
**Journal of Advances in Mathematics and Computer Science**

**37(1): 22-27, 2022; Article no.JAMCS.84338**
***ISSN: 2456-9968***
*(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)*

# Tempered Distribution Version of the Tumarkin Result

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***Authors' contributions***

*This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.*

***Article Information***

DOI: 10.9734/JAMCS/2022/v37i130428

**Open Peer Review History:**
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/84338

***Original Research Article***

***Received 15 November 2021***
***Accepted 18 January 2022***
***Published 20 January 2022***

## Abstract

We transfer a classical result of Tumarkin, about approximation of $L^p(d\sigma, R)$ functions with certain rational functions, in distributional setting. This is achieved by embedding the functions from classical spaces into the space of tempered distributions.

*Keywords: Boundary values of distributions; distributions; Tumarkin.*

## 1 Introduction

This section is devoted to known notions and results in the literature that we use in the article. Let $z_j, j \in N,$ be a sequence of complex numbers from $\Pi^+ = \{z \in \mathrm{C} | \mathrm{Im} z > 0\}$. In addition, let the following condition

$$\sum_{j=1}^{\infty} \frac{y_j}{1+|z_j|^2} < \infty, \text{where } y_j = \mathrm{Im}\{z_j\}, \quad (1)$$

is satisfied. The holomorphic function on $\Pi^+$ defined with

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$$\mathrm{B}(z) = \omega_{\mathrm{i}}(z)^m \prod_{j=1}^{\infty} \frac{|z_j^2+1|}{z_j^2+1} \omega_{z_j}(z), \tag{2}$$

where $\omega_w(z) = \frac{z-w}{z-\bar{w}}$ , for arbitrary $w \in \Pi^+$, is called Blaschke product on the upper half space with zeroes in $z_j, j \in N$.

For the definition of the desired rational functions, again one considers finite or infinite sequences of complex numbers with imaginary part not equal to 1, moreover such that some of the elements of the sequence may be the same and some of them may be ∞ (in the last case we impose its' imaginary part to be 0). For that kind of sequence, denoted again with $(z_j)_{j\in N}$ , consider the rational functions

$$\mathrm{R_j(z)} = \frac{c_0 z^p + c_1 z^{p-1} + \cdots + c_p}{(z-z_1)(z-z_2)\ldots(z-z_j)} \ , \mathrm{z} \in \Pi^+, \tag{3}$$

with poles some of the complex numbers of $(z_j)_{j\in N}$, and $c_j \in C, j \in N,$ is another sequence. Additionally, in order equation (3) to make sense in the case when some of the elements of the sequence equal ∞, one use the convention: if $z_k = \infty$, then one sets $z - z_k = 1$ in (3).

Denote by $z_j', j \in N,$ all of the elements from $(z_j)_{j\in N}$ such that $\mathrm{Im}\, z_j > 0$ , and respectively $z_j'', j \in N$, the elements from $(z_j)_{j\in N}$ such that $\mathrm{Im}\, z_j < 0$. One associate the following sums for the previously defined couple of sequences

$$\mathrm{S}_{\mathrm{k}}' = \sum_{j=1}^{k} \frac{\mathrm{Im}\, z_j'}{1+|z_j'|^2} \quad \text{and} \quad \mathrm{S}_k'' = \sum_{j=1}^{k} \frac{-\mathrm{Im}\, z_j''}{1+|z_j''|^2}.$$

On these sequences we impose the conditions

$$\overline{\lim_{k\to\infty}} \mathrm{S}_{\mathrm{k}}' < \infty, \qquad \overline{\lim_{k\to\infty}} \mathrm{S}_{\mathrm{k}}'' = \infty. \tag{4}$$

Denote by $\mathrm{B_k}$ the Blaschke product with zeros , $\mathrm{z}_1, \mathrm{z}_2, \ldots z_k$ from the sequence numbers $(z_j)_{j\in N}$ for $k \in N$. If the condition (4) is satisfied, then the function $\lim_{k\to\infty} \log|B_k(z)|$ is a subharmonic function on the upper half space and not equal to -∞. If $\mu(z) = \lim_{k\to\infty} \log|B_k(z)|$ and $u(z)$ is harmonic majorant of $\mu(z)$ on upper half space one consider another function $\phi(\mathrm{z}) = \mathrm{e}^{\mathrm{u(z)+v(z)}}$, where $\mathrm{v(z)}$ stands for the harmonic conjugate of $u(z)$.

## 2 Known Results

The previous section is actually preparation for the classical results that follow. We state them without proof. We use the notation introduced in previous section.

**Theorem 1.** [4] We assume that condition (4) is satisfied. For an arbitrary continuous function F on R there exist a sequence $(R_j)_{j\in N}$ of rational functions given in (3) which converges to F, uniformly on R if and only if F coincides on R with the boundary value of the holomorphic function $\tilde{F}$ on $\Pi^+$ given with

$$\tilde{F}(\mathrm{z}) = \frac{\psi(\mathrm{z})}{\mathrm{B(z)}\phi(\mathrm{z})} \ , \mathrm{z} \in \Pi^+, \tag{5}$$

ψ is any bounded analytic function on $\Pi^+$.

Let $\sigma$ be a nondecreasing function of bounded variation on R. The space $L^p(d\sigma, R), p > 0$ denotes the space of set of all complex valued functions F, such that $\int_R |F(x)|^p d\sigma(x) < \infty$. Here $\sigma$ stands for a nondecreasing, function of bounded variation on $R$ (note that the integral is Lebesque –Stiltjes integral).

In order to formulate the next result, one needs to impose another condition concerning the growth of $\sigma$. Indeed one needs

$$-\infty < \int_R \frac{\log \sigma^+(x)}{1+x^2}. \tag{6}$$

Finally we are able to formulate the main result which we transfer in tempered distribution setting. We state the classical result again without proof. The notation is the same as in the previous section [6].

**Theorem 2**. We assume that conditions (4) and (6) are fulfilled and let $F \in L^p(d\sigma, R), p > 0$. Then there exist a sequence $(R_j)_{j\in N}$ of rational functions given in the form (3) such that

$\lim_{j\to\infty} \int_R |F(x) - R_j(x)|^p d\sigma(x) = 0$ if and only if the function F coincide with the boundary value of a holomorphic function $\tilde{F}$ on $\Pi^+$, almost everywhere on R, where $\tilde{F}$ is given in form (5). The functions B and φ are as in Theorem 1, ψ is analytic function on the upper half space and in $N^+$.

*Distributions.* The $S = S(R^n)$ is the space of infinitely differentiable complex valued function $\varphi$ on $R^n$ which satisfy the conditions $\sup_{t\in R^n} |t^\beta D^\alpha \varphi(t)| < \infty$

for every $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n), \alpha_i, \beta_i \in N_0, i = 1,2 \ldots n$. The space $S = S(R^n)$ is a Frechetsp spase which implies that the convergence can be considered in the following way: $\varphi_\lambda \to \varphi_{\lambda_0}$, where $\varphi_\lambda \in S$, in $S$ as $\lambda \to \lambda_0$ if and only if

$$\lim_{\lambda\to\lambda_0} \sup_{t\in R^n} |t^\beta D_t^\alpha[\varphi_\lambda(t) - \varphi(t)]| = 0$$

for every $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n), \alpha_i, \beta_i \in N_0, i = 1,2 \ldots n$.

The space $S'(R^n)$ is the space of all continuous, liear functionals on $S(R^n)$.

The space $S'(R^n)$ is called the space of tempered distributions. Unless stated differently, it is imposed that the space $S'(R^n)$ is endowed with the strong topology. Also, we use the following standard conventions about distributions: $\langle T, \varphi\rangle = T(\varphi)$ for the value of the functional Tacting on the function $\varphi$, and standard embedding of locally inegrable functions: for $f(x) \in L^1_{loc}(R^n)$ one can define the linear and continuous functional $T_f$ on $S(R^n)$ with

$$\langle T_f, \varphi\rangle = \int_{R^n} f(t)\varphi(t)dt, \varphi \epsilon S.$$

By the definition, $T_f \in S'(R^n)$ and that distribution is called tempered (regular) associated to the function f.
Now we state the main result of the article.

**Theorem 3.** Let $(z_j)_{j\in N}$ is a sequence of complex numbers which satisfy (4) and $\tilde{F}$ be of the form (5) from Theorem 2. Denote with $T_F, F \in L^p(R)$, the tempered distribution in associated with the boundary value F(x) of $\tilde{F}(z)$ on $\Pi^+$.

Then there exists a sequence, $(R_j)_{j\in N}$ of rational functions of the form (3) and, respectively, a sequence $(T_{R_j})_{j\in N}$, $T_{R_j} \in S'$ generated by $R_j$, satisfying

(i) $T_{R_j} \to T_F,\ j \to \infty,\ \text{in}\ S'$
(ii) $\overline{\lim_{n\to\infty}} \int_{-\infty}^{\infty} |R_j(x)|^p |\varphi(x)|\, dx < \infty$, for all $\varphi \epsilon S$.

**Proof.**

(i) The general idea is to prove the convergence in $S'$ in weak sense and to use Banach Steinhaus theorem to obtain the strong convergence (note that the space $S$ is Montel space) [5].

We start with applying Theorem 2. For the function $F$ one obtains a sequence $(R_j)_{j\in N}$ of rational functions of the form (3) for which the following holds

$$\lim_{j\to\infty} \int_R |F(x) - R_j(x)|^p\, dx = 0.$$

Triangle inequality implies

$$\|R_j\|_{L^p(d\sigma)} = (\int_R |R_j(x)|^p d\sigma(x))^{\frac{1}{p}} = (\int_R |R_k(x) - F(x) + F(x)|^p d\sigma(x))^{\frac{1}{p}}$$

$$\leq (\int_R |R_j(x) - F(x)|^p d\sigma(x))^{\frac{1}{p}} + (\int_R |F(x)|^p d\sigma(x))^{\frac{1}{p}} < C,$$

hence $R_j(x) \in L^p(d\sigma, R)$.

Now choose arbitrary $\varphi \epsilon S$ and fix it. We denote with q the Hölder conjugate of p, i.e.

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We can estimate as follows

$$\left|\int_R f(x)\varphi(x)dx\right| \leq \int_R |f(x)\varphi(x)dx| \leq (\int_R |f(x)|^p dx)^{\frac{1}{p}} (\int_R |\varphi(x)|^q dx)^{\frac{1}{q}},$$

for arbitrary function $f \in L^p, \varphi \in S$.

Using the previous estimate one obtains

$$\left|\langle T_{R_j}, \varphi\rangle - \langle T_F, \varphi\rangle\right| = \left|\int_{-\infty}^{\infty} R_j(x)\ \ \varphi(x)dx - \int_{-\infty}^{\infty} F(x)\varphi(x)dx\right|$$

$$= \left|\int_{-\infty}^{\infty} [R_j(x) - F(x)]\varphi(x)dx\right| \leq \int_{-\infty}^{\infty} |R_j(x) - F(x)||\varphi(x)|dx$$

$$\leq \left(\int_R |R_j(x) - F(x)|^p dx\right)^{\frac{1}{p}} \left(\int_R |\varphi(x)|^q dx\right)^{\frac{1}{q}}$$

$$\leq M\left(\int_R |R_j(x) - F(x)|^p dx\right)^{\frac{1}{p}} \to 0,$$

when $j \to \infty$.

In the previous calculations we used $M = \sup\{|\varphi(x)| | x \in R\}$ which is obviously finite since the inclusion $S \subset L^q$ is continuous and dense for arbitrary $1 \leq q < \infty$. The discussion on the start of the proof implies the claim or $T_{R_j} \to T_F,\ j \to \infty\ in\ S'$ in the strong topology.

(ii) Let $\varphi\epsilon S$ be arbitrary and fixed. The Minkowski inequality implies

$$\left(\int_R \left|R_j(x)\right|^p \varphi(x)dx\right)^{\frac{1}{p}} \leq M^{\frac{1}{p}}\left(\int_R \left|R_j(x) - F(x) + F(x)\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq M^{\frac{1}{p}}\left(\int_R \left|R_j(x) - F(x)\right|^p dx\right) + M^{\frac{1}{p}}\left(\int_R |F(x)|^p dx\right)^{\frac{1}{p}} = I_1 + I_2.$$

The integral $I_1$ tends to 0 when $j \to \infty$ which implies that $\int_R \left|R_j(x)\right|^p \varphi(x)dx \leq M_\varphi^{\frac{1}{p}} \|F\|_p + C$, for arbitrary $j \in N$. The latter implies (ii).

## 3 Conclusion

We were able to consider (tempered) distribution variant of Tumarkin result concerning approximation with rational functions. We embed $L^p$ functions continuously into $S'$ in a natural way and consider analog result, but now for their representation in $S'$.

## Competing Interests

Authors have declared that no competing interests exist.

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