}(W_n - rW_{n-1} - sW_{n-2} - tW_{n-3})x^n \\
&= W_0 + W_1 x + W_2 x^2 - rW_0 x - rW_1 x^2 - sW_0 x^2 \\
&= W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2.
\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{1 - rx - sx^2 - tx^3}.$$

We next find Binet formula of generalized Tribonacci numbers $\{V_n\}$ by the use of generating function for V_n.

Theorem 1.2. *(Binet formula of generalized Tribonacci numbers)*

$$W_n = \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.6}$$

where

$$\begin{aligned}
d_1 &= W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\
d_2 &= W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\
d_3 &= W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0).
\end{aligned}$$

Proof. Let

$$h(x) = 1 - rx - sx^2 - tx^3.$$

Then for some α, β and γ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)$$

i.e.,

$$1 - rx - sx^2 - tx^3 = (1-\alpha x)(1-\beta x)(1-\gamma x) \tag{1.7}$$

Hence $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$ are the roots of $h(x)$. This gives α, β, and γ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{r}{x} - \frac{s}{x^2} - \frac{t}{x^3} = 0.$$

This implies $x^3 - rx^2 - sx - t = 0$. Now, by (1.5) and (1.7), it follows that

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)}.$$

Then we write

$$\frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)} = \frac{A_1}{(1-\alpha x)} + \frac{A_2}{(1-\beta x)} + \frac{A_3}{(1-\gamma x)}. \quad (1.8)$$

So

$$W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2 = A_1(1-\beta x)(1-\gamma x) + A_2(1-\alpha x)(1-\gamma x) + A_3(1-\alpha x)(1-\beta x).$$

If we consider $x = \frac{1}{\alpha}$, we get $W_0 + (W_1 - rW_0)\frac{1}{\alpha} + (W_2 - rW_1 - sW_0)\frac{1}{\alpha^2} = A_1(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha})$. This gives

$$A_1 = \frac{\alpha^2(W_0 + (W_1 - rW_0)\frac{1}{\alpha} + (W_2 - rW_1 - sW_0)\frac{1}{\alpha^2})}{(\alpha-\beta)(\alpha-\gamma)} = \frac{W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0)}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly, we obtain

$$A_2 = \frac{W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0)}{(\beta-\alpha)(\beta-\gamma)}, A_3 = \frac{W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0)}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus (1.8) can be written as

$$\sum_{n=0}^{\infty} W_n x^n = A_1(1-\alpha x)^{-1} + A_2(1-\beta x)^{-1} + A_3(1-\gamma x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} W_n x^n = A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n + A_3 \sum_{n=0}^{\infty} \gamma^n x^n = \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n + A_3\gamma^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n$$

where

$$\begin{aligned} A_1 &= \frac{W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0)}{(\alpha-\beta)(\alpha-\gamma)}, \\ A_2 &= \frac{W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0)}{(\beta-\alpha)(\beta-\gamma)}, \\ A_3 &= \frac{W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0)}{(\gamma-\alpha)(\gamma-\beta)}, \end{aligned}$$

and then we get (1.6).

Note that from (1.4) and (1.6) we have

$$\begin{aligned} W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0 &= W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\ W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0 &= W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\ W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0 &= W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0). \end{aligned}$$

In this paper, we investigate the generalized reverse 3-primes sequences and we investigate, in detail, three special cases which we call them reverse 3-primes, reverse Lucas 3-primes and reverse modified 3-primes sequences. In this paper we consider the case $r = 5,\ s = 3, t = 2$ and in this case we write $V_n = W_n$. A generalized reverse 3-primes sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = 5V_{n-1} + 3V_{n-2} + 2V_{n-3} \tag{1.9}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{3}{2}V_{-(n-1)} - \frac{5}{2}V_{-(n-2)} + \frac{1}{2}V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.9) holds for all integer n.

(1.3) can be used to obtain Binet formula of generalized reverse 3-primes numbers. Binet formula of generalized reverse 3-primes numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{1.10}$$

Here, α, β and γ are the roots of the cubic equation $x^3 - 5x^2 - 3x - 2 = 0$. Moreover

$$\begin{aligned}
\alpha &= \frac{5}{3} + \left(\frac{439}{54} + \sqrt{\frac{1315}{108}}\right)^{1/3} + \left(\frac{439}{54} - \sqrt{\frac{1315}{108}}\right)^{1/3} \\
\beta &= \frac{5}{3} + \omega\left(\frac{439}{54} + \sqrt{\frac{1315}{108}}\right)^{1/3} + \omega^2\left(\frac{439}{54} - \sqrt{\frac{1315}{108}}\right)^{1/3} \\
\gamma &= \frac{5}{3} + \omega^2\left(\frac{439}{54} + \sqrt{\frac{1315}{108}}\right)^{1/3} + \omega\left(\frac{439}{54} - \sqrt{\frac{1315}{108}}\right)^{1/3}
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3)$$

Note that

$$\begin{aligned}
\alpha+\beta+\gamma &= 5, \\
\alpha\beta+\alpha\gamma+\beta\gamma &= -3, \\
\alpha\beta\gamma &= 2.
\end{aligned}$$

The first few generalized reverse 3-primes numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized reverse 3-primes numbers

n	V_n	V_{-n}
0	V_0	...
1	V_1	$\frac{1}{2}V_2 - \frac{5}{2}V_1 - \frac{3}{2}V_0$
2	V_2	$\frac{17}{4}V_1 - \frac{1}{4}V_0 - \frac{3}{4}V_2$
3	$2V_0 + 3V_1 + 5V_2$	$\frac{37}{8}V_0 - \frac{1}{8}V_1 - \frac{1}{8}V_2$
4	$10V_0 + 17V_1 + 28V_2$	$\frac{37}{16}V_2 - \frac{187}{16}V_1 - \frac{113}{16}V_0$
5	$56V_0 + 94V_1 + 157V_2$	$\frac{639}{32}V_1 - \frac{35}{32}V_0 - \frac{113}{32}V_2$
6	$314V_0 + 527V_1 + 879V_2$	$\frac{1383}{64}V_0 - \frac{51}{64}V_1 - \frac{35}{64}V_2$
7	$1758V_0 + 2951V_1 + 4922V_2$	$\frac{1383}{128}V_2 - \frac{6985}{128}V_1 - \frac{4251}{128}V_0$
8	$9844V_0 + 16\,524V_1 + 27\,561V_2$	$\frac{24\,021}{256}V_1 - \frac{1217}{256}V_0 - \frac{4251}{256}V_2$
9	$55\,122V_0 + 92\,527V_1 + 154\,329V_2$	$\frac{51\,693}{512}V_0 - \frac{2417}{512}V_1 - \frac{1217}{512}V_2$
10	$308\,658V_0 + 518\,109V_1 + 864\,172V_2$	$\frac{51\,693}{1024}V_2 - \frac{260\,899}{1024}V_1 - \frac{159\,913}{1024}V_0$
11	$1728\,344V_0 + 2901\,174V_1 + 4838\,969V_2$	$\frac{902\,951}{2048}V_1 - \frac{42\,059}{2048}V_0 - \frac{159\,913}{2048}V_2$
12	$9677\,938V_0 + 16\,245\,251V_1 + 27\,096\,019V_2$	$\frac{1932\,079}{4096}V_0 - \frac{109\,531}{4096}V_1 - \frac{42\,059}{4096}V_2$
13	$54\,192\,038V_0 + 90\,965\,995V_1 + 151\,725\,346V_2$	$\frac{1932\,079}{8192}V_2 - \frac{9744\,513}{8192}V_1 - \frac{6015\,299}{8192}V_0$

Now we define three special cases of the sequence $\{V_n\}$. reverse 3-primes sequence $\{N_n\}_{n\geq 0}$, reverse Lucas 3-primes sequence $\{S_n\}_{n\geq 0}$ and reverse modified 3-primes sequence $\{U_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$N_{n+3} = 5N_{n+2} + 3N_{n+1} + 2N_n, \quad N_0 = 0, N_1 = 1, N_2 = 5,$$

$$S_{n+3} = 5S_{n+2} + 3S_{n+1} + 2S_n, \quad S_0 = 3, S_1 = 5, S_2 = 31, \tag{1.11}$$

and

$$U_{n+3} = 5U_{n+2} + 3U_{n+1} + 2U_n, \quad U_0 = 0, U_1 = 1, U_2 = 4, \tag{1.12}$$

For generalized 3-primes sequence (and it's three special cases, 3-primes, Lucas 3-primes and modified 3-primes sequences) see [15].

The sequences $\{N_n\}_{n\geq 0}$, $\{S_n\}_{n\geq 0}$ and $\{U_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$N_{-n} = -\frac{3}{2}N_{-(n-1)} - \frac{5}{2}N_{-(n-2)} + \frac{1}{2}N_{-(n-3)}, \tag{1.13}$$

$$S_{-n} = -\frac{3}{2}S_{-(n-1)} - \frac{5}{2}S_{-(n-2)} + \frac{1}{2}S_{-(n-3)} \tag{1.14}$$

and

$$U_{-n} = -\frac{3}{2}U_{-(n-1)} - \frac{5}{2}U_{-(n-2)} + \frac{1}{2}U_{-(n-3)} \tag{1.15}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.13), (1.14) and (1.15) hold for all integer n.

Note that the sequences N_n, S_n and U_n are not indexed in [16] yet. Next, we present the first few values of the reverse 3-primes, reverse Lucas 3-primes and reverse modified 3-primes numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10	11
N_n	0	1	5	28	157	879	4922	27561	154329	864172	4838969	27096019
N_{-n}		0	$\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{1}{8}$	$\frac{37}{16}$	$-\frac{113}{32}$	$-\frac{35}{64}$	$\frac{1383}{128}$	$-\frac{4251}{256}$	$-\frac{1217}{512}$	$\frac{51693}{1024}$
S_n	3	5	31	176	983	5505	30826	172611	966543	5412200	30305851	169698941
S_{-n}		$-\frac{3}{2}$	$-\frac{11}{4}$	$\frac{75}{8}$	$-\frac{127}{16}$	$-\frac{413}{32}$	$\frac{2809}{64}$	$-\frac{4805}{128}$	$-\frac{15327}{256}$	$\frac{105267}{512}$	$-\frac{181751}{1024}$	$-\frac{568725}{2048}$
U_n	0	1	4	23	129	722	4043	22639	126768	709843	3974797	22257050
U_{-n}		$-\frac{1}{2}$	$\frac{5}{4}$	$-\frac{5}{8}$	$-\frac{39}{16}$	$\frac{187}{32}$	$-\frac{191}{64}$	$-\frac{1453}{128}$	$\frac{7017}{256}$	$-\frac{7285}{512}$	$-\frac{54\,127}{1024}$	$\frac{263299}{2048}$

For all integers n, reverse 3-primes, reverse Lucas 3-primes and reverse modified 3-primes numbers (using initial conditions in (1.10)) can be expressed using Binet's formulas as

$$\begin{aligned} N_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\ S_n &= \alpha^n+\beta^n+\gamma^n, \\ U_n &= \frac{(\alpha-1)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{(\beta-1)\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{(\gamma-1)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}, \end{aligned}$$

respectively.

2 GENERATING FUNCTIONS AND OBTAINING BINET FORMULA FROM GENERATING FUNCTION

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 2.1. *Suppose that $f_{V_n}(x)=\sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized reverse 3-primes sequence $\{V_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n=\frac{V_0+(V_1-5V_0)x+(V_2-5V_1-3V_0)x^2}{1-5x-3x^2-2x^3}.$$

Proof. Take $r=5$, $s=3, t=2$ in Lemma 1.1.
The previous Lemma gives the following results as particular examples.

Corollary 2.2. *Generated functions of reverse 3-primes, reverse Lucas 3-primes and reverse modified 3-primes numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} N_n x^n &= \frac{x}{1-5x-3x^2-2x^3}, \\ \sum_{n=0}^{\infty} S_n x^n &= \frac{3-10x-3x^2}{1-5x-3x^2-2x^3}, \\ \sum_{n=0}^{\infty} U_n x^n &= \frac{x-x^2}{1-5x-3x^2-2x^3}, \end{aligned}$$

respectively.

We next find Binet formula of generalized reverse 3-primes numbers $\{V_n\}$ by the use of generating function for V_n.

Theorem 2.3. *(Binet formula of generalized reverse 3-primes numbers)*

$$V_n=\frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{2.1}$$

where

$$\begin{aligned} d_1 &= V_0\alpha^2+(V_1-5V_0)\alpha+(V_2-5V_1-3V_0), \\ d_2 &= V_0\beta^2+(V_1-5V_0)\beta+(V_2-5V_1-3V_0), \\ d_3 &= V_0\gamma^2+(V_1-5V_0)\gamma+(V_2-5V_1-3V_0). \end{aligned}$$

Proof. Take $r = 5,\ s = 3, t = 2$ in Theorem 1.2.
Note that from (1.10) and (2.1) we have

$$\begin{aligned}
V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0 &= V_0\alpha^2 + (V_1 - 5V_0)\alpha + (V_2 - 5V_1 - 3V_0), \\
V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0 &= V_0\beta^2 + (V_1 - 5V_0)\beta + (V_2 - 5V_1 - 3V_0), \\
V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0 &= V_0\gamma^2 + (V_1 - 5V_0)\gamma + (V_2 - 5V_1 - 3V_0).
\end{aligned}$$

Next, using the last Theorem, we present the Binet formulas of reverse 3-primes, reverse Lucas 3-primes and reverse modified 3-primes sequences.

Corollary 2.4. *Binet formulas of reverse 3-primes, reverse Lucas 3-primes and reverse modified 3-primes sequences are*

$$\begin{aligned}
N_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\
S_n &= \alpha^n + \beta^n + \gamma^n, \\
U_n &= \frac{(\alpha-1)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(\beta-1)\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{(\gamma-1)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)},
\end{aligned}$$

respectively.

We can find Binet formulas by using matrix method with a similar technique which is given in [17]. Take $k = i = 3$ in Corollary 3.1 in [17]. Let

$$\begin{aligned}
\Lambda &= \begin{pmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{pmatrix}, \Lambda_1 = \begin{pmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{pmatrix}, \\
\Lambda_2 &= \begin{pmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{pmatrix}, \Lambda_3 = \begin{pmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{pmatrix}.
\end{aligned}$$

Then the Binet formula for reverse 3-primes numbers is

$$\begin{aligned}
N_n &= \frac{1}{\det(\Lambda)} \sum_{j=1}^{3} N_{4-j} \det(\Lambda_j) = \frac{1}{\det(\Lambda)} (N_3 \det(\Lambda_1) + N_2 \det(\Lambda_2) + N_1 \det(\Lambda_3)) \\
&= \left(28 \begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 5 \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix} \right) / \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}.
\end{aligned}$$

Similarly, we obtain the Binet formula for reverse Lucas 3-primes and reverse modified 3-primes numbers as

$$\begin{aligned}
S_n &= \frac{1}{\det(\Lambda)} (176 \det(\Lambda_1) + 31 \det(\Lambda_2) + 5 \det(\Lambda_3)), \\
U_n &= \frac{1}{\det(\Lambda)} (23 \det(\Lambda_1) + 4 \det(\Lambda_2) + \det(\Lambda_3)),
\end{aligned}$$

respectively.

3 SIMSON FORMULAS

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized reverse 3-primes sequence $\{V_n\}_{n\geq 0}$.

Theorem 3.1 (Simson Formula of Generalized Reverse 3-primes Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} V_{n+2} & V_{n+1} & V_n \\ V_{n+1} & V_n & V_{n-1} \\ V_n & V_{n-1} & V_{n-2} \end{vmatrix} = 2^n \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}. \tag{3.1}$$

Proof. (3.1) is given in Soykan [18].
The previous theorem gives the following results as particular examples.

Corollary 3.2. *For all integers n, Simson formula of reverse 3-primes, reverse Lucas 3-primes and reverse modified 3-primes numbers are given as*

$$\begin{aligned} \begin{vmatrix} N_{n+2} & N_{n+1} & N_n \\ N_{n+1} & N_n & N_{n-1} \\ N_n & N_{n-1} & N_{n-2} \end{vmatrix} &= -2^{n-1}, \\ \begin{vmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{vmatrix} &= -1315 \times 2^{n-2}, \\ \begin{vmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{vmatrix} &= -9 \times 2^{n-2}, \end{aligned}$$

respectively.

4 SOME IDENTITIES

In this section, we obtain some identities of reverse 3-primes, reverse Lucas 3-primes and reverse modified 3-primes numbers. First, we can give a few basic relations between $\{N_n\}$ and $\{S_n\}$.

Lemma 4.1. *The following equalities are true:*

$$\begin{aligned} 2630N_n &= 81S_{n+4} - 541S_{n+3} + 503S_{n+2}, \\ 1315N_n &= -68S_{n+3} + 373S_{n+2} + 81 \times S_{n+1}, \\ 1315N_n &= 33S_{n+2} - 123S_{n+1} - 136S_n, \\ 1315N_n &= 42S_{n+1} - 37S_n + 66S_{n-1}, \\ 1315N_n &= 173S_n + 192S_{n-1} + 84S_{n-2}, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} 8S_n &= 75N_{n+4} - 397N_{n+3} - 127N_{n+2}, \\ 4S_n &= -11N_{n+3} + 49N_{n+2} + 75N_{n+1}, \\ 2S_n &= -3N_{n+2} + 21N_{n+1} - 11 \times N_n, \\ S_n &= 3N_{n+1} - 10N_n - 3N_{n-1}, \\ S_n &= 5N_n + 6N_{n-1} + 6N_{n-2}. \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (4.1). To show (4.1), writing

$$N_n = a \times S_{n+4} + b \times S_{n+3} + c \times S_{n+2}$$

and solving the system of equations

$$\begin{array}{rcl} N_0 & = & a \times S_4 + b \times S_3 + c \times S_2 \\ N_1 & = & a \times S_5 + b \times S_4 + c \times S_3 \\ N_2 & = & a \times S_6 + b \times S_5 + c \times S_4 \end{array}$$

we find that $a = \frac{81}{2630}, b = -\frac{541}{2630}, c = \frac{503}{2630}$. The other equalities can be proved similarly. Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{N_n\}$ and $\{U_n\}$.

Lemma 4.2. *The following equalities are true:*

$$\begin{array}{rcl} 18N_n & = & -7U_{n+4} + 37U_{n+3} + 13U_{n+2}, \\ 9N_n & = & U_{n+3} - 4U_{n+2} - 7U_{n+1}, \\ 9N_n & = & U_{n+2} - 4U_{n+1} + 2U_n, \\ 9N_n & = & U_{n+1} + 5U_n + 2 \times U_{n-1}, \\ 9N_n & = & 10U_n + 5U_{n-1} + 2U_{n-2}, \end{array}$$

and

$$\begin{array}{rcl} 8U_n & = & -5N_{n+4} + 35N_{n+3} - 39N_{n+2}, \\ 4U_n & = & 5N_{n+3} - 27N_{n+2} - 5N_{n+1}, \\ 2U_n & = & -N_{n+2} + 5N_{n+1} + 5N_n, \\ U_n & = & N_n - N_{n-1}. \end{array}$$

We give a few basic relations between $\{S_n\}$ and $\{U_n\}$.

Lemma 4.3. *The following equalities are true:*

$$\begin{array}{rcl} 36S_n & = & 113U_{n+4} - 551U_{n+3} - 449U_{n+2}, \\ 18S_n & = & 7U_{n+3} - 55U_{n+2} + 113U_{n+1}, \\ 9S_n & = & -10 \times U_{n+2} + 67U_{n+1} + 7U_n, \\ 9S_n & = & 17U_{n+1} - 23U_n - 20U_{n-1}, \\ 9S_n & = & 62U_n + 31U_{n-1} + 34U_{n-2}, \end{array}$$

and

$$\begin{array}{rcl} 5260U_n & = & -341S_{n+4} + 1271S_{n+3} + 3597S_{n+2}, \\ 2630U_n & = & -217S_{n+3} + 1287S_{n+2} - 341S_{n+1}, \\ 1315U_n & = & 101S_{n+2} - 496S_{n+1} - 217S_n, \\ 1315U_n & = & 9S_{n+1} + 86S_n + 202S_{n-1}, \\ 1315U_n & = & 131S_n + 229S_{n-1} + 18S_{n-2}. \end{array}$$

We now present a few special identities for the reverse modified 3-primes sequence $\{U_n\}$.

Theorem 4.4. *(Catalan's identity) For all integers n and m, the following identity holds*

$$\begin{aligned} U_{n+m}U_{n-m}-U_n^2 &= (N_{n+m}-N_{n+m-1})(N_{n-m}-N_{n-m-1})-(N_n-N_{n-1})^2 \\ &= (N_n(N_m-N_{m+1})+N_{n-1}(-N_m+N_{m-2})+N_{n-2}(-N_m+N_{m-1})) \\ &\quad (N_n(N_{-m}-N_{1-m})+N_{n-1}(-N_{-m}+N_{-m-2})+N_{n-2}(-N_{-m}+N_{-m-1})) \\ &\quad -(N_n-N_{n-1})^2 \end{aligned}$$

Proof. We use the identity

$$U_n = N_n - N_{n-1}$$

and the identity (6.6).
Note that for $m = 1$ in Catalan's identity, we get the Cassini identity for the reverse modified 3-primes sequnce

Corollary 4.5. *(Cassini's identity) For all integers numbers n and m, the following identity holds*

$$U_{n+1}U_{n-1}-U_n^2 = (N_{n+1}-N_n)(N_{n-1}-N_{n-2})-(N_n-N_{n-1})^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using $U_n = N_n - N_{n-1}$.The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of reverse modified 3-primes sequence $\{U_n\}$.

Theorem 4.6. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$U_{m+1}U_n - U_mU_{n+1} = (N_{m+1}-N_m)(N_n-N_{n-1})-(N_m-N_{m-1})(N_{n+1}-N_n).$$

(b) *(Gelin-Cesàro's identity)*

$$U_{n+2}U_{n+1}U_{n-1}U_{n-2}-U_n^4 = (N_{n+2}-N_{n+1})(N_{n+1}-N_n)(N_{n-1}-N_{n-2})(N_{n-2}-N_{n-3})-(N_n-N_{n-1})^4.$$

(c) *(Melham's identity)*

$$U_{n+1}U_{n+2}U_{n+6}-U_{n+3}^3 = (N_{n+1}-N_n)(N_{n+2}-N_{n+1})(N_{n+6}-N_{n+5})-(N_{n+3}-N_{n+2})^3.$$

Proof. Use the identity $U_n = N_n - N_{n-1}$.

5 SUM FORMULAS

5.1 Sums of Terms with Positive Subscripts

The following proposition presents some formulas of generalized reverse 3-primes numbers with positive subscripts.

Proposition 5.1. *If $r = 5, s = 3, t = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} V_k = \frac{1}{9}(V_{n+3} - 4V_{n+2} - 7V_{n+1} - V_2 + 4V_1 + 7V_0)$.

(b) $\sum_{k=0}^{n} V_{2k} = \frac{1}{45}(-2V_{2n+2} + 17V_{2n+1} + 14V_{2n} + 2V_2 - 17V_1 + 31V_0)$.

(c) $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{45}(7V_{2n+2} + 8V_{2n+1} - 4V_{2n} - 7V_2 + 37V_1 + 4V_0)$.

Proof. Take $r = 5, s = 3, t = 2$ in Theorem 2.1 in [19] (or take $x = 1, r = 5, s = 3, t = 2$ in Theorem 2.1 in [20]).

From the last proposition, we have the following corollary which gives sum formulas of reverse 3-primes numbers (take $V_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 5$).

Corollary 5.1. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} N_k = \frac{1}{9}(N_{n+3} - 4N_{n+2} - 7N_{n+1} - 1)$.

(b) $\sum_{k=0}^{n} N_{2k} = \frac{1}{45}(-2N_{2n+2} + 17N_{2n+1} + 14N_{2n} - 7)$.

(c) $\sum_{k=0}^{n} N_{2k+1} = \frac{1}{45}(7N_{2n+2} + 8N_{2n+1} - 4N_{2n} + 2)$.

Taking $V_n = S_n$ with $S_0 = 3, S_1 = 5, S_2 = 31$ in the last proposition, we have the following corollary which presents sum formulas of reverse Lucas 3-primes numbers.

Corollary 5.2. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} S_k = \frac{1}{9}(S_{n+3} - 4S_{n+2} - 7S_{n+1} + 10)$.

(b) $\sum_{k=0}^{n} S_{2k} = \frac{1}{45}(-2S_{2n+2} + 17S_{2n+1} + 14S_{2n} + 70)$.

(c) $\sum_{k=0}^{n} S_{2k+1} = \frac{1}{45}(7S_{2n+2} + 8S_{2n+1} - 4S_{2n} - 20)$.

From the last proposition, we have the following corollary which gives sum formulas of reverse modified 3-primes numbers (take $V_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = 4$).

Corollary 5.3. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} U_k = \frac{1}{9}(U_{n+3} - 4U_{n+2} - 7U_{n+1})$.

(b) $\sum_{k=0}^{n} U_{2k} = \frac{1}{45}(-2U_{2n+2} + 17U_{2n+1} + 14U_{2n} - 9)$.

(c) $\sum_{k=0}^{n} U_{2k+1} = \frac{1}{45}(7U_{2n+2} + 8U_{2n+1} - 4U_{2n} + 9)$.

The following proposition presents some formulas of generalized reverse 3-primes numbers with positive subscripts.

Proposition 5.2. *If $r = 5, s = 3, t = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} (-1)^k V_k = \frac{1}{5}((-1)^n (V_{n+3} - 6V_{n+2} + 3V_{n+1}) + V_2 - 6V_1 + 3V_0)$.

(b) $\sum_{k=0}^{n} (-1)^k V_{2k} = \frac{1}{25}((-1)^n (4V_{2n+2} - 17V_{2n+1} - 6V_{2n}) - 4V_2 + 17V_1 + 31V_0)$.

(c) $\sum_{k=0}^{n} (-1)^k V_{2k+1} = \frac{1}{25}((-1)^n (3V_{2n+2} + 6V_{2n+1} + 8V_{2n}) - 3V_2 + 19V_1 - 8V_0)$.

Proof. Take $x = -1, r = 5, s = 3, t = 2$ in Theorem 2.1 in [20].

From the last proposition, we have the following corollary which gives sum formulas of reverse 3-primes numbers (take $V_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 5$).

Corollary 5.4. *For $n \geq 0$, reverse 3-primes numbers have the following properties.*

(a) $\sum_{k=0}^{n} (-1)^k N_k = \frac{1}{5}((-1)^n (N_{n+3} - 6N_{n+2} + 3N_{n+1}) - 1)$.

(b) $\sum_{k=0}^{n} (-1)^k N_{2k} = \frac{1}{25}((-1)^n (4N_{2n+2} - 17N_{2n+1} - 6N_{2n}) - 3)$.

(c) $\sum_{k=0}^{n} (-1)^k N_{2k+1} = \frac{1}{25}((-1)^n (3N_{2n+2} + 6N_{2n+1} + 8N_{2n}) + 4)$.

Taking $V_n = S_n$ with $S_0 = 3, S_1 = 5, S_2 = 31$ in the last proposition, we have the following corollary which presents sum formulas of reverse Lucas 3-primes numbers.

Corollary 5.5. *For $n \geq 0$, reverse Lucas 3-primes numbers have the following properties.*

(a) $\sum_{k=0}^{n} (-1)^k S_k = \frac{1}{5}((-1)^n (S_{n+3} - 6S_{n+2} + 3S_{n+1}) + 10)$.

(b) $\sum_{k=0}^{n} (-1)^k S_{2k} = \frac{1}{25}((-1)^n (4S_{2n+2} - 17S_{2n+1} - 6S_{2n}) + 54)$.

(c) $\sum_{k=0}^{n} (-1)^k S_{2k+1} = \frac{1}{25}((-1)^n (3S_{2n+2} + 6S_{2n+1} + 8S_{2n}) - 22)$.

From the last proposition, we have the following corollary which gives sum formulas of reverse modified 3-primes numbers (take $V_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = 4$).

Corollary 5.6. *For $n \geq 0$, reverse modified 3-primes numbers have the following properties.*

(a) $\sum_{k=0}^{n} (-1)^k U_k = \frac{1}{5}((-1)^n (U_{n+3} - 6U_{n+2} + 3U_{n+1}) - 2)$.

(b) $\sum_{k=0}^{n} (-1)^k U_{2k} = \frac{1}{25}((-1)^n (4U_{2n+2} - 17U_{2n+1} - 6U_{2n}) + 1)$.

(c) $\sum_{k=0}^{n} (-1)^k U_{2k+1} = \frac{1}{25}((-1)^n (3U_{2n+2} + 6U_{2n+1} + 8U_{2n}) + 7)$.

5.2 Sums of Terms with Negative Subscripts

The following proposition presents some formulas of generalized reverse 3-primes numbers with negative subscripts.

Proposition 5.3. *If* $r = 5, s = 3, t = 2$ *then for* $n \geq 1$ *we have the following formulas:*

(a) $\sum_{k=1}^{n} V_{-k} = \frac{1}{9}(-10V_{-n-1} - 5V_{-n-2} - 2V_{-n-3} + V_2 - 4V_1 - 7V_0)$.

(b) $\sum_{k=1}^{n} V_{-2k} = \frac{1}{45}(-7V_{-2n+1} + 37V_{-2n} + 4V_{-2n-1} - 2V_2 + 17V_1 - 31V_0)$.

(c) $\sum_{k=1}^{n} V_{-2k+1} = \frac{1}{45}(2V_{-2n+1} - 17V_{-2n} - 14V_{-2n-1} + 7V_2 - 37V_1 - 4V_0)$.

Proof. Take $r = 5, s = 3, t = 2$ in Theorem 3.1 in [19] or (or take $x = 1, r = 5, s = 3, t = 2$ in Theorem 3.1 in [20]).

From the last proposition, we have the following corollary which gives sum formulas of reverse 3-primes numbers (take $V_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 5$).

Corollary 5.7. *For* $n \geq 1$, *reverse 3-primes numbers have the following properties.*

(a) $\sum_{k=1}^{n} N_{-k} = \frac{1}{9}(-10N_{-n-1} - 5N_{-n-2} - 2N_{-n-3} + 1)$.

(b) $\sum_{k=1}^{n} N_{-2k} = \frac{1}{45}(-7N_{-2n+1} + 37N_{-2n} + 4N_{-2n-1} + 7)$.

(c) $\sum_{k=1}^{n} N_{-2k+1} = \frac{1}{45}(2N_{-2n+1} - 17N_{-2n} - 14N_{-2n-1} - 2)$.

Taking $V_n = S_n$ with $S_0 = 3, S_1 = 5, S_2 = 31$ in the last proposition, we have the following corollary which presents sum formulas of reverse Lucas 3-primes numbers.

Corollary 5.8. *For* $n \geq 1$, *reverse Lucas 3-primes numbers have the following properties.*

(a) $\sum_{k=1}^{n} S_{-k} = \frac{1}{9}(-10S_{-n-1} - 5S_{-n-2} - 2S_{-n-3} - 10)$.

(b) $\sum_{k=1}^{n} S_{-2k} = \frac{1}{45}(-7S_{-2n+1} + 37S_{-2n} + 4S_{-2n-1} - 70)$.

(c) $\sum_{k=1}^{n} S_{-2k+1} = \frac{1}{45}(2S_{-2n+1} - 17S_{-2n} - 14S_{-2n-1} + 20)$.

From the last proposition, we have the following corollary which gives sum formulas of reverse modified 3-primes numbers (take $V_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = 4$).

Corollary 5.9. *For* $n \geq 1$, *reverse modified 3-primes numbers have the following properties.*

(a) $\sum_{k=1}^{n} U_{-k} = \frac{1}{9}(-10U_{-n-1} - 5U_{-n-2} - 2U_{-n-3})$.

(b) $\sum_{k=1}^{n} U_{-2k} = \frac{1}{45}(-7U_{-2n+1} + 37U_{-2n} + 4U_{-2n-1} + 9)$.

(c) $\sum_{k=1}^{n} U_{-2k+1} = \frac{1}{45}(2U_{-2n+1} - 17U_{-2n} - 14U_{-2n-1} - 9)$.

The following proposition presents some formulas of generalized reverse 3-primes numbers with negative subscripts.

Proposition 5.4. *If* $r = 5, s = 3, t = 2$ *then for* $n \geq 1$ *we have the following formulas:*

(a) $\sum_{k=1}^{n} (-1)^k V_{-k} = \frac{1}{5}((-1)^n (4V_{-n-1} + V_{-n-2} + 2V_{-n-3}) - V_2 + 6V_1 - 3V_0)$.

(b) $\sum_{k=1}^{n} (-1)^k V_{-2k} = \frac{1}{25}((-1)^n (-3V_{-2n+1} + 19V_{-2n} - 8V_{-2n-1}) + 4V_2 - 17V_1 - 31V_0)$.

(c) $\sum_{k=1}^{n} (-1)^k V_{-2k+1} = \frac{1}{25}((-1)^n (4V_{-2n+1} - 17V_{-2n} - 6V_{-2n-1}) + 3V_2 - 19V_1 + 8V_0)$.

Proof. Take $x = -1, r = 5, s = 3, t = 2$ in Theorem 3.1 in [20].

From the last proposition, we have the following corollary which gives sum formulas of reverse 3-primes numbers (take $V_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 5$).

Corollary 5.10. *For $n \geq 1$, reverse 3-primes numbers have the following properties.*

(a) $\sum_{k=1}^{n}(-1)^k N_{-k} = \frac{1}{5}((-1)^n (4N_{-n-1} + N_{-n-2} + 2N_{-n-3}) + 1)$.

(b) $\sum_{k=1}^{n}(-1)^k N_{-2k} = \frac{1}{25}((-1)^n (-3N_{-2n+1} + 19N_{-2n} - 8N_{-2n-1}) + 3)$.

(c) $\sum_{k=1}^{n}(-1)^k N_{-2k+1} = \frac{1}{25}((-1)^n (4N_{-2n+1} - 17N_{-2n} - 6N_{-2n-1}) - 4)$.

Taking $V_n = S_n$ with $S_0 = 3, S_1 = 5, S_2 = 31$ in the last proposition, we have the following corollary which gives sum formulas of reverse Lucas 3-primes numbers.

Corollary 5.11. *For $n \geq 1$, reverse Lucas 3-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^k S_{-k} = \frac{1}{5}((-1)^n (4S_{-n-1} + S_{-n-2} + 2S_{-n-3}) - 10)$.

(b) $\sum_{k=1}^{n}(-1)^k S_{-2k} = \frac{1}{25}((-1)^n (-3S_{-2n+1} + 19S_{-2n} - 8S_{-2n-1}) - 54)$.

(c) $\sum_{k=1}^{n}(-1)^k S_{-2k+1} = \frac{1}{25}((-1)^n (4S_{-2n+1} - 17S_{-2n} - 6S_{-2n-1}) + 22)$.

From the last proposition, we have the following corollary which gives sum formulas of reverse modified 3-primes numbers (take $V_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = 4$).

Corollary 5.12. *For $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}(-1)^k U_{-k} = \frac{1}{5}((-1)^n (4U_{-n-1} + U_{-n-2} + 2U_{-n-3}) + 2)$.

(b) $\sum_{k=1}^{n}(-1)^k U_{-2k} = \frac{1}{25}((-1)^n (-3U_{-2n+1} + 19U_{-2n} - 8U_{-2n-1}) - 1)$.

(c) $\sum_{k=1}^{n}(-1)^k U_{-2k+1} = \frac{1}{25}((-1)^n (4U_{-2n+1} - 17U_{-2n} - 6U_{-2n-1}) - 7)$.

5.3 Sums of Squares of Terms with Positive Subscripts

The following proposition presents some formulas of generalized reverse 3-primes numbers with positive subscripts.

Proposition 5.5. *If $r = 5, s = 3, t = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} V_k^2 = \frac{1}{225}(-8V_{n+3}^2 - 263V_{n+2}^2 - 257V_{n+1}^2 + 91V_{n+3}V_{n+2} + 22V_{n+3}V_{n+1} - 52V_{n+2}V_{n+1} + 8V_2^2 + 263V_1^2 + 257V_0^2 - 91V_2V_1 - 22V_2V_0 + 52V_1V_0)$.

(b) $\sum_{k=0}^{n} V_{k+1}V_k = \frac{1}{450}(11V_{n+3}^2 + 221V_{n+2}^2 + 44V_{n+1}^2 - 97V_{n+3}V_{n+2} + 26V_{n+3}V_{n+1} - 266V_{n+2}V_{n+1} - 11V_2^2 - 221V_1^2 - 44V_0^2 + 97V_2V_1 - 26V_2V_0 + 266V_1V_0)$.

(c) $\sum_{k=0}^{n} V_{k+2}V_k = \frac{1}{450}(29V_{n+3}^2 - 31V_{n+2}^2 + 116V_{n+1}^2 - 133V_{n+3}V_{n+2} - 136V_{n+3}V_{n+1} + 76V_{n+2}V_{n+1} - 29V_2^2 + 31V_1^2 - 116V_0^2 + 133V_2V_1 + 136V_2V_0 - 76V_1V_0)$.

Proof. Take $x = 1, r = 5, s = 3, t = 2$ in Theorem 4.1 in [21], see also [22].

From the last proposition, we have the following Corollary which gives sum formulas of reverse 3-primes numbers (take $V_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 5$).

Corollary 5.13. *For $n \geq 0$, reverse 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} N_k^2 = \frac{1}{225}(-8N_{n+3}^2 - 263N_{n+2}^2 - 257N_{n+1}^2 + 91N_{n+3}N_{n+2} + 22N_{n+3}N_{n+1} - 52N_{n+2}N_{n+1} + 8)$.

(b) $\sum_{k=0}^{n} N_{k+1}N_k = \frac{1}{450}(11N_{n+3}^2 + 221N_{n+2}^2 + 44N_{n+1}^2 - 97N_{n+3}N_{n+2} + 26N_{n+3}N_{n+1} - 266N_{n+2}N_{n+1} - 11)$.

(c) $\sum_{k=0}^{n} N_{k+2}N_k = \frac{1}{450}(29N_{n+3}^2 - 31N_{n+2}^2 + 116N_{n+1}^2 - 133N_{n+3}N_{n+2} - 136N_{n+3}N_{n+1} + 76N_{n+2}N_{n+1} - 29)$.

Taking $V_n = S_n$ with $S_0 = 3, S_1 = 5, S_2 = 31$ in the last Proposition, we have the following Corollary which presents sum formulas of reverse Lucas 3-primes numbers.

Corollary 5.14. *For $n \geq 0$, reverse Lucas 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} S_k^2 = \frac{1}{225}(-8S_{n+3}^2 - 263S_{n+2}^2 - 257S_{n+1}^2 + 91S_{n+3}S_{n+2} + 22S_{n+3}S_{n+1} - 52S_{n+2}S_{n+1} + 1205).$

(b) $\sum_{k=0}^{n} S_{k+1}S_k = \frac{1}{450}(11S_{n+3}^2 + 221S_{n+2}^2 + 44S_{n+1}^2 - 97S_{n+3}S_{n+2} + 26S_{n+3}S_{n+1} - 266S_{n+2}S_{n+1} + 115).$

(c) $\sum_{k=0}^{n} S_{k+2}S_k = \frac{1}{450}(29S_{n+3}^2 - 31S_{n+2}^2 + 116S_{n+1}^2 - 133S_{n+3}S_{n+2} - 136S_{n+3}S_{n+1} + 76S_{n+2}S_{n+1} + 3985).$

From the last proposition, we have the following corollary which gives sum formulas of reverse modified 3-primes numbers (take $V_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = 4$).

Corollary 5.15. *For $n \geq 0$, reverse modified 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} U_k^2 = \frac{1}{225}(-8U_{n+3}^2 - 263U_{n+2}^2 - 257U_{n+1}^2 + 91U_{n+3}U_{n+2} + 22U_{n+3}U_{n+1} - 52U_{n+2}U_{n+1} + 27).$

(b) $\sum_{k=0}^{n} U_{k+1}U_k = \frac{1}{450}(11U_{n+3}^2 + 221U_{n+2}^2 + 44U_{n+1}^2 - 97U_{n+3}U_{n+2} + 26U_{n+3}U_{n+1} - 266U_{n+2}U_{n+1} - 9).$

(c) $\sum_{k=0}^{n} U_{k+2}U_k = \frac{1}{450}(29U_{n+3}^2 - 31U_{n+2}^2 + 116U_{n+1}^2 - 133U_{n+3}U_{n+2} - 136U_{n+3}U_{n+1} + 76U_{n+2}U_{n+1} + 99).$

The following proposition presents some formulas of generalized reverse 3-primes numbers with positive subscripts.

Proposition 5.6. *If $r = 5, s = 3, t = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k V_k^2 = \frac{1}{150}((-1)^n(-V_{n+3}^2 - 79V_{n+2}^2 + 154V_{n+1}^2 + 21V_{n+3}V_{n+2} - 22V_{n+3}V_{n+1} + 56V_{n+2}V_{n+1}) - V_2^2 - 79V_1^2 + 154V_0^2 + 56V_1V_0 - 22V_2V_0 + 21V_2V_1).$

(b) $\sum_{k=0}^{n}(-1)^k V_{k+1}V_k = \frac{1}{300}((-1)^n(11V_{n+3}^2 + 119V_{n+2}^2 - 44V_{n+1}^2 - 81V_{n+3}V_{n+2} - 58V_{n+3}V_{n+1} + 284V_{n+2}V_{n+1}) + 11V_2^2 + 119V_1^2 - 44V_0^2 - 81V_2V_1 - 58V_2V_0 + 284V_1V_0).$

(c) $\sum_{k=0}^{n}(-1)^k V_{k+2}V_k = \frac{1}{100}((-1)^n(9V_{n+3}^2 - 39V_{n+2}^2 - 36V_{n+1}^2 - 39V_{n+3}V_{n+2} - 2V_{n+3}V_{n+1} - 104V_{n+2}V_{n+1}) + 9V_2^2 - 39V_1^2 - 36V_0^2 - 39V_2V_1 - 2V_2V_0 - 104V_1V_0).$

Proof. Take $x = -1, r = 5, s = 3, t = 2$ in Theorem 4.29 in [21]

From the above proposition, we have the following corollary which gives sum formulas of reverse 3-primes numbers (take $V_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 5$).

Corollary 5.16. *For $n \geq 0$, reverse 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k N_k^2 = \frac{1}{150}((-1)^n(-N_{n+3}^2 - 79N_{n+2}^2 + 154N_{n+1}^2 + 21N_{n+3}N_{n+2} - 22N_{n+3}N_{n+1} + 56N_{n+2}N_{n+1}) + 1).$

(b) $\sum_{k=0}^{n}(-1)^k N_{k+1}N_k = \frac{1}{300}((-1)^n(11N_{n+3}^2 + 119N_{n+2}^2 - 44N_{n+1}^2 - 81N_{n+3}N_{n+2} - 58N_{n+3}N_{n+1} + 284N_{n+2}N_{n+1}) - 11).$

(c) $\sum_{k=0}^{n}(-1)^k N_{k+2}N_k = \frac{1}{100}((-1)^n(9N_{n+3}^2 - 39N_{n+2}^2 - 36N_{n+1}^2 - 39N_{n+3}N_{n+2} - 2N_{n+3}N_{n+1} - 104N_{n+2}N_{n+1}) - 9).$

Taking $N_n = S_n$ with $S_0 = 3, S_1 = 5, S_2 = 31$ in the above proposition, we have the following corollary which presents sum formulas of reverse Lucas 3-primes numbers.

Corollary 5.17. *For $n \geq 0$, reverse Lucas 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^{k}S_{k}^{2}=\frac{1}{150}((-1)^{n}(-S_{n+3}^{2}-79S_{n+2}^{2}+154S_{n+1}^{2}+21S_{n+3}S_{n+2}-22S_{n+3}S_{n+1}+56S_{n+2}S_{n+1})+499)$.

(b) $\sum_{k=0}^{n}(-1)^{k}S_{k+1}S_{k}=\frac{1}{300}((-1)^{n}(11S_{n+3}^{2}+119S_{n+2}^{2}-44S_{n+1}^{2}-81S_{n+3}S_{n+2}-58S_{n+3}S_{n+1}+284 S_{n+2}S_{n+1})-539)$.

(c) $\sum_{k=0}^{n}(-1)^{k}S_{k+2}S_{k}=\frac{1}{100}((-1)^{n}(9S_{n+3}^{2}-39S_{n+2}^{2}-36S_{n+1}^{2}-39S_{n+3}S_{n+2}-2S_{n+3}S_{n+1}-104 S_{n+2}S_{n+1})-441)$.

From the above proposition, we have the following corollary which gives sum formulas of reverse modified 3-primes numbers (take $S_n=U_n$ with $U_0=0,U_1=1,U_2=4$).

Corollary 5.18. *For $n\geq 0$, reverse modified 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^{k}U_{k}^{2}=\frac{1}{150}((-1)^{n}(-U_{n+3}^{2}-79U_{n+2}^{2}+154U_{n+1}^{2}+21U_{n+3}U_{n+2}-22U_{n+3}U_{n+1}+56U_{n+2}U_{n+1})-11)$.

(b) $\sum_{k=0}^{n}(-1)^{k}U_{k+1}U_{k}=\frac{1}{300}((-1)^{n}(11U_{n+3}^{2}+119U_{n+2}^{2}-44U_{n+1}^{2}-81U_{n+3}U_{n+2}-58U_{n+3}U_{n+1}+284U_{n+2}U_{n+1})-29)$.

(c) $\sum_{k=0}^{n}(-1)^{k}U_{k+2}U_{k}=\frac{1}{100}((-1)^{n}(9U_{n+3}^{2}-39U_{n+2}^{2}-36U_{n+1}^{2}-39U_{n+3}U_{n+2}-2U_{n+3}U_{n+1}-104 U_{n+2}U_{n+1})-51)$.

6 MATRICES RELATED WITH GENERALIZED REVERSE 3-PRIMES NUMBERS

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix}=\begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n}\begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{6.1}$$

For matrix formulation (6.1), see [23]. In fact, Kalman gave the formula in the following form

$$\begin{pmatrix} W_{n} \\ W_{n+1} \\ W_{n+2} \end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^{n}\begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

We define the square matrix A of order 3 as:

$$A=\begin{pmatrix} 5 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that

$$\det A=\begin{vmatrix} 5 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}=2.$$

From (1.9) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix}=\begin{pmatrix} 5 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix} \tag{6.2}$$

and from (6.1) (or using (6.2) and induction) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix}=\begin{pmatrix} 5 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n}\begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V = N$ in (6.2) we have

$$\begin{pmatrix} N_{n+2} \\ N_{n+1} \\ N_n \end{pmatrix} = \begin{pmatrix} 5 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} N_{n+1} \\ N_n \\ N_{n-1} \end{pmatrix}. \tag{6.3}$$

We also define

$$B_n = \begin{pmatrix} N_{n+1} & 3N_n + 2N_{n-1} & 2N_n \\ N_n & 3N_{n-1} + 2N_{n-2} & 2N_{n-1} \\ N_{n-1} & 3N_{n-2} + 2N_{n-3} & 2N_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & 3V_n + 2V_{n-1} & 2V_n \\ V_n & 3V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & 3V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix}$$

Theorem 6.1. *For all integer $m, n \geq 0,$ we have*

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n.$

Proof.

(a) By expanding the vectors on the both sides of (6.3) to 3-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1} B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} 5 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_n & 3V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & 3V_{n-2} + 2V_{n-3} & 2V_{n-2} \\ V_{n-2} & 3V_{n-3} + 2V_{n-4} & 2V_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} V_{n+1} & 3V_n + 2V_{n-1} & 2V_n \\ V_n & 3V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & 3V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix} = C_n. \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1} C_1$. Now

$$C_{n+m} = A^{n+m-1} C_1 = A^{n-1} A^m C_1 = A^{n-1} C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

Some properties of matrix A^n can be given as

$$A^n = 5A^{n-1} + 3A^{n-2} + 2A^{n-3}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = 2^n$$

for all integer m and n.

Theorem 6.2. *For $m, n \geq 0$ we have*

$$V_{n+m} = V_n N_{m+1} + V_{n-1}(3N_m + 2N_{m-1}) + 2V_{n-2}N_m \quad (6.4)$$
$$= V_n N_{m+1} + (3V_{n-1} + 2V_{n-2})\, N_m + 2V_{n-1}N_{m-1} \quad (6.5)$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m.

$$\begin{pmatrix} V_{n+m+1} & 3V_{n+m} + 2V_{n+m-1} & 2V_{n+m} \\ V_{n+m} & 3V_{n+m-1} + 2V_{n+m-2} & 2V_{n+m-1} \\ V_{n+m-1} & 3V_{n+m-2} + 2V_{n+m-3} & 2V_{n+m-2} \end{pmatrix}$$
$$= \begin{pmatrix} V_{n+1} & 3V_n + 2V_{n-1} & 2V_n \\ V_n & 3V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & 3V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix} \begin{pmatrix} N_{m+1} & 3N_m + 2N_{m-1} & 2N_m \\ N_m & 3N_{m-1} + 2N_{m-2} & 2N_{m-1} \\ N_{m-1} & 3N_{m-2} + 2N_{m-3} & 2N_{m-2} \end{pmatrix}$$
$$= \begin{pmatrix} 2V_n N_{m-1} + N_{m+1}V_{n+1} + N_m(3V_n + 2V_{n-1}) & D_1 & D_4 \\ V_n N_{m+1} + N_m(3V_{n-1} + 2V_{n-2}) + 2N_{m-1}V_{n-1} & D_2 & D_5 \\ N_m(3V_{n-2} + 2V_{n-3}) + N_{m+1}V_{n-1} + 2N_{m-1}V_{n-2} & D_3 & D_6 \end{pmatrix}$$

where

$$D_1 = V_{n+1}(3N_m + 2N_{m-1}) + 2V_n(3N_{m-2} + 2N_{m-3}) + (3N_{m-1} + 2N_{m-2})(3V_n + 2V_{n-1}),$$
$$D_2 = V_n(3N_m + 2N_{m-1}) + 2V_{n-1}(3N_{m-2} + 2N_{m-3}) + (3N_{m-1} + 2N_{m-2})(3V_{n-1} + 2V_{n-2}),$$
$$D_3 = V_{n-1}(3N_m + 2N_{m-1}) + 2V_{n-2}(3N_{m-2} + 2N_{m-3}) + (3N_{m-1} + 2N_{m-2})(3V_{n-2} + 2V_{n-3}),$$
$$D_4 = 2N_m V_{n+1} + 4V_n N_{m-2} + 2N_{m-1}(3V_n + 2V_{n-1}),$$
$$D_5 = 2N_m V_n + 4N_{m-2}V_{n-1} + 2N_{m-1}(3V_{n-1} + 2V_{n-2}),$$
$$D_6 = 2N_m V_{n-1} + 4N_{m-2}V_{n-2} + 2N_{m-1}(3V_{n-2} + 2V_{n-3}).$$

From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof.

Remark 6.1. By induction, it can be proved that for all integers $m, n \leq 0$, (6.4) holds. So for all integers m, n, (6.4) is true.

Corollary 6.3. *For all integers m, n, we have*

$$N_{n+m} = N_n N_{m+1} + N_{n-1}(3N_m + 2N_{m-1}) + 2N_{n-2}N_m, \quad (6.6)$$
$$S_{n+m} = S_n N_{m+1} + S_{n-1}(3N_m + 2N_{m-1}) + 2S_{n-2}N_m, \quad (6.7)$$
$$U_{n+m} = U_n N_{m+1} + U_{n-1}(3N_m + 2N_{m-1}) + 2U_{n-2}N_m. \quad (6.8)$$

7 CONCLUSIONS

Sequences of integer number such as Fibonacci, Lucas, Pell, Jacobsthal are the most well-known second order recurrence sequences. The Fibonacci numbers are perhaps most famous for appearing in the rabbit breeding problem, introduced by Leonardo de Pisa in 1202 in his book called Liber Abaci. The Fibonacci and Lucas sequences are sources of many nice and interesting identities. For rich applications of these second order sequences in science and nature, one can see the citations in [24].

As a third order sequence, we introduce the generalized reverse 3-primes sequence (it's three special cases, namely, reverse 3-primes, reverse Lucas 3-primes and reverse modified 3-primes sequences) and we present Binet's formulas, generating functions, Simson formulas, the summation formulas, some identities and matrices for these sequences.

Third order sequences have many applications. We now present one of them. The ratio of two consecutive Padovan numbers converges to the plastic ratio, α_P (which is given in (7) below), which have many applications to such as architecture, see [25]. Padovan numbers is defined by the third-order recurrence relations

$$P_{n+3} = P_{n+1} + P_n, \quad P_0 = 1, P_1 = 1, P_2 = 1.$$

The characteristic equation associated with Padovan sequence is $x^3 - x - 1 = 0$ with roots α_P, β_P and γ_P in which

$$\alpha_P = \left(\tfrac{1}{2} + \sqrt{\tfrac{23}{108}}\right)^{1/3} + \left(\tfrac{1}{2} - \sqrt{\tfrac{23}{108}}\right)^{1/3} \simeq 1.324717957$$

24(7.1)is called plastic number (or plastic ratio or plastic constant or silver number) and

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \alpha_P.$$

The plastic number is used in art and architecture. Richard Padovan studied on plastic number in Architecture and Mathematics in [26, 27].

We now present some other applications of third order sequences.

- For the applications of Padovan numbers and Tribonacci numbers to coding theory see [28] and [29], respectively.
- For the application of Padovan numbers to Gaussian numbers, see [30].
- For the application of Pell-Padovan numbers to quaternions and groups see [31] and [32], respectively.
- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions see [33] and [34], respectively.
- For the application of Tribonacci numbers to special matrices, see [13].

COMPETING INTERESTS

Author has declared that no competing interests exist.

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