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Relativistic Velocity Addition on a Space-Time Diagram

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Abstract

We derive the addition of velocities in special relativity from the Minkowski's space-time diagram. We only need to draw some world lines on the diagram, measure the lengths and divide the two lengths for obtaining the velocity. We also give the theoretical background for this method. This method is so simple that it is worth for undergraduate students to acquire the addition of velocities in special relativity.

Keywords

Velocity Addition, Special Relativity, Space-Time Diagram

1. Introduction

Special relativity is the key issue for understanding modern physics. However, since our daily experiences are much smaller than the speed of light, it is necessary to look for devices which deduce the correct results of special relativity. In the region of Newtonian mechanics, we often use the space-time diagram for analyzing the kinematics of the motion of objects. We conclude that the gradient of the curve on the space-time diagram is considered the velocity of objects. In the region of special relativity, Minkowski's space-time diagrams have been used as an educational tool [1] [2]. They are indispensable tools for students to grab the meaning of special relativity. In fact, the orthogonal and oblique coordinate axes show transparent relations between events in one inertial frame and another inertial frame. It is also useful to use Minkowski's diagrams in momentum space for obtaining the results of special relativity [3]. Especially, they serve as powerful tools for analyzing the collision problems [4] [5] [6].

Recent paper [7] deals with the addition of velocities in terms of Minkowski's diagram in momentum space. By working in momentum space with mass-shell hyperbolas, it clearly shows the addition of velocities in special relativity. In this paper, we try to deal with the same problem in Minkowski's space-time diagrams. Although the usage of Minkowski's space-time diagram has been discussed in many ways, the method shown in this article is seldom seen in texts for undergraduate physics course. This method is quite simple, because we only draw some world lines, measure the lengths and make the divisions of them. It is a certain extension of Newtonian mechanics for obtaining the velocities from space-time diagrams. Therefore, it is worth for undergraduate students to know the relationship between Newtonian mechanics and special relativity.

This paper is organized in the following way. In Section 2, we introduce the space-time diagram and draw world lines of the moving frame and the object on this diagram. After measuring the lengths, we derive the addition of velocities. In Section 3, we show the theoretical background for the method which is introduced in the previous section. Section 4 deals with other examples which illuminate this method. Section 5 is devoted to a conclusion.

2. Space-Time Diagram and Velocities

Let us formulate the problem and fix our notation. Let *K* be a particular inertial frame of reference and *K'* be the moving frame with respect to *K* in the $+x$-direction with speed $V = +0.5c$, where *c* is the speed of light. The observer in *K* measures the moving object in the $+x$-direction with speed $v = +0.8c$. This situation is depicted in **Figure 1**.

The problem is, what is the speed *v'* of the object from the observer in moving frame *K'*. Of course, we observe $v' = v - V = +0.3c$ in Newtonian sense. But it is not correct. On the contrary, we have the following formula of special relativity

$$v' = \frac{v - V}{1 - vV/c^2}, \tag{1}$$

and we obtain $v' = +0.5c$ with the substitution of $v = +0.8c$ and $V = +0.5c$ in Equation (1).

Now, let us introduce the Minkowski's diagram with the space *x*-axis in the horizontal axis, while the time $w = ct$ -axis in the vertical axis, as shown in **Figure 2**.

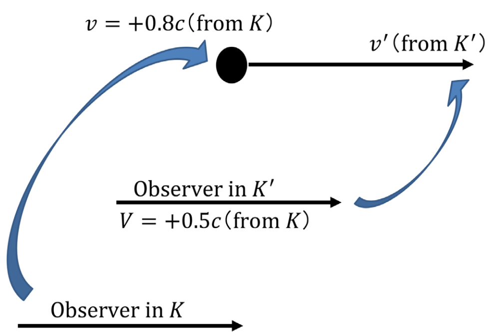

Figure 1. An observer in inertial frame *K* sees another observer in moving frame *K'* in the $+x$-direction with speed $V = +0.5c$ and the object in the same direction with speed $v = +0.8c$. The problem is how does the observer in *K'* detect the speed *v'* of the object.

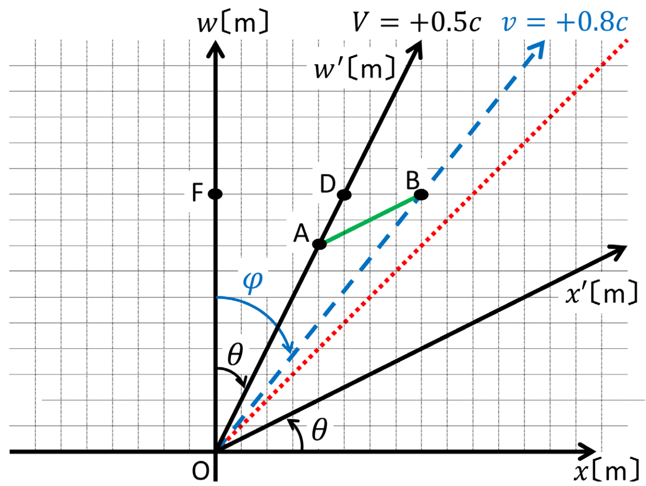

Figure 2. The Minkowski's space-time diagram of the situation in **Figure 1**.

This orthogonal $w-x$ diagram is used by an observer in an inertial frame *K*. An observer in the moving frame *K'* uses the oblique $w'-x'$ axes which are inclined to the angle θ with respect to the $w-x$ axes. These axes are drawn by the solid lines. The object with speed $v=+0.8c$ from the inertial frame *K* is depicted by the dashed line. In **Figure 2**, these velocities *v* and *V* are related to the angles from the vertical *w*-axis as follows,

$$v/c=\frac{FB}{OF}=\tan\varphi=+0.8,\quad V/c=\frac{FD}{OF}=\tan\theta=+0.5. \tag{2}$$

These relationships are the same with which we calculate in terms of Newtonian mechanics. The dotted line shows the world line of light which is moving in the $+x$ -direction with the velocity $+c$.

Now, let us consider the velocity v'/c of which the observer in *K'* evaluates the speed of the object [8]. In **Figure 2**, the line *AB* is parallel to the *x'*-axis which is inclined to θ from the *x*-axis. The velocity *v'* is given by

$$v'/c=\frac{AB}{OA}, \tag{3}$$

which is a certain extension of obtaining velocities from the space-time diagram in Newtonian mechanics. We measure the lengths *OA* and *AB* with a ruler, put these values into Equation (3) and obtain $v'/c=+0.50$. This is coincident with the value which we derive from Equation (1) with $v=+0.8c$ and $V=+0.5c$.

As we see in **Figure 3**, if we extend the line segment *AB* to the point *C*, which is on the world line of light. Hence, the observer in the moving frame *K'* sees light. Since the length *OA* is equal to *AC* in **Figure 3**, then the observer in the moving frame *K'* calculates $v'/c=\frac{AC}{OA}=1$. This is one of the postulates of special relativity, that is, the speed of light is always equal to *c* regardless of the motion of the observer or the source. The equality of the lengths *OA* and *AC* demonstrates this postulate.

Note that we easily see in **Figure 3** that the point *B* is the midpoint of the line

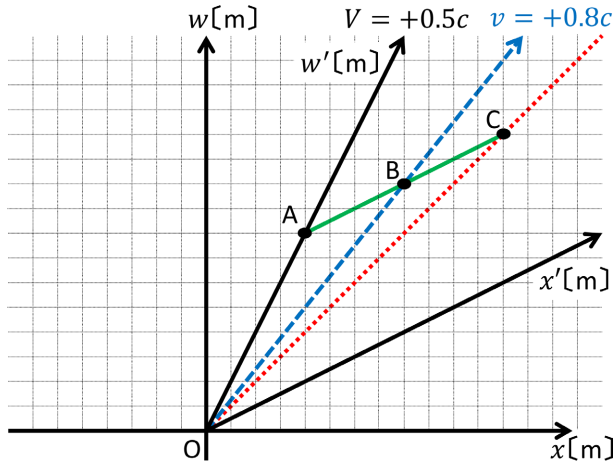

Figure 3. The observer in the frame K' measures the speed of light.

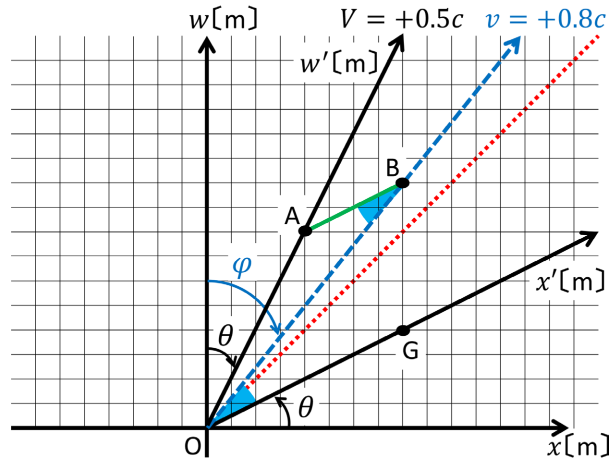

Figure 4. The angle ABO is equal to the angle BOG, because the line segment AB is parallel to the x'-axis.

segment AC. That is why the observer in K' evaluates the speed of the object as $+0.5c$.

3. Theoretical Background

In this section, let us deduce the relativistic velocity addition formula (1) from Minkowski's space-time diagram of **Figure 4**.

When we apply the theorem of sine to the triangle OAB, we obtain

$$\frac{AB}{\sin(\varphi-\theta)} = \frac{OA}{\sin\left(\frac{\pi}{2}-\varphi-\theta\right)}. \tag{4}$$

From this expression, we calculate Equation (3) and use the addition theorem of sine [8]

$$\frac{v'}{c} = \frac{AB}{OA} = \frac{\sin(\varphi-\theta)}{\cos(\varphi+\theta)} = \frac{\sin\varphi\cos\theta-\cos\varphi\sin\theta}{\cos\varphi\cos\theta-\sin\varphi\sin\theta} = \frac{\tan\varphi-\tan\theta}{1-\tan\varphi\tan\theta}. \tag{5}$$

Substituting the relations in Equation (2), we obtain the addition of velocities

$$\frac{v'}{c}=\frac{v/c-V/c}{1-vV/c^2}, \tag{6}$$

which is coincide with Equation (1).

4. Other Examples

Now, let us consider other examples which illuminate the validity of this method. We consider the observer moving in the -*x*-direction with the speed $V=-0.5c$ with respect to the inertial frame *K*. The orthogonal $w-x$ axes in **Figure 5** express the space-time diagram of the inertial frame *K*. In contrast to **Figure 2**, the oblique $w'-x'$ axes are inclined to the opposite direction with the angle θ. In **Figure 5**, the world line of the object with the speed $v=+0.5c$ from the inertial frame *K* is depicted by the dashed line. And the world line of light with the speed $+c$ is also depicted with the dotted line.

The line segment AC is parallel to the *x'*-axis which is inclined to the *x*-axis with the angle θ. In order to obtain the velocities of the object and light from the moving frame *K'*, we measure OA, AB and AC by a ruler. Thus, we just divide these lengths as

$$\frac{v'}{c}=\frac{AB}{OA}=0.8,\quad \frac{v'}{c}=\frac{AC}{OA}=1. \tag{7}$$

The first equation in Equation (7) is also obtained by substituting $v=+0.5c$ and $V=-0.5c$ in Equation (1). The second equation in Equation (7) is realized from the equality of lengths of OA and AC. Moreover, we see in **Figure 5** that the line segment AB is 80% of the line segment AC. This is consistent with the first equation in Equation (7).

5. Conclusion

We present the relativistic addition of velocities on the Minkowski's space-time diagram. It is shown that we draw some world lines, measure the lengths of them

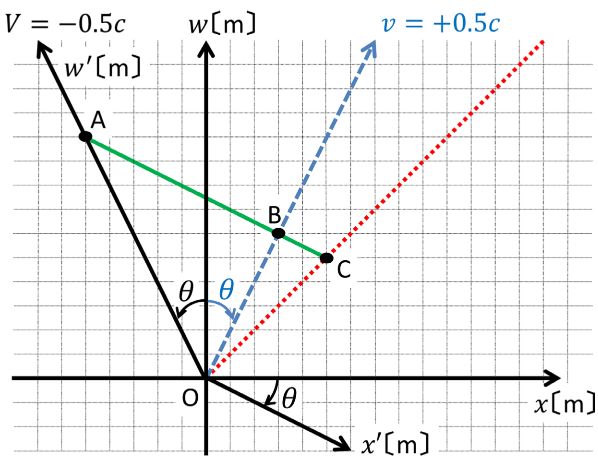

Figure 5. The observer in the frame *K'* is moving -*x*-direction with $V=-0.5c$ with respect to the frame *K*. The observer in the inertial frame *K* sees the object with its velocity $v=+0.5c$ and light with $+c$.

and make the division of these values to obtain the addition of velocities.

In order to obtain the addition of velocities, we also used the momentum space [7], the velocity space [9], a geometric circle [10] and a euclidian space-time [11]. Compared to these methods, the space-time diagram gives a more simple way to obtain them. Besides, this method is a certain extension of Newtonian mechanics.

In the classroom, students should use the graph paper to draw the space-time diagram, like the text [12]. It is easy for students to draw the lines and measure the length on the graph paper.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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