Hindawi
Advances in High Energy Physics
Volume 2020, Article ID 5193692, 8 pages
https://doi.org/10.1155/2020/5193692

Research Article

Non-Abelian Gravitoelectromagnetism and Applications at Finite Temperature

A. F. Santos,[1] **J. Ramos**,[2] **and Faqir C. Khanna**[3]

[1] *Instituto de Física, Universidade Federal de Mato Grosso, 78060-900, Cuiabá, Mato Grosso, Brazil*
[2] *Faculty of Science, Burman University, Lacombe, Alberta, Canada T4L 2E5*
[3] *Department of Physics and Astronomy, University of Victoria, BC, Canada V8P 5C2*

Correspondence should be addressed to A. F. Santos; alesandroferreira@fisica.ufmt.br

Received 22 January 2020; Revised 9 March 2020; Accepted 18 March 2020; Published 3 April 2020

Academic Editor: Michele Arzano

Studies about a formal analogy between the gravitational and the electromagnetic fields lead to the notion of Gravitoelectromagnetism (GEM) to describe gravitation. In fact, the GEM equations correspond to the weak-field approximation of the gravitation field. Here, a non-abelian extension of the GEM theory is considered. Using the Thermo Field Dynamics (TFD) formalism to introduce temperature effects, some interesting physical phenomena are investigated. The non-abelian GEM Stefan-Boltzmann law and the Casimir effect at zero and finite temperatures for this non-abelian field are calculated.

1. Introduction

The Standard Model (SM) is a non-abelian gauge theory with symmetry group $\mathscr{U}(1)\times\mathscr{SU}(2)\times\mathscr{SU}(3)$. SM describes theoretically and experimentally three of the four fundamental forces of nature, i.e., the electromagnetic, weak, and strong forces. The electromagnetism is a $\mathscr{U}(1)$ abelian gauge theory which has been tested to a high precision. The generalization of an abelian gauge theory to the non-abelian gauge theory was proposed by Yang and Mills [1]. The last one describes the electroweak unification and quantum chromodynamics. The electroweak interaction is described by an $\mathscr{SU}(2)\times\mathscr{U}(1)$ group and while the $\mathscr{SU}(3)$ group satisfies the quantum chromodynamics [2–4].

Gravity is not a part of SM. This implies that the SM is not a fundamental theory that describes all fundamental interactions of nature. In this paper, an extension of non-abelian gravity is discussed. Some applications of such a theory are developed. The gravitational theory studied here is the Gravitoelectromagnetism (GEM). GEM is an approach based on describing gravity in a way analogous to the electromagnetism [5–7]. Several studies about the GEM theory have been developed [8–14]. These ideas arise from the analogy between equations for the Newton and Coulomb laws and the interest has increased with the discovery of the Lense-Thirring effect, where a rotating mass generates a gravitomagnetic field [15–17]. Some experiments that study this effect have been developed, such as LAGEOS (Laser Geodynamics Satellites) and LAGEOS 2 [18], the Gravity Probe B [19], and the mission LARES (Laser Relativity Satellite) [20, 21].

The GEM theory may be analyzed by three different approaches: (i) using the similarity between the linearized Einstein and Maxwell equations [22], (ii) a theory based on an approach using tidal tensors [23], and (iii) the decomposition of the Weyl tensor (C_{ijkl}) into $\mathscr{B}_{ij}=1/2\epsilon_{ikl}C^{kl}_{0j}$ and $\mathscr{E}_{ij}=-C_{0i0j}$, the gravitomagnetic and gravitoelectric components, respectively [24]. In this paper, the Weyl tensor approach is used. A Lagrangian formulation for GEM is developed [25], and a gauge transformation in GEM is studied [26]. Here, an extension to non-abelian GEM fields is introduced. Applications of the non-abelian GEM at finite temperature are investigated. The temperature effects are introduced using Thermo Field Dynamics (TFD) formalism.

There are two ways to introduce the temperature effect: (i) using the imaginary time formalism [27] and (ii) using the real-time formalism [28–36]. In this paper, TFD formalism is chosen. It is a real-time finite temperature formalism. In this formalism, a thermal state is developed where the main objective is to interpret the statistical average of an arbitrary operator as an expectation value in a thermal vacuum. Two elements are necessary to construct this thermal state: (i) doubling of the original Hilbert space and (ii) the use of Bogoliubov transformations. These are two Hilbert spaces, the original space S and the tilde space $\tilde{S}$, which are related by a mapping, called the tilde conjugation rules, while the Bogoliubov transformation consists in a rotation involving these two spaces that ultimately introduce the temperature effects.

The Stefan-Boltzmann law and the Casimir effect for the non-abelian GEM field at finite temperature are calculated. The Stefan-Boltzmann law describes the power radiated from a black body in terms of its temperature. The Casimir effect, proposed by H. Casimir [37], is a quantum phenomenon that appears due to vacuum fluctuations of any quantum field. The results in this case may be at zero or finite temperatures.

This paper is organised as follows. In section II, a brief introduction to the abelian GEM Lagrangian formalism is presented. In section III, an extension to non-abelian GEM field is developed. The energy-momentum tensor associated to the non-abelian gauge field is calculated. In section IV, the TFD formalism is introduced. In section V, some applications considering the non-abelian GEM field at finite temperature are analysed. (i) The Stefan-Boltzmann law is calculated. (ii) The Casimir effect at zero temperature is obtained, and (iii) the Casimir effect at finite temperature is calculated. In section VI, some concluding remarks are presented.

2. Lagrangian Formulation of Abelian Gem

In this section, an introduction to the Lagrangian formulation of abelian GEM is presented. The GEM field equations, Maxwell-like equations, are

$$\begin{aligned} \partial^i \mathcal{E}^{ij} &= -4\pi G \rho^j, \\ \partial^i \mathcal{B}^{ij} &= 0, \\ \epsilon^{\langle ikl}\partial^k \mathcal{B}^{lj\rangle} + \frac{1}{c}\frac{\partial \mathcal{E}^{ij}}{\partial t} &= -\frac{4\pi G}{c} J^{ij}, \\ \epsilon^{\langle ikl}\partial^k \mathcal{E}^{lj\rangle} + \frac{1}{c}\frac{\partial \mathcal{B}^{ij}}{\partial t} &= 0, \end{aligned} \tag{1}$$

where G is the gravitational constant, ε^{ikl} is the Levi-Civita symbol, ρ^j is the vector mass density, J^{ij} is the mass current density, and c is the speed of light. The quantities $\mathcal{E}^{ij}$, $\mathcal{B}^{ij}$, and J^{ij} are the gravitoelectric field, the gravitomagnetic field, and the mass current density, respectively. The symbol $\langle \cdots \rangle$ denotes symmetrization of the first and last indices, i.e., i and j.

The fields $\mathcal{E}^{ij}$ and $\mathcal{B}^{ij}$ are expressed in terms of a symmetric rank-2 tensor field, $\tilde{\mathcal{A}}$, with components $\mathcal{A}^{ij}$, such that

$$\begin{aligned} \mathcal{B} &= \operatorname{curl} \tilde{\mathcal{A}}, \\ \mathcal{E} + \frac{1}{c}\frac{\partial \tilde{\mathcal{A}}}{\partial t} &= -\operatorname{grad} \varphi, \end{aligned} \tag{2}$$

where φ is the GEM counterpart of the electromagnetic (EM) scalar potential ϕ.

Defining $\mathcal{F}^{\mu\nu\alpha}$ as the gravitoelectromagnetic tensor, the GEM field equations become

$$\begin{aligned} \partial_\mu \mathcal{F}^{\mu\nu\alpha} &= \frac{4\pi G}{c} \mathcal{J}^{\nu\alpha}, \\ \partial_\mu \mathcal{G}^{\mu\langle\nu\alpha\rangle} &= 0, \end{aligned} \tag{3}$$

where $\mathcal{J}^{\nu\alpha}$ depends on quantities ρ^i and J^{ij} that are the mass and the current density, respectively. In addition, the gravitoelectromagnetic tensor is defined as

$$\mathcal{F}^{\mu\nu\alpha} = \partial^\mu \mathcal{A}^{\nu\alpha} - \partial^\nu \mathcal{A}^{\mu\alpha}, \tag{4}$$

and the dual GEM tensor is defined as

$$\mathcal{G}^{\mu\nu\alpha} = \frac{1}{2}\epsilon^{\mu\nu\gamma\sigma}\eta^{\alpha\rho} - \mathcal{F}_{\gamma\sigma\rho}. \tag{5}$$

Using these definitions, the GEM Lagrangian density is given as [25].

$$\mathcal{L}_G = -\frac{1}{16\pi}\mathcal{F}_{\mu\nu\alpha}\mathcal{F}^{\mu\nu\alpha} - \frac{G}{c}\mathcal{J}^{\nu\alpha}\mathcal{A}_{\nu\alpha}. \tag{6}$$

This Lagrangian allows considering several gravitational applications involving the graviton, such as interactions with other fundamental particles. This makes it possible to study several related topics.

In this way, the GEM theory is described by two fields $\mathcal{E}^{ij}$ and $\mathcal{B}^{ij}$, which are symmetric and traceless tensors of the second order. These fields can be expressed in terms of the symmetric gravitoelectromagnetic potential $A_{\mu\nu}$ [25, 26], analogous to that of electromagnetism A_μ. Thus, $A_{\mu\nu}$ is the fundamental field in GEM and naturally, it has two indices [25, 26].

It is important to note that GEM equations correspond to the weak-field approximation of General Relativity. They do not describe strong fields and, therefore, do not include the full Einstein equations. To be more specific, the abelian GEM corresponds to the linear part of Einstein equations and the non-abelian GEM corresponds up to the second order in the weak-field approach.

3. Non-Abelian Gem

Let us consider an extension of the GEM field to include the non-abelian gauge transformations [38]. Then, in this section, the Lagrangian for the non-abelian GEM field is presented and the energy-momentum tensor associated to the non-abelian field is calculated.

In order to obtain the non-abelian gauge transformation for the GEM field, let us investigate the Dirac Lagrangian under global and local gauge transformations. The free Dirac Lagrangian is given as

$$\mathscr{L}_D = -i\bar{\psi}(x)\gamma_\mu\partial^\mu\psi(x) + m\bar{\psi}(x)\psi(x), \tag{7}$$

where $\psi(x)$ is a two-component column vector. This Lagrangian is invariant under global $\mathcal{SU}(2)$ gauge transformation given as

$$\psi'(x) = U\psi(x), \tag{8}$$

with U being a 2×2 unitary matrix that is written as $U = e^{iH}$, where H is a hermitian matrix. To study local gauge transformation, more details are necessary.

Let us assume that the local gauge transformation is

$$\psi'(x) = U(x)\psi(x) = e^{igH(x)}\psi(x), \tag{9}$$

where g is the coupling constant and $H(x)$ is the hermitian 2×2 matrix given by

$$H(x) = \sigma\cdot\mathbf{a}(x), \tag{10}$$

with $\mathbf{a}(x)$ being real functions of x and σ are Pauli matrices. The Pauli matrices $\sigma^i(\mathrm{i}=1,2,3)$ are the generators of the non-abelian group $\mathcal{SU}(2)$ satisfying the commutation relations $[\sigma^i,\sigma^j] = 2i\varepsilon^{ijk}\sigma^k$. In a more compact form, $\mathbf{a}(x)$ is written as

$$\mathbf{a}(x) = p_\alpha\mathbf{b}^\alpha(x), \tag{11}$$

where $\mathbf{b}^\alpha(x)$ are vectors associated to each of the four directions in Minkowski spacetime and p_α are the components of the one-form $\tilde{p}$. Then, the local gauge transformation becomes

$$\psi'(x) = e^{igp_\alpha\sigma\cdot\mathbf{b}^\alpha(x)}\psi(x). \tag{12}$$

The Dirac Lagrangian is not invariant under this local gauge transformation since the derivative $\partial^\mu\psi'(x)$ introduces a new term in the Lagrangian. In order to obtain an invariant Lagrangian, a covariant derivative is defined as

$$D^\mu = \partial^\mu - igp_\alpha\sigma\cdot\mathbf{A}^{\mu\alpha}(x), \tag{13}$$

where the tensor gauge field $\mathbf{A}^{\mu\alpha}(x)$ has three components $\mathbf{A}^{\mu\alpha}(x) = (A_1^{\mu\alpha}(x), A_2^{\mu\alpha}(x), A_3^{\mu\alpha}(x))$ and it transforms as

$$\mathbf{A}'^{\mu\alpha}_k(x) = A_k^{\mu\alpha}(x) + \partial^\mu b_k^\alpha + 2g\varepsilon^{ijk}p_\beta A_i^{\mu\beta}b_j^\alpha. \tag{14}$$

An important note, there is one tensor gauge field $A_i^{\mu\alpha}(x)$ for each generator σ^i of the group $\mathcal{SU}(2)$. Moreover, in the definition of the covariant derivative D^μ (Equation (13)), the gauge field $A_{\mu\nu}$ should appear to keep the local gauge invariance like in electromagnetism. In order to have it, the oneform p_α is introduced [26]. The one form makes the phase function to split into phase factors each associated with one of the four directions in spacetime.

Using these results and replacing the derivative ∂^μ by the covariant derivative D^μ, the Dirac Lagrangian is gauge invariant, i.e.,

$$\mathscr{L}_D = -i\bar{\psi}(x)\gamma_\mu D^\mu\psi(x) + m\bar{\psi}(x)\psi(x). \tag{15}$$

In this formulation, three new gauge tensor fields are introduced. To write a full Lagrangian invariant under local gauge transformation, a kinetic term of $\mathbf{A}^{\mu\alpha}(x)$ must be constructed. To do that, an analogue of the electromagnetic tensor $F_{\mu\nu}$ is constructed. For obtaining the antisymmetric third-rank tensor of the gauge field, let us consider a covariant derivative (13). Then,

$$[D^\mu, D^\nu] = -igp_\alpha\sigma\cdot\mathbf{F}^{\mu\nu\alpha}, \tag{16}$$

where

$$F_k^{\mu\nu\alpha} = \partial^\mu A_k^{\nu\alpha} - \partial^\nu A_k^{\mu\alpha} + 2g\varepsilon^{ijk}p_\beta A_i^{\mu\alpha}A_j^{\nu\beta}. \tag{17}$$

Then, the full Lagrangian that is invariant under local $\mathcal{SU}(2)$ gauge transformations is

$$\mathscr{L}_D = -i\bar{\psi}(x)\gamma_\mu D^\mu\psi(x) + m\bar{\psi}(x)\psi(x) - \frac{1}{16\pi}\mathbf{F}_{\mu\nu\alpha}\cdot\mathbf{F}^{\mu\nu\alpha}. \tag{18}$$

This Lagrangian describes two equal mass Dirac fields interacting with three massless tensor gauge fields.

In conclusion, GEM is an approach based on formulating gravity in analogy to electromagnetism. In this way, GEM becomes a gauge field theory of gravity in contrast with the geometric theory of General Relativity. Then, it is expected that $\mathcal{SU}(2)$ be the gauge symmetry group. It is the Weyl tensors $\mathscr{E}^{ij}$ and $\mathscr{B}^{ij}$ that keep the connection of GEM to gravity.

Now, let us determine the energy-momentum tensor associated with the non-abelian GEM field.

3.1. Energy-Momentum Tensor for Non-Abelian GEM. Hereafter, the Lagrangian density for the free non-abelian GEM field is considered, i.e.,

$$\mathscr{L} = -\frac{1}{16\pi}F^a_{\mu\nu\alpha}F^{\mu\nu\alpha a} \tag{19}$$

The index a is summed over the generators of the gauge group and for an $\mathcal{SU}(N)$ group, one has $a, b, c = 1\cdots N^2-1$. Here, as a first application of the non-abelian GEM, the self-interaction between the tensor gauge fields is ignored.

Using the energy-momentum tensor definition,

$$T^{\mu\nu} = \frac{\partial\mathscr{L}}{\partial(\partial_\mu A^a_{\lambda\alpha})}\partial^\nu A^a_{\lambda\alpha} - \eta^{\mu\nu}\mathscr{L}, \tag{20}$$

the energy-momentum tensor associated with the non-abelian GEM field is

$$T^{\mu\nu} = \frac{1}{4}\left[-\mathcal{F}^{\mu a}_{\lambda\alpha}\mathcal{F}^{\nu\lambda\alpha a} + \frac{1}{4}\eta^{\mu\nu}\mathcal{F}^{a}_{\rho\sigma\theta}\mathcal{F}^{\rho\sigma\theta a}\right]. \quad (21)$$

To avoid a product of field operators at the same space-time point, the energy-momentum tensor is written as

$$T^{\mu\nu}(x) = \frac{1}{4\pi}\lim_{x'\to x}\left\{\tau\left[-\mathcal{F}^{\mu a}_{\lambda\alpha}(x)\mathcal{F}^{\nu\lambda\alpha a}\left(x'\right) + \frac{1}{4}\eta^{\mu\nu}\mathcal{F}^{a}_{\rho\sigma\theta}(x)\mathcal{F}^{\rho\sigma\theta a}\left(x'\right)\right]\right\} \quad (22)$$

where τ is the time order operator.

The quantization of the non-abelian GEM field requires that

$$\pi^{\kappa\lambda a} = \frac{\partial\mathcal{L}}{\partial(\partial_0 A^a_{\kappa\lambda})} = -\frac{1}{4\pi}F^{0\kappa\lambda a}. \quad (23)$$

the commutation relation0 and div $\tilde{A} = \partial_i A^{ij} = 0$, the covariant quantization is carried out and the commutation relation is

$$\left[A^{ija}(\mathrm{x},t), \pi^{klb}\left(\mathrm{x}',t\right)\right] = \frac{i}{2}\left[\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk} - \frac{1}{\nabla^2}\left(\delta^{jl}\partial^i\partial^k - -\delta^{jk}\partial^i\partial^l - \delta^{il}\partial^j\partial^k + \delta^{ik}\partial^j\partial^l\right)\right]\delta^3\left(\mathrm{x}-\mathrm{x}'\right)\delta^{ab}. \quad (24)$$

Other commutation relations are zero.

In order to write the energy-momentum tensor, let us consider

$$\tau\left[\mathcal{F}^{\alpha\kappa\gamma a}(x)\mathcal{F}^{\mu\nu\rho a}\left(x'\right)\right] = \mathcal{F}^{\alpha\kappa\gamma a}(x)\mathcal{F}^{\mu\nu\rho a}\left(x'\right)\theta\left(x_0 - x_0'\right) + \mathcal{F}^{\mu\nu\rho a}\left(x'\right)\mathcal{F}^{\alpha\kappa\gamma a}(x)\theta\left(x_0 - x_0'\right), \quad (25)$$

with $\theta(x_0 - x_{0'})$ being the step function. In the calculations that follow, we use the commutation relation, Equation (24), and

$$\partial^\mu\theta\left(x_0 - {x_0}'\right) = n_0^\mu\delta\left(x_0 - {x_0}'\right), \quad (26)$$

where $n_0^\mu = (1,0,0,0)$ is a time-like vector.

Using these definitions, the energy-momentum tensor for the non-abelian GEM field becomes

$$T^{\mu\nu}(x) = -\frac{1}{4\pi}\lim_{x'\to x}\left\{\Delta^{\mu\nu,\lambda\varepsilon\omega\upsilon}\left(x,x'\right)\tau\left[A^a_{\lambda\varepsilon}(x)A^a_{\omega\upsilon}\left(x'\right)\right]\right\}, \quad (27)$$

where

$$\Delta^{\mu\nu,\lambda\varepsilon\omega\upsilon}\left(x,x'\right) = \Gamma^{\mu}{}_{\rho\alpha,}{}^{\nu\rho\alpha,\lambda\varepsilon\omega\upsilon}\left(x,x'\right) - \frac{1}{4}\eta^{\mu\nu}\Gamma^{\rho\sigma\theta,}{}_{\rho\sigma\theta,}{}^{\lambda\varepsilon\omega\upsilon}\left(x,x'\right), \quad (28)$$

with

$$\Gamma^{\alpha\kappa\gamma,\mu\nu\rho,\lambda\varepsilon\omega\upsilon}\left(x,x'\right) = \left(g^{\kappa\lambda}g^{\varepsilon\gamma}\partial^\alpha - g^{\alpha\lambda}g^{\varepsilon\gamma}\partial^\kappa\right)\cdot\left(g^{\nu\omega}g^{\rho\upsilon}\partial'^\mu - g^{\mu\omega}g^{\rho\upsilon}\partial'^\nu\right). \quad (29)$$

The vacuum expectation value of the energy-momentum tensor leads to the expression

$$\langle T^{\mu\nu}(x)\rangle = \langle 0 \mid T^{\mu\nu}(x) \mid 0\rangle = -\frac{1}{4\pi}\lim_{x'\to x}\left\{\Delta^{\mu\nu,\lambda\varepsilon\omega\upsilon}\left(x,x'\right)\left\langle 0\left|\tau\left[A^a_{\lambda\varepsilon}(x)A^a_{\omega\upsilon}\left(x'\right)\right]\right|0\right\rangle\right\}, \quad (30)$$

where the graviton propagator is

$$\left\langle 0\left|\tau\left[A^a_{\lambda\varepsilon}(x)A^a_{\omega\upsilon}\left(x'\right)\right]\right|0\right\rangle = \delta^{ab}\left\langle 0\left|\tau\left[A^a_{\lambda\varepsilon}(x)A^b_{\omega\upsilon}\left(x'\right)\right]\right|0\right\rangle = i\delta^{ab}D^{ab}_{\lambda\varepsilon\omega\upsilon}\left(x-x'\right), \quad (31)$$

with

$$D^{ab}_{\lambda\varepsilon\omega\upsilon} = \frac{1}{2}\delta^{ab}(g_{\lambda\omega}g_{\varepsilon\upsilon} + g_{\lambda\upsilon}g_{\varepsilon\omega} - g_{\lambda\varepsilon}g_{\omega\upsilon})G_0\left(x-x'\right), \quad (32)$$

and $G_0(x-x')$ is the massless scalar field propagator. Then, the vacuum expectation value of $T^{\mu\nu}(x)$ becomes

$$\langle T^{\mu\nu}(x)\rangle = -\frac{3i}{8\pi}\lim_{x'\to x}\left\{\Gamma^{\mu\nu}\left(x,x'\right)G_0\left(x-x'\right)\right\}, \quad (33)$$

with

$$\Gamma^{\mu\nu}\left(x,x'\right) = 8\left(\partial^\mu\partial'^\nu - \frac{1}{4}\eta^{\mu\nu}\partial^\rho\partial'_\rho\right). \quad (34)$$

Now, the main objective is to study the effects due to temperature and spatial compactification in Equation (33). To achieve such an objective, the Thermo Field Dynamics formalism is used.

4. Thermo Field Dynamics (TFD) Formalism

Here, the Thermo Field Dynamics (TFD) formalism is introduced. TFD is a quantum field theory at finite temperature [31–36]. In this formalism, the statistical average of any operator is equal to its expected value in a thermal vacuum. For this equality to be true, two main elements are required, i.e., (i) doubling of the original Hilbert space and (ii) the Bogoliubov transformation.

This doubling is defined as $\mathcal{S}_T = \mathcal{S} \otimes \tilde{\mathcal{S}}$, where $\tilde{\mathcal{S}}$ and $\mathcal{S}$ are the tilde and original Hilbert space, respectively. The Bogoliubov transformation corresponds to a rotation of the tilde and non-tilde variables which introduces the thermal effects. To understand this doubling of Hilbert space, let us consider

$$\begin{pmatrix} d(\alpha) \\ \tilde{d}^\dagger(\alpha) \end{pmatrix} = \mathcal{B}(\alpha) \begin{pmatrix} d(k) \\ \tilde{d}^\dagger(k) \end{pmatrix}, \tag{35}$$

where $\mathcal{B}(\alpha)$ is the Bogoliubov transformation given as

$$\mathcal{B}(\alpha) = \begin{pmatrix} u(\alpha) & -v(\alpha) \\ -v(\alpha) & u(\alpha) \end{pmatrix} \tag{36}$$

with

$$v^2(\alpha) = (e^{\alpha\omega} - 1)^{-1}, \; u^2(\alpha) = 1 + v^2(\alpha). \tag{37}$$

The parameter α is the compactification parameter defined by $\alpha = (\alpha_0, \alpha_1, \cdots \alpha_{D-1})$ and ω is energy. The temperature effect is described by the choice $\alpha_0 \equiv \beta$ and $\alpha_1, \cdots \alpha_{D-1} = 0$. In this case, with $\alpha = \beta$, the quantities $v^2(\beta)$ and $u^2(\beta)$ are related to the Bose distribution.

In order to introduce an application of TFD formalism, let us consider the free scalar field propagator. Then, in a doublet notation, it is given as

$$G_0^{(ab)}\left(x - x' ; \alpha\right) = i\left\langle 0, \tilde{0} \right| \tau\left[\phi^a(x;\alpha)\phi^b\left(x';\alpha\right)\right]\left| 0, \tilde{0}\right\rangle, \tag{38}$$

where $\phi(x;\alpha) = \mathcal{B}(\alpha)\phi(x)\mathcal{B}^{-1}(\alpha)$ and $a, b = 1, 2$. Then,

$$G_0^{(ab)}\left(x - x' ; \alpha\right) = i\int \frac{d^4k}{(2\pi)^4} e^{-ik\left(x-x'\right)} G_0^{(ab)}(k;\alpha), \tag{39}$$

where

$$G_0^{(11)}(k;\alpha) \equiv G_0(k;\alpha) = G_0(k) + v^2(k;\alpha)[G_0(k) - G_0^*(k)], \tag{40}$$

with

$$G_0(k) = \frac{1}{k^2 - m^2 + i\varepsilon}, \qquad G_0(k) - G_0^*(k)] = 2\pi i \delta\left(k^2 - m^2\right). \tag{41}$$

The parameter $v^2(k;\alpha)$ is the generalized Bogoliubov transformation [39]. It is defined as

$$v^2(k;\alpha) = \sum_{s=1}^{d} \sum_{\{\sigma_s\}} 2^{s-1} \sum_{l_{\sigma_1},\cdots,l_{\sigma_s}=1}^{\infty} (-\eta)^{s+\sum_{r=1}^{s} l_{\sigma_r}} \exp\left[-\sum_{j=1}^{s} \alpha_{\sigma_j} l_{\sigma_j} k^{\sigma_j}\right], \tag{42}$$

with d being the number of compactified dimensions, $\eta = 1$ (-1) for fermions (bosons), $\{\sigma_s\}$ denotes the set of all combinations with s elements and k is the 4-momentum.

For the doubled notation, the vacuum expectation value of the energy-momentum tensor of the non-abelian GEM is

$$\left\langle T^{\mu\nu(ab)}(x;\alpha)\right\rangle = -\frac{3i}{8\pi} \lim_{x'\to x} \left\{\Gamma^{\mu\nu}\left(x, x'\right) G_0^{(ab)}\left(x - x';\alpha\right)\right\}. \tag{43}$$

In order to obtain a physical (renormalized) energy-momentum tensor, the standard Casimir prescription is used. Then,

$$\mathcal{T}^{\mu\nu(ab)}(x;\alpha) = \left\langle T^{\mu\nu(ab)}(x;\alpha)\right\rangle - \left\langle T^{\mu\nu(ab)}(x)\right\rangle. \tag{44}$$

In this form, a measurable physical quantity is given as

$$\mathcal{T}^{\mu\nu(ab)}(x;\alpha) = -\frac{3i}{8\pi} \lim_{x'\to x} \left\{\Gamma^{\mu\nu}\left(x, x'\right) \bar{G}_0^{(ab)}\left(x - x';\alpha\right)\right\}, \tag{45}$$

where

$$\bar{G}_0^{(ab)}\left(x - x';\alpha\right) = G_0^{(ab)}\left(x - x';\alpha\right) - G_0^{(ab)}\left(x - x'\right). \tag{46}$$

In the next section, some applications for different choices of parameter α are developed.

5. Some Applications

In this section, applications, which consider the temperature effects and spatial compactifications, are calculated.

5.1. Stefan-Boltzmann Law. As a first application, consider the thermal effect that appears for $\alpha = (\beta, 0, 0, 0)$. In this case, the generalized Bogoliubov transformation becomes

$$v^2(\beta) = \sum_{j_0=1}^{\infty} e^{-\beta k^0 j_0}. \tag{47}$$

Then, the Green function is given as

$$\begin{aligned} \bar{G}_0^{(11)}\left(x - x';\alpha\right) &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\left(x-x'\right)} \sum_{j_0=1}^{\infty} e^{-\beta k^0 j_0} [G_0(k) - G_0^*(k)], \\ &= 2\sum_{j_0=1}^{\infty} G_0\left(x - x' - i\beta j_0 n_0\right), \end{aligned} \tag{48}$$

where $n_0^\mu = (1,0,0,0)$. Then, the energy-momentum tensor at finite temperature is

$$T^{\mu\nu(11)}(x;\beta) = -\frac{6i}{\pi}\lim_{x'\to x}\left\{\sum_{j_0=1}^{\infty}\left(\partial^\mu\partial'^\nu - \frac{1}{4}g^{\mu\nu}\partial^\rho\partial'_\rho\right)G_0\left(x-x'-i\beta j_0 n_0\right)\right\}. \quad (49)$$

Using the Riemann Zeta function, i.e.,

$$\zeta(4) = \sum_{j_0=1}^{\infty}\frac{1}{j_0^4} = \frac{\pi^4}{90}, \quad (50)$$

the Stefan-Boltzmann law for the non-abelian GEM field is obtained as

$$E(T) \equiv \mathcal{T}^{00(11)}(x;\beta) = \frac{\pi}{10}T^4. \quad (51)$$

Note that the energy density of the non-abelian gauge fields is similar to the abelian field case.

Here, the numeric value is multiplied by the group generator number.

5.2. Casimir Effect at Zero Temperature. Here, $\alpha = (0,0,0,iL)$ is chosen and the Bogoliubov transformation is

$$v^2(L) = \sum_{l_3=1}^{\infty}e^{-iLk^3 l_3}. \quad (52)$$

The Green function is

$$\bar{G}_0^{(11)}\left(x-x';L\right) = 2\sum_{l_3=1}^{\infty}G_0\left(x-x'-Ll_3 z\right). \quad (53)$$

A sum over l_3, for $L=2d$, defines the nontrivial part of the Green function with the Dirichlet boundary condition. With these conditions, the energy-momentum tensor becomes

$$\mathcal{T}^{\mu\nu(11)}(x;d) = -\frac{6i}{\pi}\lim_{x\to x'}\left\{\sum_{l_3=1}^{\infty}\left(\partial^\mu\partial'^\mu - \frac{1}{4}g^{\mu\nu}\partial^\rho\partial'_\rho\right)G_0\left(x-x'-2dl_3 z\right)\right\}. \quad (54)$$

For $\mu=\nu=0$, the Casimir energy to the non-abelian field case is

$$E(d) \equiv \mathcal{T}^{00(11)}(x;d) = -\frac{\pi}{480d^4}, \quad (55)$$

and for $\mu=\nu=3$, the Casimir pressure for the non-abelian GEM field is

$$P(d) \equiv \mathcal{T}^{33(11)}(x;d) = -\frac{\pi}{160d^4}. \quad (56)$$

The negative sign shows that the Casimir force between the plates is attractive, similar to the case of the electromagnetic field and of the abelian GEM field.

5.3. Casimir Effect at Finite Temperature. For $\alpha = (\beta,0,0,i2d)$, the temperature effects and spatial compactifications are considered. In this case, the Bogoliubov transformation becomes

$$\begin{aligned} v^2\left(k^0,k^3;\beta,d\right) &= v^2\left(k^0;\beta\right) + v^2\left(k^3;d\right) + 2v^2\left(k^0;\beta\right)v^2\left(k^3;d\right) \\ &= \sum_{j_0=1}^{\infty}e^{-\beta k^0 j_0} + \sum_{l_3=1}^{\infty}e^{-iLk^3 l_3} + 2\sum_{j_0,l_3=1}^{\infty}e^{-\beta k^0 j_0 - iLk^3 l_3}. \end{aligned} \quad (57)$$

The Green function, corresponding to the first two terms, is given in Equation (48) and in Equation (53), respectively. The Green function associated with the third term is

$$\bar{G}_0^{(11)}\left(x-x';\beta,d\right) = 2\int\frac{d^4k}{(2\pi)^4}e^{-ik(x-x')}\sum_{j_0,l_3=1}^{\infty}e^{-\beta k^0 j_0 - iLk^3 l_3}\left[G_0(k) - G_0^*(k)\right] = 4\sum_{j_0,l_3=1}^{\infty}G_0\left(x-x'-i\beta j_0 n - 2dl_3 z\right). \quad (58)$$

Then, the Casimir energy and pressure at finite temperature are given, respectively, by

$$\begin{aligned} E(\beta,d) = \mathcal{T}^{00(11)}(\beta,d) &= \frac{\pi}{10\beta^4} - \frac{\pi}{480d^4} \\ &+ \frac{6}{\pi^3}\sum_{j_0,l_3=1}^{\infty}\frac{3(\beta j_0)^2 - (2dl_3)^2}{\left[(\beta j_0)^2 + (2dl_3)^2\right]^3}, \end{aligned}$$

$$\begin{aligned} P(\beta,d) = \mathcal{T}^{33(11)}(\beta,d) &= \frac{\pi}{30\beta^4} - \frac{\pi}{160d^4} \\ &+ \frac{6}{\pi^3}\sum_{j_0,l_3=1}^{\infty}\frac{(\beta j_0)^2 - 3(2dl_3)^2}{\left[(\beta j_0)^2 + (2dl_3)^2\right]^3}. \end{aligned} \quad (59)$$

Note that the first and second terms are the Stefan-Boltzmann law and Casimir effect at zero temperature, respectively, while the third term corresponds to the Casimir effect at finite temperature.

In the last case, both effects, temperature and spatial compactification, are present.

6. Conclusion

The non-abelian GEM field is investigated. First, the Lagrangian formulation for the abelian GEM field is presented. Then, using the principle of local gauge invariance, an extension of the non-abelian GEM field is constructed. The symmetry group for the non-abelian GEM is group $\mathcal{SU}(2)$. The abelian and non-abelian GEMs have a correspondence with the weak-field approach of General Relativity. The abelian GEM has a structure equivalent to the weak-field approximation of first-order and non-abelian Weyl GEM is equivalent to the weak-field approximation up to the second order. For simplicity, the self-interaction terms of the non-abelian gauge field are ignored. Then, the energy-momentum tensor is calculated. The TFD formalism is used to introduce thermal effects. This formalism requires two basic ingredients: the doubling of the Hilbert space and the Bogoliubov transformation. With this formalism, the vacuum expectation value of the energy-momentum tensor is obtained and thus, some applications for the non-abelian GEM field are investigated. The Stefan-Boltzmann law and the Casimir effect at finite temperature are calculated. Our results show that the non-abelian quantities are similar to the abelian quantities. The main difference consists in the fact that the non-abelian results are equal to the abelian result multiplied by the number of gauge fields. These results are similar to the case of the electromagnetic field. For example, the non-abelian GEM Casimir effect is attractive as the electromagnetic case. In addition, calculations involving the $\mathcal{SU}(2)$ group and GEM have not been done in the literature. This work is the first to introduce this type of approach.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work by A. F. S. is supported by CNPq projects 308611/2017-9 and 430194/2018-8.

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